A SUBSTITUTE MATRIX FOR PHOTOGRAMMETRICALLY DETERMINED POINT FIELDS

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1. INTRODUCTION

The precision of a geodetically or photogrammetrically determined point field is usually presented by the covariance matrix of the coordinates of the points. It can be stated that this matrix is never produced in photogrammetric point deteremination methods.
Of course this is understandable because the production of such a matrix will require the manipulation i.e., inversions and multiplications of matrices of very large order and will necessitate tremendous storage requirements.

On the other hand this covariance matrix is needed
- when new measurements are available e.g., angles, distances, etc. which are used in a second step of an adjustment in phases for the production of the final result
- in the densification of an existing network, for assessing the precision and reliability of the new coordinates and for the testing of the given points
- especially nowadays when point fields of different levels of precision are integrated in data base systems.

The problem can be circumvented by describing the precision of a point field by an artificial covariance matrix, a matrix which will be generated and used when needed, as a "substitute matrix" [12].
Thus, the substitute matrix should have the following properties.
a. It should represent the precision of the point field as good as possible, in comparison with the real matrix.
b. It should be easy to generate, with minimum storage requirements.

In this paper a search for a substitute matrix is carried out for photogrammetrically determined point fields.

2. THEORETICAL CONSIDERATIONS

The above two properties, which the substitute matrix should possess, give rise to the following questions:
- how can a substitute matrix, from now on called the H matrix, be generated?
- what criteria should be used for the comparison of the H matrix with the real one, which from now on will be called the G matrix?

2.1. The substitute matrix, S-transformations

The generation of a substitute matrix is based on the ideas of Baarda [2], for the construction of an artificial covariance matrix, which is also called the criterion matrix.
Baarda's criterion matrix for a planimetric point field, describes desirable properties of precision i.e., it gives circular point and relative standard ellipses, whereas variances of distance ratios and angles are only dependent on the shape and size of the triangles from which they are taken and not on the actual position of the triangle in the point field.

In a fictituous coordinate system, i.e., no S-base has been specified in the sense of [2], the submatrix pertaining to the points i,j has the form.

	x_i	y_i	x_j	x_j
x_i	d^2	0	$d^2 - d_{ij}^2$	0
y_i	0	d^2	0	$d^2 - d_{ij}^2$
x_j	$d^2 - d_{ij}^2$	0	d^2	0
y_j	0	$d^2 - d_{ij}^2$	0	d^2

(2.1.1)

With an anologous extension to an arbitrary number of points where, d is a paramater; the radius of the circular point standard ellipses.
d^2_{ij} is a monotonic non decreasing function of the distance l_{ij} between the points i,j. This function will further be called the "choice function".
If two points of the point field - these two points will be called the S-base [2] - are kept fixed, i.e., a coordinate system is introduced, then the precision of the coordiantes of the points with respect to the S-base can be expressed, by transforming the previously defined matrix by the use of a S-transformation, a differential similarity transformation. An extensive theory of S-transformations can be found in [2], [10], [11].

An S-transformation applies to the matrix defined in (2.1.1.) results in the elimination of the parameter d [2]. The substitute matrix is thus completely defined by the function d^2_{ij}.
It should be clear, however, that the transformed matrix expresses the precision of the point field with respect to the chosen S-base. It is therefore obvious that the comparison of the two covariance matrices can only be made after the two matrices have been transformed to the same S-base.

2.1.1. The choice functions

For the generation of the substitute matrix three choice functions d^2_{ij} have been considered.

1. Linear choice function

$$d_{ij}^2 = \Delta d^2 + c_1 l_{ij} \qquad (2.1.2)$$

This function is extensively used in the Netherlands for the generation of a criterion covariance matrix for geodetic point fields [1], [2].

2. Logarithmic choice function

$$d_{ij}^2 = \Delta d^2 + C_1^2 \; C_2 \; \ln \; (1 + \frac{l_{ij}}{C_2}) \qquad (2.1.3)$$

The function (without the term Δd^2) has been proposed in [3] as a covariance function for the use in geodetic networks.

