GENERALIZED RELATIVE ORIENTATION IN CLOSE-RANGE PHOTOGRAMMETRY - A survey of methods

Ingolf Hådem
The University of Trondheim
Norway
Commission V

ABSTRACT

First, some general considerations are given concerning potential strategies for least squares nonlinear parameter estimation where no or only coarse approximations are available. This is followed by a survey of methods of generalized relative orientation. The use of such methods for obtaining approximations for the bundle adjustment is also described.

INTRODUCTION

In photogrammetry and geodesy there has been a growing interest in designing a general numerical algorithm for finding the approximations needed for solving nonlinear problems iteratively [9] p. 281. This is particulary true in the field of industrial photogrammetry where the traditional method of deriving approximations sometimes involves operations such as extra field measurements, which might hamper and delay the process seriously [6].

This paper deals with potential numerical algorithms for the relative orientation of pairs of photographs assuming no restrictions in the size of the unknown parameters, and which require no (or only coarse) approximations ($bx \neq 0$ and known inner orientation are however assumed). First, two general strategies for solving nonlinear problems principally without the knowledge of á priori approximations are considered: a) parameter transformation as a means of making the equations directly solvable for finding approximations, and b) the trial and error method with different sets of conjectured approximations for an iterative solution technique (e.g. continuation). Some difficulties such as singularity and multiple solutions are indicated and some mathematical models and techniques of generalized relative orientation are surveyed. (In [5,10] such methods are reported to have found practical applications). Finally an algorithm for finding initial values for the bundle method is outlined.

GENERAL STRATEGIES FOR SOLVING NONLINEAR PROBLEMS WITHOUT THE KNOWLEDGE OF Á PRIORI APPROXIMATIONS

Parameter transformation might lead to equations by means of which good approximations can be directly found. Let

$$f_i(\mathbf{x}) = 0, \quad i = 1,.. \; n, \text{ or } \mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{1}$$

be n nonlinear equations in m unknowns x_i, $(n \geq m)$. With

$$\mathbf{y} = \mathbf{g}(\mathbf{x}), \quad m_1 \text{ parameters } y_i, \quad (m_1 \geq m)$$

it may be feasible to reformulate (1) into

$$\mathbf{A}\mathbf{y} + \boldsymbol{\ell} = \mathbf{0}, \quad n_1 \text{ equations}, (n_1 \geq m_1) \tag{2a}$$

$$\mathbf{w}(\mathbf{y}) = \mathbf{0}, \quad (m_1 - m) \text{ nonlinear equations} \tag{2b}$$

so that good approximations of y_i can be found by means of (2a).

Example: To find the rotation α between the two coordinate systems $(U,Z),(U,Z)'$ with common points, from the equations:

$$U - U' \cos\alpha + Z' \sin\alpha = 0$$

$$z - U' \sin\alpha - Z' \cos\alpha = 0$$

Introducing $y_1 = \cos\alpha$, $y_2 = \sin\alpha$ (i.e.: $\alpha = \text{arctg}(y_2/y_1)$) we get linear equations with regard to $\mathbf{y}$, and constraint $y_1^2 + y_2^2 = 1$. In [13] this strategy is used to generalize relative orientation with minimum 8 relative orientation points.

The trial and error method to find the actual solution $\mathbf{x}^*$ in a nonlinear problem might involve following steps:

- Guess different sets $\mathbf{x}^0$ distributed in a region that probably covers $\mathbf{x}^*$.
- For each set compute $S^0 = \sum_{i=1}^{n} f_i^2(\mathbf{x}^0) P_i$ with P_i properly chosen.
- Try an iterative method with the set $\mathbf{x}^0$ which has given the least S^0.
- If no success, try with the set $\mathbf{x}^0$ which has given the next least S^0, and so on.

