ON THE RELATIONSHIP BETWEEN GAUSSIAN AND BINOMIAL MODELS FOR IMAGE REGISTRATION BY SEQUENTIAL TESTS OF HYPHOTESES

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ABSTRACT

In a previous paper (Mascarenhas and Pereira, 1983), a method for treating translational image registration problems by sequential test of hyphoteses was presented. Two types of statistical models were used then to describe the images to be registered, namely, a gaussian and a binomial model. The proposed method successfully registered a LANDSAT image against noisy versions of itself, different channels of the same image, as well as images taken six months apart. In this paper, relationships between both models are established. First, by assuming that both images to be registered are gaussian and one image is essentially a noisy version of the other, and that both images are thresholded at the mean value, a derivation is made of the probability curve of the binary error being equal to one, at the registration point, versus the signal to noise ratio. Second, assuming that the cross--correlation between the signals in both images is markovian and separable, the set of probability curves of the binary error being equal to one versus the distance from registration point is derived. Third, under the same assumptions of the previous case, the set of curves relating the error variance versus the displacement is also obtained. The curves, for the second and third cases are described for different values of the correlation coefficients in both directions and of the signal to noise ratio.

I - INTRODUCTION

In image processing there are several situations in which is necessary to match two or more images of the same scene in a way that there is a point by point correspondence between them. Such process is known as image registration.

When the image geometries are the same, or the differences between them are not significant, a translational registration algorithm is commonly used.

Several translational registration methods have been developed. They basically differ by the similarity measure used and by the image information handled. The similarity measures are mathematical criteria that, by maximization or minimization (depending on the criterion) over all the candidates to registration, define the best match between images.

Examples of similarity measures are: correlation, absolute value of the error between images, the mean square error and the error variance. The image information commonly used are: the original image data, the image gradient and binary images obtained by thresholding original data (or gradient) at the mean, the median value or at some other specified threshold.

Some registration methods are based on statistical modelling of the images involved. Kaneko (1976) models the images with gaussian distributions. Barnea and Silverman (1972), in the SSDA, model the absolute value of error between images with an exponential distribution.

In a recent work (Mascarenhas and Pereira, 1983), two translational registration methods were developed using the theory of sequential tests of hyphoteses. In the first method, the images are modelled with a gaussian distribution and the sequential test is applied to the error variance between images. In the second, where images thresholded at the mean value are used, the test is applied to the absolute value of the cumulative binary error, modelled with a binomial distribution. Both methods were applied to LANDSAT images, tests having been run for: different channels of the same image, images obtained at different dates and noisy versions of one image. The tests results showed successful registration for all cases.

In this work, relationships between the two models above mentioned are established. One considers that the images to be registered is a noisy version of the reference image. At item 2 probability curves of the binary error being equal to one versus signal to noise ratio for the registration point are generated. At item 3 a generalization is made, where a set of curves relating the same error probability to the displacement from registration is obtained. For this purpose, one assumes that the cross-correlation function between images is markovian and separable. Assuming the same model for the cross--correlation function, at item 4 the curves relating the error variance with displacement from registration are generated.

II - BINARY ERROR AT THE REGISTRATION POINT

Some basic hyphoteses assumed for this item and for the itens 3 and 4 refer to the images involved. One assumes that the reference image is composed of a signal (S) with gaussian distribution $N(0,\sigma^2)$ and the image to be registered is a version of the first but corrupted by aditive withe noise (N) independent from and with gaussian distribution $N(0,\sigma_N^2)$. Briefly:

reference image = S, (1)
image to be registered = S + N. (2)

S and N being independent, their joint probability density function is given by:

$$f_{S,N}\ (s,n) = \frac{1}{2\pi\sigma\sigma_N} \exp\left[-\frac{1}{2}\left(\frac{s^2}{\sigma^2} + \frac{n^2}{\sigma_n^2}\right)\right] = f_s(s).f_N(n). \qquad (3)$$

Figure 1 displays the ellipse that corresponds to a constant value of this density. This curve has its main axes parallel to the coordinate axes.