3. Exponential choice function

$$d_{ij}^2 = \Delta d^2 + C_1 \; (1-\exp(-C_2^2 l_{ij}^2)) \qquad (2.1.4)$$

Similar functions are used as covariance functions, mostly in DEM, in connection with linear least squares filtering and prediction [13] where:
C_1, C_2 are parameters of the choice functions, the values of which are sought through an experimental process.
l_{ij} is the distance between points i,j.

Δd^2 is a parameter which has been introduced to express the uncertainty of the point definition [2], i.e., the transition from a physical point to a mathematical point. In the experiments, if not otherwise stated, its value has been taken as 10 cm^2. This choice is related to the measuring precision of model coordinates and the average local redundance [5] of tie points.

2.2. Criteria for comparison

It seems to be reasonable to assume, in the sense of an S-system, that the variance of any function F of coordinates calculated using the real covariance matrix G, is smaller or equal to the variance of the function, calculated using the substitute matrix H.
This leads to

$$\sigma_{FF}^G \leq \sigma_{FF}^H$$

or

$$\Lambda G \Lambda^* \leq \Lambda H \Lambda^*$$

where
Λ is row vector, with the coefficients of the linearized form of F.
* indicates transposition.

We may also write

$$\Lambda(G-H)\Lambda^* \leq 0$$

hereby requiring the matrix (G-H), to be negative semi-definite [2].

This is fulfilled if:

$\{\lambda\}_{max} < 1$

where

$\{\lambda\}_{max}$ is the maximum eigen value of the general eigen value problem
$\det|G-\lambda H| = 0$

In our case the upper bound of $\{\lambda\}_{max}$

$$\boxed{\{\lambda\}_{max} = 1} \qquad (2.2.1)$$

will be used as a first criterion.

Further, the two matrices G and H can be diagonalized simultaneously [2]. [6]

$$V^*HV = I \qquad (2.2.2)$$
$$V^*GV = D$$

where
V is the matrix formed by the eigen vectors of the general eigen value problem $\det|G-\lambda H| = 0$.
D is a diagonal matrix with the eigenvalues of the same eigen value problem as diagonal elements.
I is the unit matrix.

With the transformation in (2.2.2.) the standard hyperellipsoid belonging to the covariance matrix H, becomes a hypersphere with radius equal to 1, whilst the hyperellipsoid belonging to the covariance matrix G becomes one with semi-axes equal to the square root of the eigen values.
With the condition $\{\lambda\}_{max} = 1$, the situation is depicted in fig. 1.