Note: if $\mathbf{x}$ can be grouped into $\mathbf{x}_1, \mathbf{x}_2, \ldots$ so that (1) becomes

$$\mathbf{g}(\mathbf{x}_1)\mathbf{x}_2 + \boldsymbol{\ell}_1 = \mathbf{0} \quad \text{(linear in } \mathbf{x}_2\text{)}$$

$$\mathbf{h}(\mathbf{x}_1,\mathbf{x}_2)\mathbf{x}_3 + \boldsymbol{\ell}_2 = \mathbf{0} \quad \text{(linear in } \mathbf{x}_3\text{)}$$

$$\ldots\ldots$$

$$\mathbf{w}(\mathbf{x}_1,\mathbf{x}_2,\ldots) = \mathbf{0}$$

the derivation of $\mathbf{x}^0$ requires guessed values for only $\mathbf{x}_1$.

This strategy is used in [10] for relative orientation. The method is slow. Another approach could be to search for good approximations on the basis that certain conditions must be satisfied to guarantee the convergence of Newton's iteration. [8].

SOME ITERATIVE METHODS FOR LEAST SQUARES NONLINEAR ESTIMATION

Gauss Newton's method, [3] p. 164, gives fast convergence with $\mathbf{x}^0$ near $\mathbf{x}^*$, but is sensitive to poor $\mathbf{x}^0$. Let us write (1) as

$$\mathbf{f}(\mathbf{z};\mathbf{x}) = \mathbf{0}$$

where $\mathbf{z}$ = expectation of "observation" $\mathbf{Z}$ with weight coefficient matrix $\mathbf{Q}_{ZZ}$. Introducing $\mathbf{z} = \mathbf{Z} + \mathbf{v}_Z$, ($-\mathbf{v}_Z$: observational error) the error equations become

$$\mathbf{f}((\mathbf{Z} + \mathbf{v}_Z)\ ;\ \mathbf{x}) = \mathbf{0}$$

Because the left side $\approx \mathbf{f}(\mathbf{Z};\mathbf{x}) + \mathbf{A}\,\mathbf{v}_Z$, we can consider

$$\mathbf{v} = \mathbf{f}(\mathbf{Z};\mathbf{x})$$

as the error equations where $\mathbf{v}$ is the error of an equivalent observation with weight coefficient matrix ([9] p. 114):

$$\mathbf{Q} = \mathbf{A}\,\mathbf{Q}_{ZZ}\,\mathbf{A}^T, \quad \mathbf{A} = (\partial \mathbf{f}(\mathbf{z};\mathbf{x})/\partial \mathbf{z})_{\mathbf{z}=\mathbf{Z}} \qquad (3)$$

Thus, the least squares estimates $\mathbf{x}^*$ shal fulfil

$$S \equiv \mathbf{f}(\mathbf{Z};\mathbf{x})^T\,\mathbf{Q}^{-1}\mathbf{f}(\mathbf{Z};\mathbf{x}) = \text{min.}, \text{ or } \partial S/\partial \mathbf{x} = \mathbf{0} \qquad (4)$$

Gauss-Newton's method determines $d\mathbf{x}$ in the recursive formula

$$\mathbf{x}^{q+1} = \mathbf{x}^q + d\mathbf{x}, \quad q = 0,1.. \tag{5}$$

on the basis of solving the normal equations

$$\mathbf{B}^T \mathbf{P}\, \mathbf{B}d\mathbf{x} + \mathbf{B}^T \mathbf{P}\, \mathbf{f}(\mathbf{Z},\mathbf{x}^q), \quad \mathbf{P}^{-1} = \mathbf{Q}_{\mathbf{x}=\mathbf{x}^q} \text{ , cf. (3)} \tag{6}$$

which are derived by (4) using the approximation.

$$\mathbf{f}(\mathbf{Z};\mathbf{x}) \approx \mathbf{f}(\mathbf{Z};\mathbf{x}^q) + \mathbf{B}\, d\mathbf{x}\ , \quad \mathbf{B} = (\partial \mathbf{f}(\mathbf{Z};\mathbf{x})/\partial \mathbf{x})_{\mathbf{x}=\mathbf{x}^q}$$

