INTERNATIONAL INSTITUTE
FOR
AERIAL SURVEY
AND
EARTH SCIENCES (ITC)
ENSCHEDE,THE NETHERLANDS

S–TRANSFORMATIONS AND ARTIFICIAL COVARIANCE MATRICES IN PHOTOGRAMMETRY

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ISP COMMISSION III, PRESENTED PAPER

FOURTEENTH CONGRESS OF THE INTERNATIONAL SOCIETY FOR PHOTOGRAMMETRY, HAMBURG 1980

S-transformations and criterion matrices in photogrammetry

At present the densification of geodetic point fields, by means of aero-triangulation seems to reach a precision comparable to the precision of terrestrial methods. This fact requires another attitude of photogrammetrists with respect to ground control, which in the future cannot be considered as non-stochastic anymore.
Criterion matrices (artificial covariance matrices) could be useful then to describe the precision of given ground control points and to evaluate the precision of point field densifications. S-transformations will give a good tool to link up photogrammetric blocks with given geodetic point fields. This paper will discuss the concept of criterion matrices and S-transformations and their possible application in photogrammetry.

Die Genauigkeit, die sich bei der Verdichtung von Festpunktnetzen mittels Aerotriangulation mittlerweilen erreichen lässt, steht derjenigen aus terrestrichen Methoden kaum nach. Dies verlangt nach einer anderen Einstellung der Photogrammetristen gegenüber den Passpunkte, die künftig nicht mehr als fehlerfrei betrachtet werden können.
Kriterium Matrizen (künstliche Koverianz Matrizen) könnten dazu dienen, um die Genauigkeit der gegeben Festpunkte zu beschreiben und um die Genauigkeit der Netzverdichtung zu beurteilen. S-Transformationen bieten eine gute Möglichkeit photogrammetrische Blocks mit dem gegebenen Punktfeld zu verbinden. Der vorliegende Artikel erläutert das Konsept der Kriterium Matrizen und S-Transformation sowie deren mögliche Anwendung in der Photogrammetrie.

La densification des résaux géodésiques, au moyen de l'aérotriangulation, semble atteindre actuellement une précision comparable à celle des méthodes terrestres. Ce fait nécessite une autre attitude des photogrammètres quant au canevas terrestre qui ne pourra plus être considéré comme non-stochastique.
Des matrices-critère (matrices de covariance artificielles) pourraient alors être utiles pour décrire la précision des points d'appui donnés et pour évaluer la précision de densifications de points terrestres. Les transformations "S-systèmes" peuvent donner un bon outil pour relier des blocs photogrammétriques avec des résaux géodésiques donnés. Cet article discutera le concept des matrices-critère et des transformations "S-systèmes" et leur application possible en photogrammétrie.

1. A sketch of the problem

The densification of geodetic pointfields has tradionally been based on the assumption that it could start from given points, which are not stochastic. These points were determined by a higher order survey, for which the obtained precision was considered much better than that of the densification survey.
Experience with modern survey equipment and methods, shows however, that in many cases this assumption cannot be maintained any longer. Moreover, given coordinates should be entered as stochastic variates (observations) in the adjustment of lower order networks. This complicates the adjustment considerably and a revision of computing methods seems to be necessary. An adjustment in two steps seems to be a proper answer to these problems. In the first step the network is adjusted without given points. This means that the strength of the "free" network is analysed and the search for gross observational errors is not effected by given coordinates. The coordinates of points in the network are then computed in a local system (see [3]). The connection to given coordinates in the second step gives the opportunity to test these independently of the observations adjusted and checked in the first step.
In photogrammetry one meets a similar situation. In the traditional approach ground control is entered in the adjustment of aerotriangulation blocks as not stochastic. The development of aerotriangulation during the seventies made clear, however, that this approach is not always justified. The precision of photogrammetric point determination reaches in some cases the same level as the precision of some terrestrial surveys. The coordinates of ground control points should then be entered as stochastic variates in the block adjustment. In most block adjustment programmes, this is not possible, so the computing procedure should be modified.