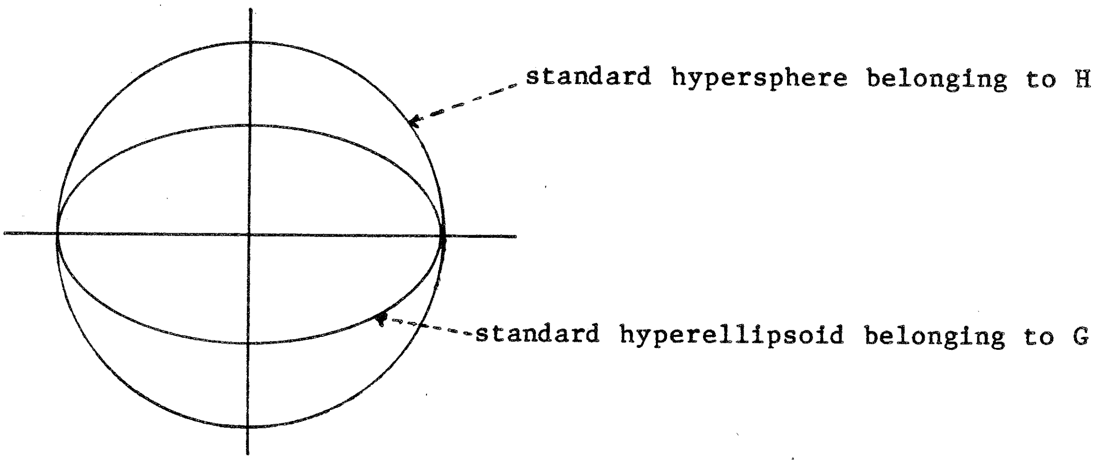

Figure 1

By changing the values of the parameters of the choice function, the matrix H changes and therefore the results of the eigen value problem change.
The shapes of the two hyperellipsoids will agree better with each other as the ratios of the eigen values approach 1.
Therefore a possible criterion of "fit" could be

$$\boxed{\frac{\{\lambda\}\ max}{\{\lambda\}\ min} \text{ as small as possible, approaching 1}} \qquad (2.2.3)$$

This is the second criterion used.

In terms of variances this criterion means:

$$\{\lambda\}_{min}\ \sigma^{H}_{FF} < \sigma^{G}_{FF} < \{\lambda\}_{max}\ \sigma^{H}_{FF}$$

Note: The general eigen value problem is invariant with respect to a non-singular transformation. In view of the fact that the comparison of the two matrices should be made after a transformation to the same S-base, it is obvious that the choice of the S-base is not important for the comparison.

3. EXPERIMENTS

A series of experiments have been carried out with simulated planimetric model blocks [8].
For each experiment the covariance matrix of the point field is produced through the adjustment algorithm.
Also a substitute matrix for the point field is generated.
The two matrices are transformed into the same S-base, and compared using the results of the general eigen value problem.
The parameter values of the choice functions which generate the best substitute matrix, according to the criteria formed in paragraph 2, were obtained by successive trials.
The following block configurations with the indicated control distributions were treated:

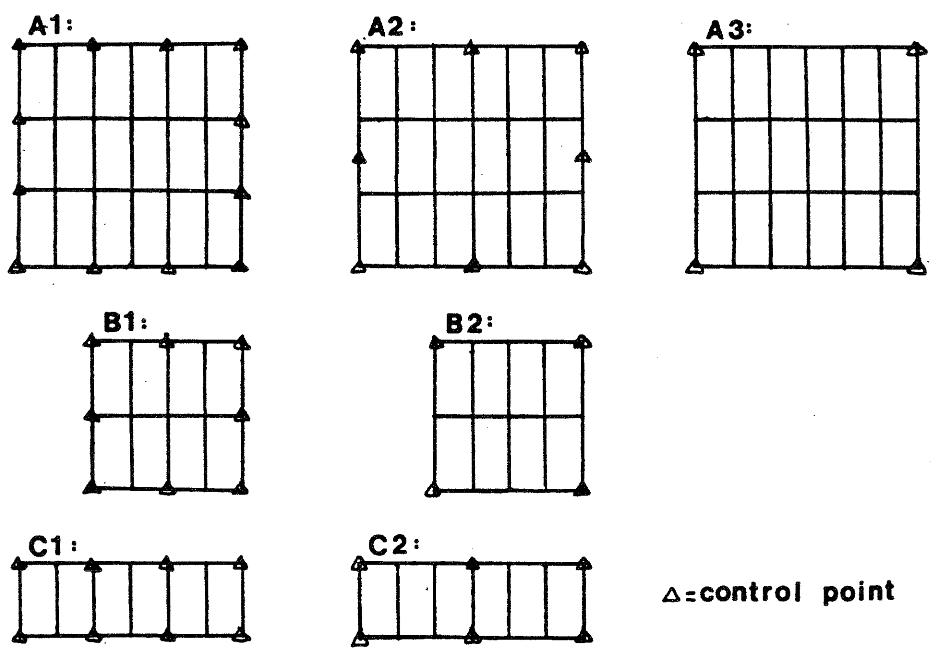

Figure 2

For each model of the above blocks the following cases were considered.
- four single tie points (4 corners)
- four double tie points
- six single tie points (6 standard positions)
- six double tie points

Also the case where an extra point is added in the center of the model was investigated. This point does not participate in the adjustment. It can be considered as a detail point (cadastral) point, a situation usually encountered in photogrammetry.
The following stochastic models were considered for the observations.