The iteration is chosen on $\mathbf{x}$ only, cf. [9] p.276. $\mathbf{v}_z^{q+1}$ is obtained from ([9] p. 115):

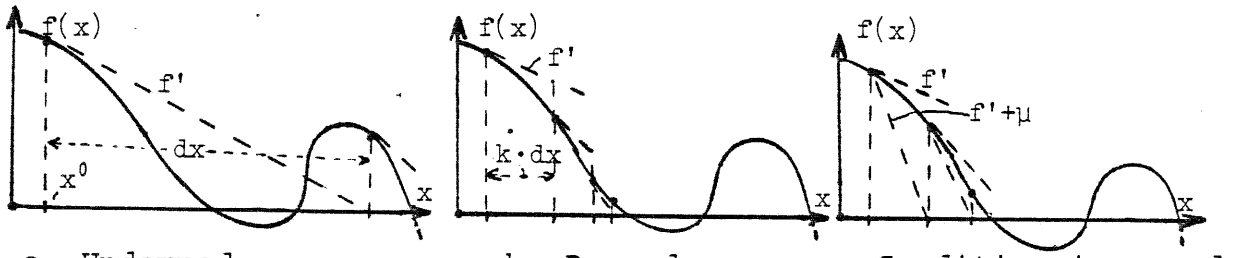
$$-\mathbf{v}_z^{q+1} = \mathbf{Q}_{zz}\mathbf{A}^T \mathbf{P}(\mathbf{f}(\mathbf{Z};\mathbf{x}^q) + \mathbf{B}\, d\mathbf{x})$$

Remarks and comments

- Determinating $d\mathbf{x}$ in (5) by solving

$$\mathbf{f}(\mathbf{Z};\mathbf{x}^q) + \mathbf{B}\, d\mathbf{x} = \mathbf{0} \tag{7}$$

  when m = n, agrees with Newton's method. [4] p. 8.
- Different criteria for the termination of the iteration process are discussed in [2] p. 240, [9] p. 289, [14] p. 6.
- Convergence is guaranteed with $\mathbf{x}^0$ sufficiently close to $\mathbf{x}^*$.



a. Undamped b. Damped c. Condition improved

Fig. 1. Examples of $\{\mathbf{x}^q\}$. (The basic method is Newton's)

- From a coarse $\mathbf{x}^0$, the iteration may diverge, repeat itself in cycles, behave quite unregulary near singular points, or converge (often after many iterations) to some undesired solution.
- The probability for convergence from an arbitary $\mathbf{x}^0$ is greater if $d\mathbf{x}$ in (5) is substituted by a damped term

  $k^q\, d\mathbf{x}, \quad k^q < 1$

  with a sequence of damping factors, $\{k^q\}$, properly chosen, cf. [4] p. 11. See fig 1b. (A so called damped method).
- The condition is improved if a scalar matrix is added

  $\mu^q \mathbf{I}$

  to $\mathbf{B}^T\mathbf{P}\mathbf{B}$ in (6) (or to $\mathbf{B}$ in (7), using Newton's method) with $\{\mu^q\}$ properly chosen, cf. [3] p. 166. See fig 1c. (A so called condition improved method).