The method given for terrestrial networks can be used here as well, this will be explained in sections 3.1-2. The terrestrial coordinates are now considered as being stochastic, this fact raises the question of what variance-covariance matrix should be used. In many cases the original matrix is not available anymore, that is why an artificial matrix should be used instead. The latter is supposed to describe the precision of the points "sufficiently". Grafarend [4,5], Baarda [1] and Molenaar [8] give suggestions for these artifical matrices.
As S-transformations and artifical covariance matrices appear to be indispensible in the proposed method of computation, their meaning will be explained in the following sections.

2. S-transformations and criterion matrices

We shall first explain the meaning of S-transformations and artificial covariance matrices in a planimetric coordinate system, because they are easy to understand. Then follows a sketch of the generalisation to three-dimensional space as given in [8] .

2.1. S-transformation in planimetry

Consider a planimetric independent model block. The adjustment of such a block can be made without ground control. For the coordinates of all points in the block approximate values will be introduced to initiate the computations. If the coordinates of two points are kept as fixed, then corrections for the others will follow from the adjustment. The two fixed points are called "S-base" [1] . The result of this procedure is a set of coordinates which gives the positioning of the points in the block relat-

ively with respect to the S-base. The variance-covariance matrix of these coordinates expresses the precision of this relative positioning.
If another S-base is chosen, another set of coordinates will be found and another variance-covariance matrix. So one should not speak about "absolute" coordinates and "absolute" precision, because they are always relative with respect to an S-base. It is better to talk about "S-systems". S-systems are specified by the choice of an S-base. It is possible to transform directly from one S-system to another, by means of S-transformations.
Suppose we want to transform from an S-system with base (u,v) to a system with base (r,s) and in both systems we have the same approximate values for the coordinates :

$$(1.1) \qquad \begin{bmatrix} x_i^{o\,(u,v)} \\ y_i^{o\,(u,v)} \end{bmatrix} = \begin{bmatrix} x_i^{o\,(r,s)} \\ y_i^{o\,(r,s)} \end{bmatrix} = \begin{bmatrix} x_i^{o} \\ y_i^{o} \end{bmatrix}$$

- index i indicates point number
- superscript (u,v) (r,s) indicates S-system
- superscript ^{o} indicates approx. values.

The expectational values for the coordinates are not equal in both systems, but they do not differ very much :

$$(1.2) \qquad \begin{bmatrix} \tilde{x}_i^{(u,v)} \\ \tilde{y}_i^{(u,v)} \end{bmatrix} \approx \begin{bmatrix} \tilde{x}_i^{(r,s)} \\ \tilde{y}_i^{(r,s)} \end{bmatrix}$$

$\sim$ denotes mathematical expectation

The transformation from (u,v) - to (r,s) - system is then :

$$(2.1) \qquad \begin{bmatrix} \tilde{x}_i^{(r,s)} \\ \tilde{y}_i^{(r,s)} \end{bmatrix} = \begin{bmatrix} \tilde{a} & -\tilde{b} \\ \tilde{b} & \tilde{a} \end{bmatrix} \begin{bmatrix} \tilde{x}_i^{(u,v)} \\ \tilde{y}_i^{(u,v)} \end{bmatrix} + \begin{bmatrix} \tilde{d}_x \\ \tilde{d}_y \end{bmatrix}$$

and

$$(2.2) \qquad \begin{bmatrix} x_i^{o\,(r,s)} \\ y_i^{o\,(r,s)} \end{bmatrix} = \begin{bmatrix} a^o & -b^o \\ b^o & a^o \end{bmatrix} \begin{bmatrix} x_i^{o\,(u,v)} \\ y_i^{o\,(u,v)} \end{bmatrix} + \begin{bmatrix} d_x^o \\ d_y^o \end{bmatrix}$$