For the model points:
- equal precision of 10 cm on the ground for x and y coordinates with no correlation between them. These will be referred to as uncorrelated model points.
- a full covariance matrix is used for model points. The determination of this matrix is based on the work of Ligterink [9], taking only the observational errors into account. These errors originate from inner orientation, relative orientation and from the measurements of model coordiantes. The measured points are considered to be signalized. These will be referred to as correlated model points.

The latter could be a better model than the former, but it should still be considered with care, since correlation between models is not taken into account [4].

Ground control points
- not stochastic
- equal precision of 5 cm for the x and y coordinates, and no correlation between them.

The generated independent models have 20% side overlap and 60% forward overlap. The scale of the photography has been assumed to be 1:10,000 and the principal distance to be 152 mm.

3.1. Analysis of the results

The values of the parameters of the choice functions which generate the best substitute matrices are summarized in tables I and II. The ground control is not stochastic.
The first vertical column in these tables refers to the type of choice function used. The second column refers to the block configuration e.g., A1, A2, etc. (see also fig. 2).
The first horizontal line indicates the tie point configuration.
The second line indicates the parameters of the choice functions.
In all the cases $\{\lambda\}_{max} = 1$.

From these tables the following conclusions can be drawn:
- the exponential and logarithmic choice functions give comparable results, with the logarithmic function slightly better in most of the cases. The linear choice function gives very large values for the ratio $\{\lambda\}_{max}/\{\lambda\}_{min}$.
- when double points are used, the ratio $\{\lambda\}_{max}/\{\lambda\}_{min}$ becomes smaller compared with the corresponding cases of single points. Further, the ratio increases with a decrease in the number of control points in the block (relaxed control).
- when six points per model are used, the ratio becomes larger compared to the ratio of the corresponding cases with four points per model. This may be caused by the non homogeneity in the

precision of the point field (some points appear in four models, others in only two)
- in the case of correlated model points, the ratio $\{\lambda\}_{max}/\{\lambda\}_{min}$ is larger compared to the ratio of the corresponding cases, where no correlation is considered.

When the ground control points are considered as stochastic, no substantial changes in the ratio $\{\lambda\}_{max}/\{\lambda\}_{min}$ is observed (see [8]).
Also, in the cases of blocks with single points in the models, the ratio $\{\lambda\}_{max}/\{\lambda\}_{min}$ becomes larger due to the fact that the non homogeneity in the precision of the point field increases.

3.2. Partial point fields

In these series of experiments, the objective was to investigate whether the parameters found for the substitute matrix for the whole point field, can be used for the generation of a substitute matrix for parts of the point field.
The results in [8] reveal that in the case of homogeneous point fields (full perimeter control, four points per model), the exponential choice function seems to be the most adequate for the generation of a substitute matrix for the partial point fields.
However, in all the other cases the matrices generated for the partial point fields are suitable as an upper bound for precision, since, $\{\lambda\}_{max}$ is always smaller than 1.

When the single points in the models (cadastral points) were treated as partial point fields, it was clear that they belong to another level of precision and therefore other values for the parameters have to be found.

3.3. Consistency

If we allow a small increase in the ratio $\{\lambda\}_{max}/\{\lambda\}_{min}$, the parameter values for the exponential choice function shows a consistency which was not observed for the parameters of the logarithmic choice function.
The increase in the ratio was never larger than 5% of the values given in the tables I and II.
The parameter C_2 assumes a value equal to 0.9 when four single or double tie points are used, and a value 1.1 when six single or double points are used, inrespective of the control distribution used.
In the case of a block with full perimeter control the values of C_1, of the exponential function, are:

	uncorrelated	correlated
four single points/model	95	60
four double "	85	55
six single "	130	45
six double "	120	40

The fact that C_1 has smaller values in the cases where double points are used (more precise point fields) show that this parameter plays the role of a scale factor in the exponential choice function.