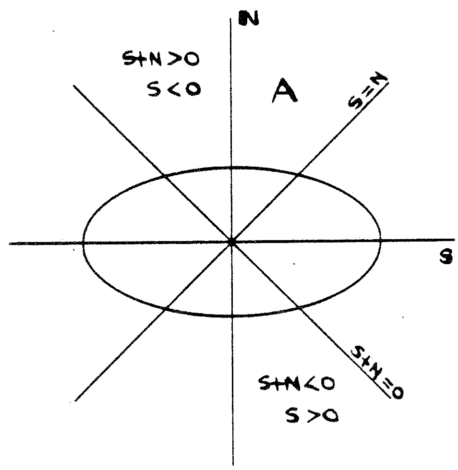

Fig. 1 - Crossection of the joint probability density function $f_{SN}\ (s,n)$.

At the registration point, the following expression is valid:

$$P[\text{error} = 1] = P[S + N < 0,\ S > 0] + P[S + N > 0,\ S < 0], \qquad (4)$$

$$= 2\ P[A], \qquad (5)$$

where the event (A) is indicated in Figure 1. Therefore,

$$P\ [\text{error} = 1] = 2 \int_0^\infty f_S\ (s)\ ds \int_s^\infty f_N\ (n)\ dn. \qquad (6)$$

By performing two changes of variables namely:

$$t = \frac{s}{\sigma}, \qquad (7)$$

$$v = \frac{n}{\sigma_N}, \qquad (8)$$

a radial symmetry is obtained and it can be easily shown that:

$$P[\text{error} = 1] = 2\,\frac{\alpha}{2\pi}\,, \tag{9}$$

$$\text{where } \alpha = \pi/2 - \text{arctg}\,(\sigma/\sigma_N), \tag{10}$$

and the final expression for the probability of error being equal to one at the registration position versus SNR (σ^2/σ_N^2) is given by:

$$P[\text{error} = 1] = \frac{1}{2} - \frac{1}{\pi}\,\text{arctg}(\sqrt{\text{SNR}}), \tag{11}$$

which is displayed in Figure 2.

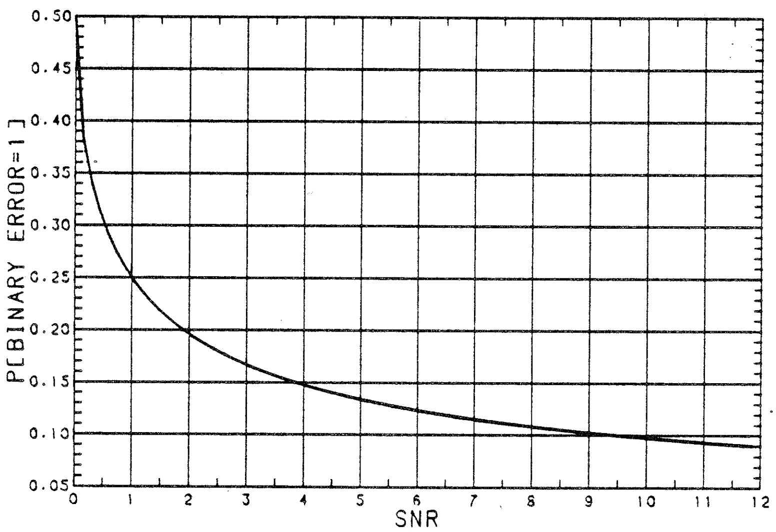

Fig. 2 - Probability of binary error equal to one versus SNR

It can be observed, as it should be expected, that the curve is monotonically decreasing starting at the value 0.5 for SNR = 0 and decreasing to zero for SNR $\rightarrow \infty$.