The continuation method solves a nonlinear problem stepwise from arbitary $\mathbf{x}^0$ [1],[2] p. 252, [4] p. 18, [15]. Each niveau solves a nonlinear sub-problem, e.g., using Gauss-Newton's method as

will be shown below:

Let $S(\mathbf{x})$ be the minimization function (4) and let us assume that a new function $S'(\mathbf{x},t)$ can be defined so that $S(\mathbf{x},1) \equiv S(\mathbf{x})$ and

$$S'(\mathbf{x},0) = \min.$$

has the known solution $\mathbf{x} = \mathbf{x}^0$. Thus we can solve the problem

$$S'(\mathbf{x},t) = \min.$$

for an increasing sequence of values of t

$$t_0 = 0 < t_1 < t_2 \ldots < t_m = 1$$

giving a corresponding sequence of continually better approximations

$$\mathbf{x}^0 \equiv \mathbf{x}(t_0),\ \mathbf{x}(t_1),\ \mathbf{x}(t_2) \ldots \mathbf{x}(t_m) \equiv \mathbf{x}^*$$

Each niveau p uses for instance Gauss_Newton's method with initial approximations: $\mathbf{x}_p^0 = \mathbf{x}(t_{p-1})$ or better: $\mathbf{x}_p^0$ = extrapolated values ($p \geq 2$) from previous solutions, see [2] p. 252. The step-size Δt should be automatically adjusted so that the total number of iterations become small.

A design of S^0 is given in [7]. If weights P_i of independent equivalent observations are introduced, we can write:

$$S' \equiv t \sum_{i=1}^{n} (f_i(\mathbf{Z};\mathbf{x}))^2 P_i + (1-t) \sum_{i=1}^{n} [f_i(\mathbf{Z};\mathbf{x}^0)(1-t) - f_i(\mathbf{Z};\mathbf{x})]^2 P_i \quad (8)$$

Note that the following error equations give a minimization function $\equiv$ (8):

$$-v_i = f_i(\mathbf{Z};\mathbf{x}), \quad \text{weight} = t\,P_i, \quad i = 1,..n \quad (9a)$$

$$-v_i' = f_i(\mathbf{Z};\mathbf{x}^0)(1-t) - f_i(\mathbf{Z};\mathbf{x}), \quad \text{weight} = (1-t)P_i, \quad i=1,..n \quad (9b)$$

One alternative to (9b) is:

$$-v_i'' = x_i - x_i^0, \quad \text{weight} = P_i''(1-t), \quad i = 1,..m \quad (9c)$$

where P_i'' are properly chosen scale factors. On niveaus p>1, one might use $\mathbf{x}(t_{p-1})$ instead of $\mathbf{x}^0$.

If singularity occurs on a niveau, some measure must be taken, for example if (9a), (9b) are used, adding (9c) with t of (9c) as an additional parameter. In [1,4,15] there is a discussion of optimal step-size control, singularity and multiple solutions, etc.

A general remark on multiple solutions

m not-overdetermined polynomial equations of orders k_i give $(k_1 \cdot k_2 \cdot \ldots \cdot k_m)$ complex and real solutions. Addition of equations might reduce the number of solutions, because in the extended system of basic equations

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}, \quad \text{m equations} \quad (10a)$$

$$\mathbf{w}(\mathbf{x}) = \mathbf{0}, \quad \text{additional equations} \quad (10b)$$

there might be some $\mathbf{x} = \mathbf{x}^*$ that fulfils (10a) but gives a contradiction $\mathbf{V}^* = \mathbf{w}(\mathbf{x}^*)$ (disregarding observational errors.) Dependent

on $\mathbf{x}^0$ one might therefore find a local undesired minimum of S, cf. (4), on the basis of system (10). Testing the size of S* might reveal such a situation. Fig. 2 shows the case that (10a) are one linear equation and one equation for a circle, and (10b) is one additional linear equation. A point $\mathbf{x}^0$ estimated from the two linear equations might be so coarse due to bad geometry (small β) and inevitable errors (e) that the iteration converges to the false solution.