Some further experiments with the exponential function showed that changing the scale of photography affects the ratio $\{\lambda\}_{max}/\{\lambda\}_{min}$. This can be compensated if the parameter C_2 is divided by the ratio of the scale numbers. Also, the effect of changing the precision of the observations is compensated by multiplying the parameter C_1 by the ratio of the assumed variances.

4. GENERAL REMARKS

- the linear choice function gives considerably worse results as compared with the other two functions. Its use in photogrammetry can not be recommended.
- the logarithmic and exponential choice functions can be used to generate a substitute matrix for photogrammetrically determined point fields.
- the parameters of the exponential choice function have some very specific properties namely:
  The parameter C_1 causes a scaling effect and its values are related to the precision of the block.
  The parameter C_2 controls the ratio $\{\lambda\}_{max}/\{\lambda\}_{min}$ i.e., how adequately the substitute matrix replaces the real covariance matrix. Its value depends only upon the tie point configuration (four or six points per model).
  Also, the effect of a change in the scale of photography or in the precision of the photogrammetric measurements can be dealt with.
  All these facts suggest that the exponential choice function is the most suitable to be used for the generation of a substitute matrix for photogrammetrically determined point fields.

Further investigations are to be carried out with the exponential choice function, which may lead to show the interrelationship of the parameter C_1 with the spacing of the control points.
Also, the role of the parameter Δd^2 will be analyzed with respect to the type of points and the local redundancy [5].
In the near future, similar investigations will be carried out to investigate the height precision of photogrammetrically determined point fields.

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BLE I. UNCORRELATED MODEL POINTS

****** PER MODEL :		4 SINGLE POINTS				4 DOUBLE POINTS				6 SINGLE POINTS				6 DOUBLE POINTS			
hoice function	case	Δd^2	C1	C2	$\frac{\lambda_{max}}{\lambda_{min}}$	Δd^2	C1	C2	$\frac{\lambda_{max}}{\lambda_{min}}$	Δd^2	C1	C2	$\frac{\lambda_{max}}{\lambda_{min}}$	Δd^2	C1	C2	$\frac{\lambda_{max}}{\lambda_{min}}$
LOGARITHMIC	A1	10	13	0.2	2.63	10	16	0.1	2.41	10	21	0.1	3.74	10	25	.05	3.75
	A2	10	14.5	0.7	4.14	10	15	0.3	3.43	10	16	0.3	3.84	10	21	0.1	3.91
	A3	10	18	0.7	5.94	10	17	0.3	4.41	10	23	0.4	7.87	10	21	0.2	5.83
	B1	10	10.5	0.5	2.04	10	10	0.5	2.08	10	15	0.3	3.16	10	17	0.15	3.41
	B2	10	12	1	2.92	10	12	0.5	2.72	10	15.5	0.5	3.86	10	15.5	0.3	3.73
	C1	10	120	.001	1.48	10	105	.001	1.38	10	24	0.05	2.05	10	31	.025	2.13
	C2	10	22	0.1	2.17	10	23	.05	1.74	10	18	0.2	2.77	10	40	0.04	5.05
EXPONENTIAL	A1	10	95	0.85	2.22	10	85	0.9	2.46	10	130	1.2	3.72	10	130	1.2	4.19
	A2	10	240	0.8	4.34	10	180	0.8	4.52	10	190	1.1	4.76	10	180	1.1	5.32
	A3	10	300	0.9	6.25	10	200	0.9	5.48	10	350	1	8.95	10	240	1.1	7.42
	B1	10	95	0.8	1.90	10	80	0.8	1.94	10	130	1.1	3.30	10	125	1.2	3.60
	B2	10	170	1	3.19	10	120	1	2.95	10	190	1.1	4.37	10	160	1.2	4.36
	C1	10	110	0.8	1.43	10	90	0.85	1.33	10	120	1.2	2.36	10	120	1.2	2.38
	C2	10	150	1	1.97	10	110	1.2	1.69	10	150	1.1	3.01	10	250	1.1	5.55
LINEAR	A1	10	70		8.95	10	55		9.09	10	120		23.34	10	110		27.12
	A2	10	120		11.23	10	80		11.46	10	130		18.72	10	120		23.59
	A3	10	200		17.90	10	115		16.21	10	240		29.13	10	140		27.75
	B1	10	60		2.95	10	50		3.13	10	110		9.42	10	105		10.93
	B2	10	110		5.65	10	90		6.26	10	160		11.23	10	120		12.82
	C1	10	70		3.30	10	55		3.18	10	100		5.95	10	90		6.26
	C2	10	110		4.87	10	80		4.59	10	110		6.47	10	150		11.18