III - BINARY ERROR VERSUS DISPLACEMENT FROM REGISTRATION

Let X and Y be pixels in each of the two images to be registered, separated by a distance i on the vertical direction and j on the horizontal direction (i=j=0 at the registration point). Assume, furthermore, that the correlation coefficient between X and Y is ρ. Under these conditions the joint probability density function of X and Y is given by

$$f_{x,y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{(1-\rho^2).2}\left(\frac{x^2}{\sigma_x^2} - \frac{2\rho xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right]. \qquad (12)$$

By thresholding both gaussian images at their mean values (which are made equal to zero without any loss of generality), the probability of the binary error being equal to one between the thresholded X and Y is given by:

$$P[\text{error} = 1] = P[X < 0, Y > 0] + P[X > 0, Y < 0], \qquad (13)$$

which is given by:

$$P[\text{error} = 1] = 2 \int_0^\infty dy \int_0^\infty f_{x,y}(x,y)\,dx. \qquad (14)$$

In order to perform the double integration, three successive changes of variables are made:

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 1/\sigma_x & 0 \\ 0 & 1/\sigma_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad (15)$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \cos\beta & \operatorname{sen}\beta \\ -\operatorname{sen}\beta & \cos\beta \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}, \qquad (16)$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+\rho}} & 0 \\ 0 & \frac{1}{\sqrt{1-\rho}} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \qquad (17)$$

where $\beta = \pi/4$.

Figure 3 illustrates the previous transformation.

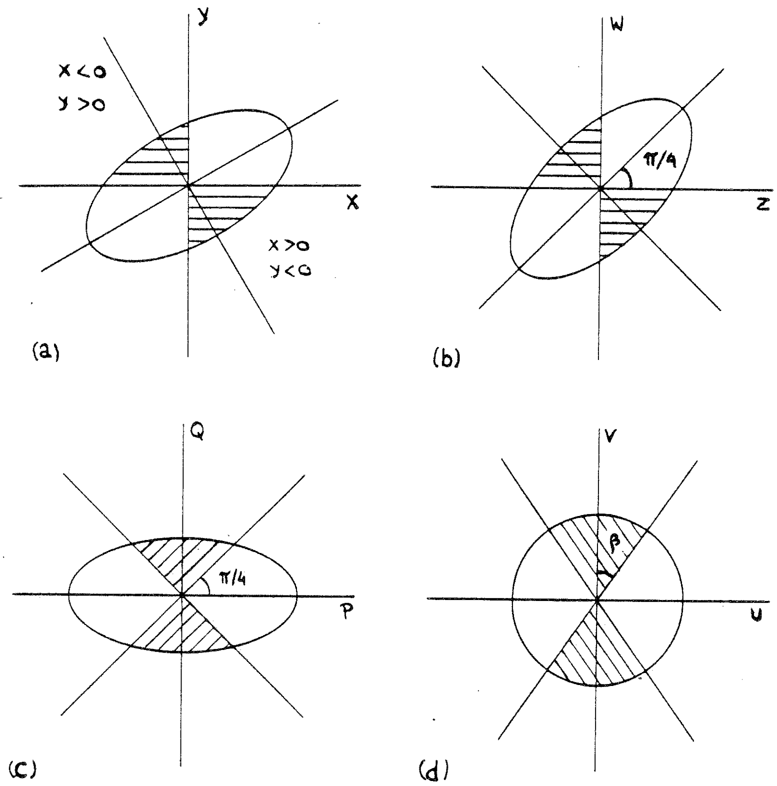

Fig. 3 - Illustration of the changes of variables

As a result obe obtains:

$$P[\text{error} = 1] = \frac{2\beta}{\pi} = 2 \text{ arctg} \left(\sqrt{\frac{1-\rho}{1-\rho}} \right) \tag{18}$$

Now, assuming that the autocorrelation function of the signal is markovian and separable, one obtains:

$$\begin{aligned} E[XY] &= E[S^{0,0} (s^{i,j} + N^{i,j})] = E[S^{0,0} S^{i,j}], \\ &= \sigma^2 . \rho_V^i \, . \, \rho_H^j \, , \end{aligned} \tag{19}$$

since signal and noise are orthogonal. Therefore, the correlation coefficient ρ will be given by:

$$\rho = \frac{E[XY]}{\sigma_x \sigma_y} = \frac{\sigma^2 \, \rho_V^i \, \rho_H^j}{\sqrt{\sigma^2 (\sigma^2 + \sigma_N^2)}} = \frac{\rho_V^i \, \rho_H^j}{\sqrt{1 + \frac{1}{SNR}}} \, . \tag{20}$$

By using Equation 18 and 20, it is possible to generate the set of curves od the probability of binary error equal to one versus distance from registration. Three examples of these curves are shown in Figure 4.