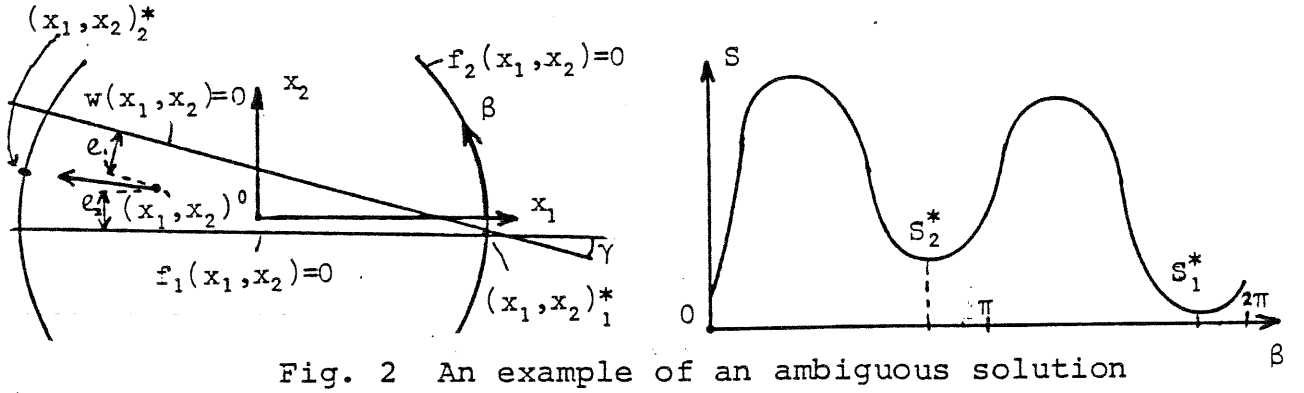


Fig. 2 An example of an ambiguous solution

METHODS OF GENERALIZED RELATIVE ORIENTATION

The traditional basis is the coplanarity condition, see fig. 3:

$$\begin{vmatrix} bx & by & bz \\ Tx & Ty & Tz \\ Tx' & Ty' & Tz' \end{vmatrix} = 0 \qquad (11)$$

where

$\{bx, by, bz\}^T$ (= **b**) : basis

$\{Tx, Ty, Tz\}^T$ = **Rx** (= **p**)

$\{Tx', Ty', Tz'\}^T$ = **R'x'** (= **p'**)

R,R': left, right photo rotation

$\{x,y,-c\}^T$ (= **x**): left image point

$\{x',y',-c\}^T$ (= **x'**): right image point

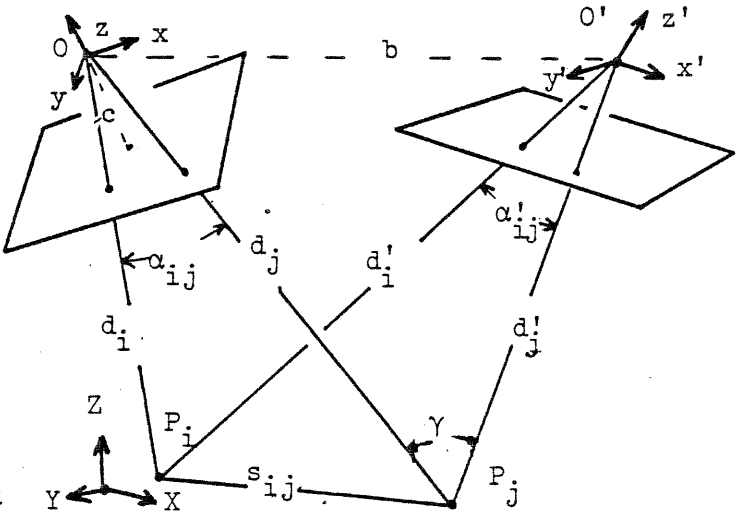


Fig. 3. Geometry of relative orientation

With weight = 1 of measured x,y,x',y', the weights of the independent equivalent observations can be derived [11]. Cf.(3).