TABLE II. CORRELATED MODEL POINTS

****** PER MODEL :		4 SINGLE POINTS				4 DOUBLE POINTS				6 SINGLE POINTS				6 DOUBLE POINTS			
Choice function	case	Δd^2	C1	C2	$\frac{\lambda_{max}}{\lambda_{min}}$	Δd^2	C1	C2	$\frac{\lambda_{max}}{\lambda_{min}}$	Δd^2	C1	C2	$\frac{\lambda_{max}}{\lambda_{min}}$	Δd^2	C1	C2	$\frac{\lambda_{max}}{\lambda_{min}}$
LOGARITHMIC	A1	10	9.5	0.4	6.94	10	9.5	0.3	7.25	10	7	0.5	6.76	10	7	0.4	6.60
	A2	10	12	0.5	13.02	10	12.5	0.4	13.52	10	9	0.6	9.40	10	9	0.4	9.57
	A3	10	15	0.5	16.80	10	15	0.3	15.99	10	17	0.5	28.48	10	18.5	0.3	28.33
	B1	10	9	0.3	3.76	10	8.5	0.3	3.94	10	6	1.2	5.54	10	7	0.4	6.23
	B2	10	10	0.7	8.41	10	10	0.5	8.81	10	12.5	0.7	16.80	10	13	0.5	17.37
	C1	10	105	.001	4.86	10	103	.001	5.02	10	7	1.1	6.92	10	28	1	1.28
	C2	10	13.5	0.3	7.99	10	18	0.1	7.88	10	13	0.2	10.39	10	12	0.2	9.72
EXPONENTIAL	A1	10	65	0.9	6.08	10	55	0.9	6.71	10	45	1.1	8.27	10	41	1.1	7.82
	A2	10	135	0.9	11.18	10	115	0.9	12.28	10	95	1	14.83	10	70	1.1	13.12
	A3	10	180	0.9	14.66	10	145	0.9	15.04	10	230	1.1	35.19	10	210	1.1	35.02
	B1	10	50	0.8	3.19	10	50	0.9	3.46	10	40	1	6.64	10	35	1.1	6.25
	B2	10	95	1	7.75	10	85	1	7.73	10	135	1.1	19.31	10	120	1.1	19.42
	C1	10	92	0.9	4.86	10	80	0.85	5.02	10	60	1	7.54				
	C2	10	120	0.9	7.36	10	100	1	7.19	10	70	1.4	12.1	10	65	2	10.44
LINEAR	A1	10	40		18.55	10	35		20.02	10	30		21.35	10	28		23.57
	A2	10	80		33.05	10	70		38.21	10	60		36.70	10	45		40.26
	A3	10	120		50.77	10	95		55.84	10	160		96.40	10	140		111.3
	B1	10	30		6.33	10	27		6.71	10	30		9.2	10	25		10.21
	B2	10	70		16.32	10	60		18.19	10	110		39.59	10	100		45.86
	C1	10	55		8.75	10	50		8.77	10	40		8.03	10	35		9.30
	C2	10	85		16.10	10	75		16.17	10	70		16.31	10	65		16.27