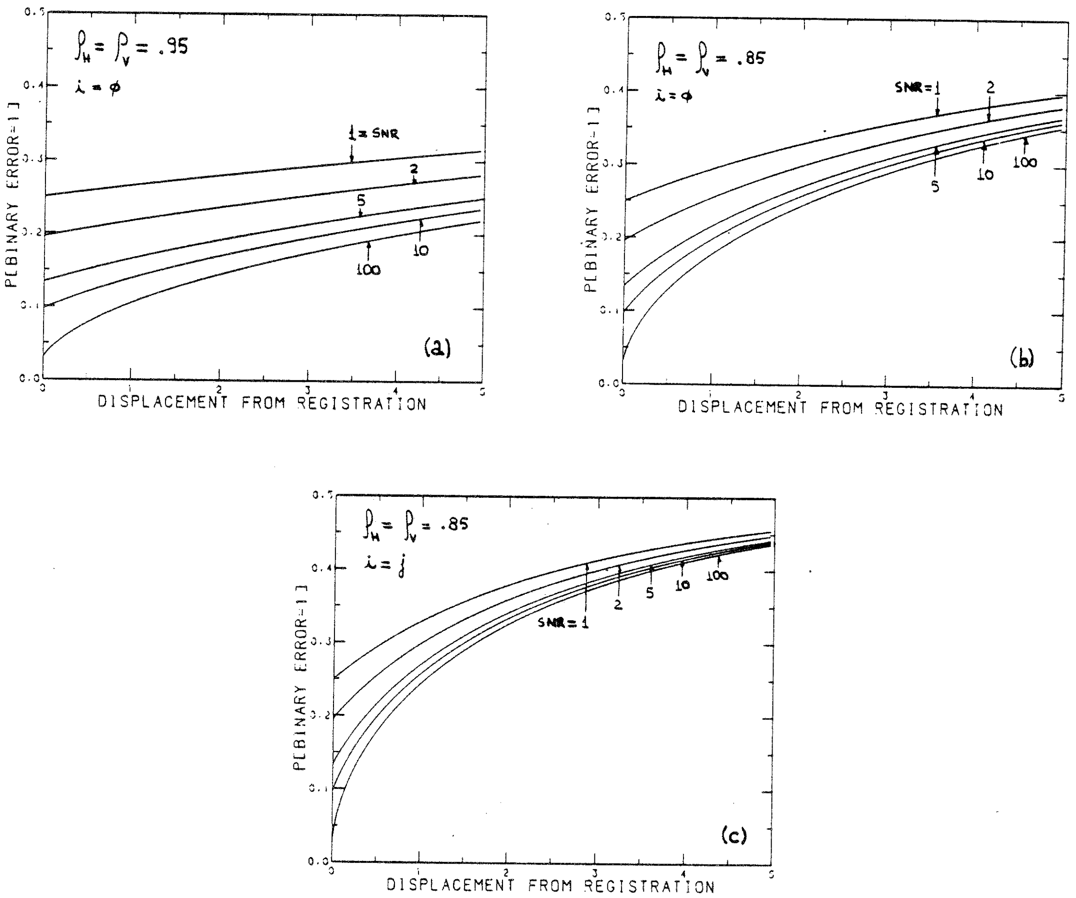

Fig. 4 a.b.c - Binary error versus displacement from registration

By examining Figures 4a and 4b, one can observe the faster increase of that probability with smaller values of ρ. Since the hyphoteses of separability of the autocorrelation also implies a smaller ρ in the diagonal direction, as compared to the horizontal or vertical, the faster increase of the curves of Figure 4c, as compared to Figure 4b, is clear. The influence of SNR can also be observed on these curves.