The asymmetric case considers R',by,bz as unknowns, with R,bx freely chosen. Note: (11) is linear in by,bz/given R'. The following trial and error method [10] uses **R=I** and the 24 different quesses of R':

$$\mathbf{R'}^0 = \begin{Bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{Bmatrix}$$

where ε_{ij} = 0 or 1 or -1

with the condition that

$|\mathbf{R'}^0| = 1$

For each R'^0 corresponding $(by,bz)^0$ and $S^0 = (\mathbf{v}^T\mathbf{v})^0$, ($v_i$:contradictions in (11)) are computed with $bx = bx^0$ and by interchanging X/Y or X/Z, if necessary, to get $bx^0 > by$ and $bx^0 > bz$. Then the least squares iterative method is tried, first with $(\mathbf{R}', bx, by, bz)^0$ having given the smallest S^0, and so on.

A solution is found actual if the following is valid:

a. All $k_i, k_i' > 0$ after having found a pair of k_i, k_i' per model-point from:
$$k_i\mathbf{p} - k_i'\mathbf{p}' - \mathbf{b} = 0$$

Note: Introducing $\mathbf{b} = -\mathbf{b}$, the signs of k_i, k_i' are changed. If unequal signs, a rotation 200^g of a photo round the basis may give equal signs of all the new k_i, k_i', [13] p. 436.

b. $S^* < S_{tol}$ ("tol" means "specified tolerance").

c. Relations between distances, or angles in the model lie within specified tolerances, e.g., $|\gamma| > \gamma_{tol}$, fig. 3.

On the basis of results from testing this method of relative orientation on simulated and real cases (5-9 model points in spatial distributions), the number of trial and error cycles, n_{cy}, can be indicated as follows:

$n_{cy} < 20$ when using the Gauss-Newton's method and $n_{it}=15$ (n_{it}: the pre-set maximum number of iterations).

$n_{cy} < 10$ when using cotinuation as described above, with $n_{it}=25$, $\Delta t = 0.1$ and 2-3 iterations per niveau.

$n_{cy} < 10$ when using a damped or a condition improving method (see above) with $n_{it} = 20$, and a pre-set linearly decending $\{k^{q+1}\}$ or $\{\mu\}$ of 10-15 steps.

Further investigations on the application of step-size control to reduce n_{cy} are desired. A numerical example with data from [13] p. 438 follows:

c = 88.10 mm

	x(mm)	y
1	0.24	102.27
2	0.35	-1.46
3	81.65	93.61
4	75.82	-57.83
5	88.46	3.90

Solution 1 ($k_i<0$, $k_i'>0$)

$$\mathbf{R}' = \begin{vmatrix} -0.69 & 0.72 & 0.41 \\ 0.24 & 0.29 & -0.92 \\ -0.67 & -0.62 & -0.38 \end{vmatrix} \qquad \mathbf{b} = \begin{vmatrix} 0.33 & 1.00 & -0.06 \end{vmatrix}$$

Solution 2 ($k_4, k_4'>0$; $k_i, k_i'<0$; $i\neq 4$)

$$\mathbf{R}' = \begin{vmatrix} 0.71 & 0.01 & -0.70 \\ -0.21 & 0.95 & -0.21 \\ 0.66 & 0.30 & 0.68 \end{vmatrix} \qquad \mathbf{b} = \begin{vmatrix} -0.09 & 1.00 & -0.44 \end{vmatrix}$$

	x'	y'
1	-66.24	14.68
2	-82.22	-54.88
3	0.14	25.57
4	6.79	-69.48
5	5.44	19.33

Solution 3, ($k_i, k_i'<0$)

$$\mathbf{R}' = \begin{vmatrix} 0.86 & 0.30 & -0.41 \\ -0.48 & 0.75 & -0.45 \\ 0.16 & 0.58 & 0.79 \end{vmatrix} \qquad \mathbf{b} = \begin{vmatrix} 1.00 & -0.75 & 0.76 \end{vmatrix}$$

Solution 4, ($k_i, k_i'<0$)

$$\mathbf{R}' = \begin{vmatrix} 0.69 & 0.05 & -0.72 \\ -0.19 & 0.98 & -0.11 \\ 0.70 & 0.22 & 0.68 \end{vmatrix} \qquad \mathbf{b} = \begin{vmatrix} -0.14 & 1.00 & -0.31 \end{vmatrix}$$

Solutions fulfilling a can be derived from solutions 3 and 4 (introducing $\mathbf{b} = -\mathbf{b}$) but not from solutions 1 and 2. (Cf. note above).