From (1.1) we find $a^o=1$, $b^o=0$, $d_x^o=d_y^o=0$ so when we neglect second order terms in the linearisation of (2.1), then the difference equation of (2.1) and (2.2) is :

$$(3) \qquad \begin{bmatrix} \Delta\tilde{x}_i^{(r,s)} \\ \Delta\tilde{y}_i^{(r,s)} \end{bmatrix} = \begin{bmatrix} \Delta\tilde{x}_i^{(u,v)} \\ \Delta\tilde{y}_i^{(u,v)} \end{bmatrix} + \begin{bmatrix} x_i^o & -y_i^o \\ x_i^o & x_i^o \end{bmatrix} \begin{bmatrix} \Delta\tilde{a} \\ \Delta\tilde{b} \end{bmatrix} + \begin{bmatrix} \Delta\tilde{d}_x \\ \Delta\tilde{d}_y \end{bmatrix}$$

with $x = x - x^o$ etc.

For the coordinate computation in the (r,s)-system (x_r^o, y_r^o) and (x_s^o, y_s^o) are kept as fixed and thus :

$$(\tilde{x}_r^{(r,s)}, \tilde{y}_r^{(r,s)}) = (x_r^o, y_r^o) \text{ and } (\tilde{x}_s^{(r,s)}, \tilde{y}_s^{(r,s)}) = (x_s^o, y_s^o)$$

hence :

$$(4) \quad \begin{array}{l} (\Delta\tilde{x}_r^{(r,s)}, \Delta\tilde{y}_r^{(r,s)}) = (0,0) \\ (\Delta\tilde{x}_s^{(r,s)}, \Delta\tilde{y}_s^{(r,s)}) = (0,0) \end{array}$$

With (4) the transformation elements can be eliminated in (3).
The result is :

$$\begin{bmatrix} \Delta\tilde{x}_{ri}^{(r,s)} \\ \Delta\tilde{y}_{ri}^{(r,s)} \end{bmatrix} = \begin{bmatrix} \Delta\tilde{x}_{ri}^{(u,v)} \\ \Delta\tilde{y}_{ri}^{(u,v)} \end{bmatrix} - \begin{bmatrix} x_{ri}^o & -y_{ri}^o \\ y_{ri}^o & x_{ri}^o \end{bmatrix} \begin{bmatrix} x_{rs}^o & -y_{rs}^o \\ y_{rs} & x_{rs}^o \end{bmatrix}^{-1} \begin{bmatrix} \Delta\tilde{x}_{rs}^{(u,v)} \\ \Delta\tilde{y}_{rs}^{(u,v)} \end{bmatrix}$$

where $x_{ri} = x_i - x_r$ and $y_{ri} = y_i - y_r$ etc.

Some rewriting using (4) leads to :

(5.1)

$$\begin{bmatrix} \Delta\tilde{x}_i^{(r,s)} \\ \Delta\tilde{y}_i^{(r,s)} \end{bmatrix} = \begin{bmatrix} \Delta\tilde{x}_i^{(u,v)} \\ \Delta\tilde{y}_i^{(u,v)} \end{bmatrix} - \begin{bmatrix} x_{si}^o & -y_{si}^o \\ y_{si}^o & x_{si} \end{bmatrix} \begin{bmatrix} x_{sr}^o & -y_{sr}^o \\ y_{sr}^o & x_{sr}^o \end{bmatrix}^{-1} \begin{bmatrix} \Delta\tilde{x}_r^{(u,v)} \\ \Delta\tilde{y}_r^{(u,v)} \end{bmatrix}$$

$$- \begin{bmatrix} x_{ri}^o & -y_{rs}^o \\ y_{ri}^o & x_{ri}^o \end{bmatrix} \begin{bmatrix} x_{rs}^o & -y_{rs}^o \\ y_{rs}^o & x_{rs}^o \end{bmatrix}^{-1} \begin{bmatrix} \Delta\tilde{x}_s^{(u,v)} \\ \Delta\tilde{y}_s^{(u,v)} \end{bmatrix}$$

Then :

$$(5.2) \quad \begin{bmatrix} \tilde{x}_i^{(r,s)} \\ \tilde{y}_i^{(r,s)} \end{bmatrix} = \begin{bmatrix} x_i^o \\ y_i^o \end{bmatrix} + \begin{bmatrix} \Delta\tilde{x}_i^{(r,s)} \\ \Delta\tilde{y}_i^{(r,s)} \end{bmatrix}$$

So (5.1) and (5.2) give the transformation from (u,v) - to (r,s)-system for coordinates, and the coefficients of (5.1) should be used to transform the variance-covariance matrix.
One should notice that when developing (5.1) we made no reference to the S-base of (u,v)-system. This means that the transformation can be applied to any S-system. If no base has been specified in an S-system, it is

called a (a)-system.
An important fact is that (5.1) has been derived from a differential similarity transformation. This implies that angles and ratios of length are invariant with respect to S-transformations.