IV - ERROR VARIANCE VERSUS DISPLACEMENT FROM REGISTRATION

Under the same assumptions of the previous paragraphs, it is possible to derive the error variance of the gaussian model as a function of displacement from registration:

$$\mathrm{var(error)} = E[S^{i,j} + N^{i,j} - S^{0,0}]. \tag{21}$$

After some manipulations this equation can be put in the form:

$$\begin{aligned}\mathrm{var(error)} &= 2(1 - \rho^{i+j})\sigma^2 + \sigma_H^2 = \\ &= \sigma^2[2(1-\rho^{i+j}) + 1/\mathrm{SNR}].\end{aligned} \tag{22}$$

Figure 5 illustrates this result and observations that are analogous to the previous paragraphs, can be made. Furthermore, one can notice that for $i=j=0$, $\mathrm{var(error)}\rho^2 = 1/\mathrm{SNR}$, and for $i,j \to \infty$. $\mathrm{var(error)}/\rho^2 = 2+1/\mathrm{SNR}$, as should be expected.

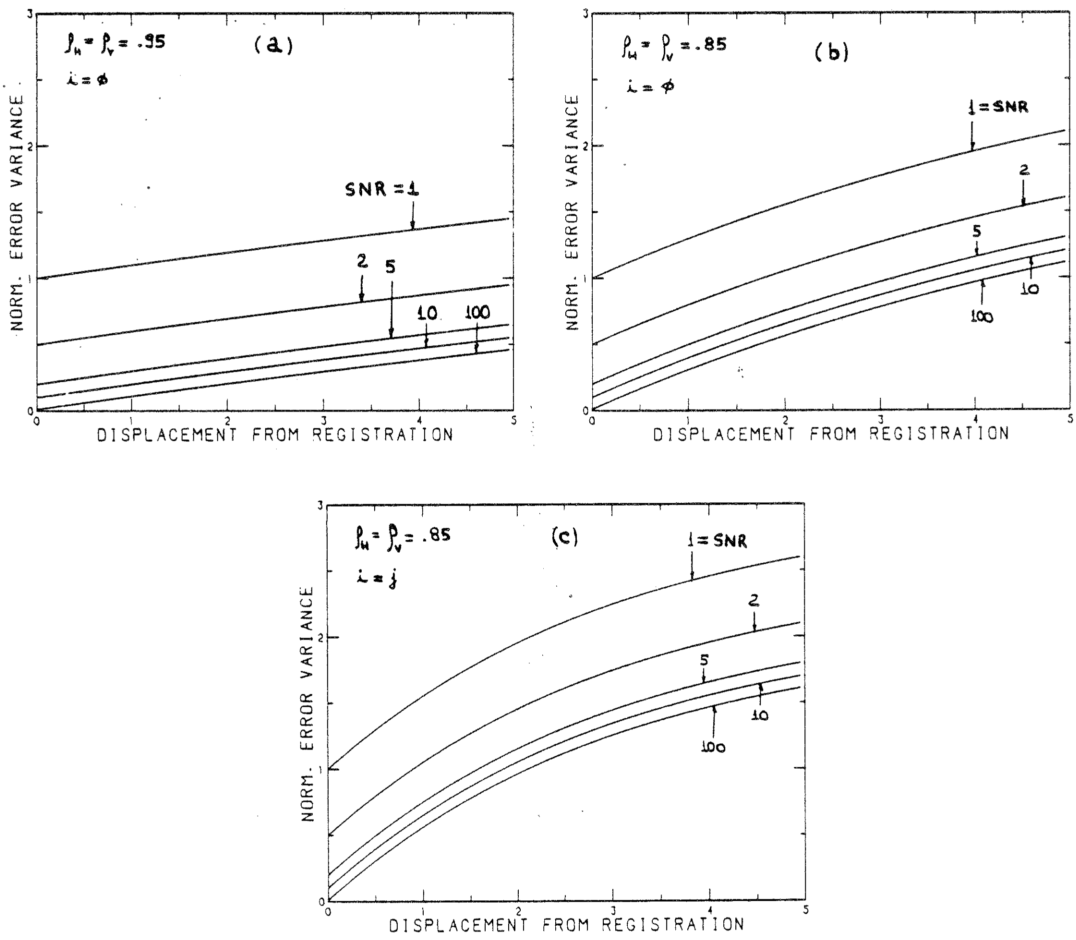

Fig. 5 a.b.c - Normalized error variance versus displacement from registration

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