The symmetrical case might choose $bx = bx^0$, $by = bz = 0$ and $a_{23} = 0$ ($\omega = 0$ or 200g) so that (11) becomes

$$bx^0(\mathbf{a}_2\mathbf{x} \cdot \mathbf{a}_3\mathbf{x}' - \mathbf{a}_2'\mathbf{x}' \cdot \mathbf{a}_3\mathbf{x}) = 0 \qquad (12)$$

when introducing

$$\mathbf{a}_i = \{a_{i1}, a_{i2}, a_{i3}\} \; ; \quad i = 1,2,3$$

with constraints

$$\sum_{j=1}^{3} a_{ij} \cdot a_{kj} = 0 \; ; \; (i,k) = (1,2), (1,3), (2,3) \qquad (13a)$$

$$\Sigma a_{ij}^2 = 1 \; ; \quad i = 1,2,3 \qquad (13b)$$

and correspondingly for a'_{ij}. (12) is linear in $\mathbf{a}_2, \mathbf{a}_3$ / given $\mathbf{a}_2', \mathbf{a}_3'$. Thus we might design the following trial and error method:

- Derive different sets $(\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_2', \mathbf{a}_3')^0$ by guessing $(\mathbf{a}_2', \mathbf{a}_3')^0$, for example:

$$\left\{\begin{matrix}\mathbf{a}_2'^0\\ \mathbf{a}_3'^0\end{matrix}\right\} = \left[\left\{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\end{matrix}\right\}, \left\{\begin{matrix}1 & 0 & 0\\ 0 & 0 & 1\end{matrix}\right\}, \left\{\begin{matrix}0 & 1 & 0\\ 1 & 0 & 0\end{matrix}\right\}, \left\{\begin{matrix}0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right\}\right]$$

  and deriving corresponding $(\mathbf{a}_2, \mathbf{a}_3)^0$ from (12) and (13b).

- Try a relative orientation (the unknowns chosen can be the 11 elements of $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_2', \mathbf{a}_3'$ with 6 constraints), first with the set $(\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_2', \mathbf{a}_3')^0$ that has given the smallest contradictions in (13), etc.

- $\mathbf{a}_1, \mathbf{a}_1'$ are computed from the solution of $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_2', \mathbf{a}_3'$, using the constraints of orthogonality. Ambiguity in signs of the estimates is obtained. The signs are determined so that $|\mathbf{R}| = |\mathbf{R}'| = +1$ and also - if possible - so that $\underline{a}$ (see above) is fulfilled.

Hout/Stefanovic's method [5] uses the nine elements u_{ij} of $\mathbf{U}_{3,3}$ as unknowns. These are defined by

$$\mathbf{U} = \mathbf{R}^T \left\{\begin{matrix}0, & -bz, & by\\ bz, & 0, & -bx\\ -by, & bx, & 0\end{matrix}\right\} \mathbf{R}'$$

and the basic equations for n model points are:

$$\mathbf{Cu} \equiv \mathbf{x}^T \mathbf{U}\mathbf{x}' = \mathbf{0}, \qquad \text{n equations} \qquad (14a)$$

$$\mathbf{U}^T(\mathbf{U}\mathbf{U}^T - b(\mathbf{u})\mathbf{I}) = \mathbf{0}, \qquad \text{3 constraints} \qquad (14b)$$

where $b(\mathbf{u})$ is half the trace of $\mathbf{U}\mathbf{U}^T$.