2.2. Criterion matrices

In this section we will concentrate on the ideas of Baarda [1], which proved to be very useful in the adjustment of densification networks, whereas the author does not know any practical applications of the work of Grafarend.
The artificial covariance matrix, also called "criterion matrix", of Baarda has the following characteristics : It gives circular point and relative standard ellipses, whereas variances of distance ratios and angles are only dependent on the shape and size of the triangles from which they are taken and not on the actual position of the triangle in the network.
In an (a)-system, that is in a fictious coordinate system without S-base the submatrix for two points is :

	x_i	y_i	x_j	y_j
x_i	d^2	0	$d^2-d^2_{ij}$	0
y_i	0	d^2	0	$d^2-d^2_{ij}$
x_j	$d^2-d^2_{ij}$	0	d^2	0
y_j	0	$d^2-d^2_{ij}$	0	d^2

- d^2 = a parameter which will be eliminated by an S-transformation
- $d^2_{ij} = (c_0 + c_1\, l_{ij})\ \mathrm{cm}^2$
- l_{ij} = distance between points i and j in km.
- c_0 and c_1 are the parameters which describe the precision of the point field.

By means of (5.1) this matrix can be transformed to a real S-system.
The use of this matrix is twofold. First : it serves during the reconnaissance phase of a network as a criterion for the precision of the coordinates to be computed. The matrix G obtained from the adjustment of the network is compared with the criterion matrix H via the generalised eigen value problem : $|G - \lambda \cdot H| \leq 0$ in which both G and H are given with respect to the same S-base.
If all eigen values are $0 < \lambda_{min}$, $\lambda_{max} \leq 1$ then variances computed from G are always less than or equal to variances computed from H, thus H gives an upperbound for the precision of the network (see [1] §8, [2]).
Second : the matrix H can replace G in future computations. This means that G has not to be stored with the coordinates, but the matrix H can be generated instead when required.
The use of the matrix as a criterion for precision seems to be more meaningful for terrestrial networks than for photogrammetric blocks, as for the latter the possibilities for improving the structure are limited.
For the adjustment of aerotriangulation blocks one could make use of this matrix to describe the precision of ground control. This is important when use is made of old point fields for which the information about the exact structure of the variance-covariance matrix is not available anymore.
Values have to be chosen then for the parameters c_0 and c_1 depending on the type of network from which the coordinates have been computed. Once the choice has been made the coordinates of the given points enter the adjustment as observations with the generated covariance matrix. The advantage of this matrix over a diagonal matrix is that it takes into account the correlation between points, which in principle always exists. This correlation is dependent on the distance between points.

The effect of the matrix on the final computed coordinates is in general not so large. The matrix is important however, for the evaluation of the precision and reliability of the final results and for the testing of the given points.

2.3. Three-dimensional space

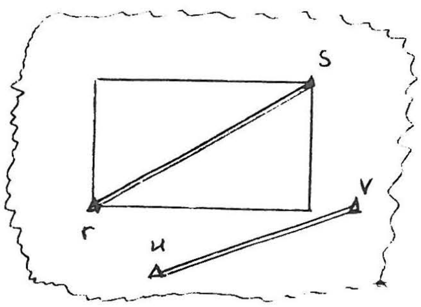
The two-dimensional approach can be applied in many cases, but the development of photogrammetry in the seventies as a method for spatial point determination requires a three-dimensional S-transformation and criterion matrix. These can be obtained as a generalisation of the planimetric solution. This has been elaborated in [8], a short sketch will be given in this section.
A cartesian x, y, z-system must be defined by means of seven parameters. These are the coordinates of two points plus one more parameter. For the latter one is apt to chose a coordinate (in most cases the z-coordinate) of a third point. An analysis of the coordinate computation in R_3 leads however to a principal choice, which is somewhat different.
When two points in R_3 are kept as fixed, the plane containing these two plus a third point can rotate freely on the line connecting the first two. Only when the direction of the normal vector to this plane is kept as fixed as well, the position of the third point can be determined by means of an observed length ratio and angle. The position of other points can be determined relatively with respect to the first group of three by means of length ratios and angles. If r and s are the first two points and t is the third, then this S-system is called (r, s; t)-system.
When the normal vector is parallel or nearly parallel to the z-axis of the coordinate system, then the choice of the z-coordinate of the third point as seventh element is an allowable approximation. Moreover, in nearly flat terrain one can use the x, y, z-coordinates of two points and the z-coordinate of a third point as an S-base.
The formulae for S-transformations in R_3 between systems with S-bases of a rigorous type are much more complicated than those in R_2.
They will not be given here. For systems with the less rigorous type of S-base, the S-transformation can be derived according to the method of section <u>2.1.</u> One should keep in mind, however, that the latter is only applicable under certain restrictions.
From experience with photogrammetric blocks and terrestrial networks we know that the relative positioning of points in the vertical sense is less precise than in horizontal sense. We also learned that the vertical positioning is to a great extent stochastically independent of the horizontal positioning. These considerations were important for the construction of a three-dimensional criterion matrix in [8]. Another important feature was the fact that large pointfields are approximately curved over a sphere. The matrix developed in [8] consists of two independent submatrices defined in a fictitious coordinate system : one for spherical heights and one for "horizontal" positioning over the sphere. These submatrices are designed so that the precision of the relative positioning of points is only dependent on that relative positioning itself, and not on the actual coordinates of the points. In this way a covariance matrix has been obtained for pointfields with a "homogeneous and isotropic inner precision". Inner precision means the precision of the angles and length ratios (and spherical distances) which desribe the internal geometry of the network completely. An S-transformation will transform the matrix from the fictitious coordinate system to an operational S-system.
Earth curvature is negligible for pointfields which cover small areas. In that case the three-dimensional criterion matrix of [8] can be simpli-

fied to a combination of the matrix for heights and the matrix for the complex plane as given in [1] .

3. Application in photogrammetry

3.1. Connection of photogrammetric blocks and ground control

Section 2.1. referred to the use of S-systems for block adjustment.



This will be elaborated in this section. Suppose terrestrial (x,y)-coordinates have been computed for points in a network with (u,v) as the S-base. A point densification will be made for a part of the network by means of aerotriangulation. According to the method proposed in section 2.1. an internal block adjustment will be made first independent of ground control. To be able to compute coordinates, points r and s will be used as an S-base with their terrestrial coordinates, which are then considered as not stochastic. For other ground control points in the block we obtain now two sets of coordinates.

$$\begin{bmatrix} \underline{x}_i^{(p)(r,s)} \\ \underline{y}^{(p)(r,s)} \end{bmatrix}$$ the photogrammetric coordinates computed with respect to base (r,s)

and

$$\begin{bmatrix} \underline{x}^{(t)(u,v)} \\ \underline{y}^{(t)(u,v)} \end{bmatrix}$$ the terrestrial coordinates with respect to base (u,v)

These two sets of coordinates have been computed in different S-systems. Therefore we find for expectational values :

$$\begin{bmatrix} \tilde{x}_i^{(p)(r,s)} \\ \tilde{y}_i^{(p)(r,s)} \end{bmatrix} \neq \begin{bmatrix} \tilde{x}_i^{(t)(u,v)} \\ \tilde{y}_i^{(t)(u,v)} \end{bmatrix}$$

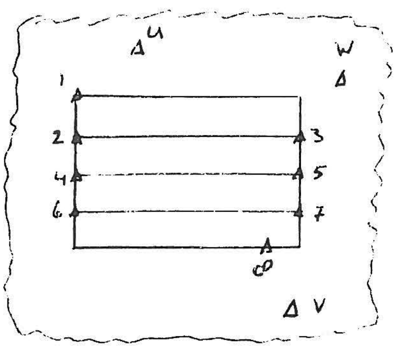
But if we apply (5.1) to transform the terrestrial coordinates from (u,v)-system to (r,s)-system, we obtain the condition equations :

$$\begin{bmatrix} \tilde{x}_i^{(p)(r,s)} \\ \tilde{y}_i^{(p)(r,s)} \end{bmatrix} = \begin{bmatrix} \tilde{x}_i^{(t)(r,s)} \\ \tilde{y}_i^{(t)(r,s)} \end{bmatrix}$$

if we suppose that there are no systematic deformations in the block. The connection of the block with the terrestrial coordinate system can be completed by means of an adjustment according to standard problem I based on these condition equations. From this adjustment all photogrammetric

points will get a correction due to their correlation with $\underline{x}_i^{(p)^{(r,s)}}$ and $\underline{y}_i^{(p)^{(r,s)}}$. The terrestrial coordinates will also get a correction unless these are made zero for practical reasons in a so-called "pseudo least squares" solution (see [1], § 7).
The condition equations make a simple test possible for the detection of gross errors in the terrestrial coordinates. The solution according to standard problem I allows an adjustment in steps i.e.: step 1 an adjustment using ground control points at the periphery of the block and step 2 an adjustment using points in the centre of the block. So the practice of using test blocks does not have to be changed, but the tests for finding deformations in the blocks can be made more rigorous according to [7].
Some experiments on this method of adjustment have been made by Jørgensen [6]. Although the method has been explained for planimetric blocks, it can be applied for three-dimensional blocks as well.

3.2. Stripwise block adjustment



Another possible procedure for block adjustment is the following :
Suppose we have a bundle block which will be used for point densification in a three-dimensional network with S-base (u,v;w). In the first step of computation the terrestrial points are transformed from base (u,v;w) to (2,3;1). The first strip is adjusted and coordinates are computed with respect to S-base (2,3;1). Then a connection is made between the terrestrial coordinates and the strip coordinates according to the method of section 3.1.
After this step the strip I coordinates and terrestrial coordinates form one system ((2,3;1)-system) which is then transformed to base (4,5;2). The same procedure is repeated for strip II and soon until the last strip has been adjusted. The final result is a rigorous bundle block adjustment where the block coordinates form one system with the terrestrial coordinates. If required, the whole system can be transformed back to the original base (u,v;w).
In this procedure only a small set of equations has to be solved (one strip) per step of the adjustment, on the other hand sufficient background memory should be available for the variance-covariance matrix of the computed coordinates to be able to find the corrections for these coordinates after each step of adjustment. In this way one could think of a block adjustment which runs parallel with the measuring process.

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January 1980

References

1. Baarda, W. : S-transformations and criterion matrices.
   - Netherlands Geodetic Commission, vols. nrs. 1, 1973.

2. Bouloucos, T. : The reliability of the observations of points in planimetric independent model blocks.
   - I.T.C.-Journal 1979-2.

3. Bouw, T. Molenaar, M. : On the evaluation of ground control data.
   - I.S.P. Congress Hamburg 1980, pres. paper com. III.

4. Grafarend, E. : Genauigkeitsmasse geodätischer Netze.
   - Deutsche geodätische Kommission, Reihe A, Heft nr. 75, 1972.

5. Grafarend, E. : Geodetic applications of stochastic processes.
   - Physics of the Earth and planetary Interiors, 12-1976.

6. Jørgensen, J. : A comparison of the precision and reliability of terrestrial coordinates with those obtained from photogrammetry.
   - M.Sc. thesis P2 course, I.T.C. Enschede, 1980.

7. Molenaar, M. e.a. : Essay on emperical accuracy studies in aerial triangulation.
   - I.T.C.-Journal 1978-1.

8. Molenaar, M. : A further inquiry into the theory of S-transformations and criterion matrices.
   - in preparation, I.T.C. Enschede.