The method estimates u (one u_{ij} chosen) by means of (14) in the standard case (camera convergencies < 50g, by $\approx$ bz $\approx$ 0) which converges when u_{23}, u_{32} have the biggest absolute values among u_{ij}, and opposite signs. Non-standard cases are reduced to the standard one by interchanging rows and columns in $\mathbf{U}$ (and $\mathbf{C}$) and changing signs, after having found the two u_{ij} that probably have the biggest absolute values, as follows (rank $(\mathbf{C}) = m$):

- m basic unknowns are expressed in terms of not-basic ones, applying Cholesky's decomposition with pivoting on $\mathbf{C}^T\mathbf{C}$:

$$\mathbf{u}_{\text{basic}} = \mathbf{f}(\mathbf{u}_{\text{not basic}}) \qquad (15)$$

- If $m = 8$, $\mathbf{u}^0$ is derived by (15). (One u_{ij} freely chosen).

- If m < 8: Different sets u^0 are derived by allocating $u_{\text{not basic}}$ different logical combinations of proper values - e.g. two $u_{ij} = 1$, two $u_{ij} = 0$ when m = 5 - and computing corresponding u_{basic} by (15). The two u_{ij} that after the mentioned re-arrangement become u_{23}, u_{32}, are then taken as those with biggest $|u^0|$ in that set u^0 that gives least contradictions in (14b).

The method presumes that traditional parameters forming the actual model can be derived from $\mathbf{u}^*$. If no success (cf. a,b,c, above) one could continue and try the set u^0 that gives the next least contradictions and so on. The trial and error work depends on m. The point distribution (the geometry) and the observational accuracy must, however, be regarded when determining the rank. A more global method than Gauss-Newton's could be advantageous.

The relative orientation with object distances as unknowns is based on following equations (see fig. 3):

$$s_{ij} = \sqrt{d_i^2 + d_j^2 - 2d_i d_j \cdot \cos\alpha_{ij}} \equiv f_s(d_i, d_j, \alpha_{ij}) \qquad (16a)$$

$$s_{ij} = \sqrt{d_i'^2 + d_j'^2 - 2d_i' d_j' \cdot \cos\alpha_{ij}'} \equiv f_s(d_i', d_j', \alpha_{ij}' \qquad (16b)$$

where

s_{ij}: Distance between object points P_i and P_j

$d_i, d_j; (d_i', d_j')$: Distances from P_i and P_j to the projection centre O; (O').

$$\cos\alpha_{ij} = (x_i x_j + y_i y_j + c^2) / (\sqrt{x_i^2 + y_i^2 + c^2} \cdot \sqrt{x_j^2 + y_j^2 + c^2})$$

$\cos\alpha_{ij}'$ = corresponding expression in terms of x', y', c.

Note: There are 3(n-2) independent equations (13a)-(13b), with (2n-1) unknowns d_i, d_i'; one $d_i^{(1)}$ chosen. (n: number of model points).

An algorithm determining different sets $(\mathbf{d}, \mathbf{d}')^0$ which could be used in a trial and error method, would be:

- Guess different sets $(d_1, d_1')^0$ under the restriction that they must give positive and real $(d_2, d_2')^0$ using the following formulas derived from (16) with $s_{12} = 1$:

$$d_2 = d_1 \cdot \cos\alpha_{12} + \varepsilon\sqrt{-d_1^2 \sin^2\alpha_{12} + 1} \quad ; \quad \varepsilon = \pm 1$$

$$d_2' = d_1' \cdot \cos\alpha_{12}' + \varepsilon'\sqrt{-d_1'^2 \cdot \sin^2\alpha_{12}' + 1} \quad ; \quad \varepsilon' = \pm 1$$

Note: $\alpha_{12} > 100^g \rightarrow \varepsilon = 1$, $\alpha_{12}' > 100^g \rightarrow \varepsilon' = 1$

- If possible, determine corresponding sets positive and real $(d_k, d_k')^0$, k = 3,4,..,n, from the equations:

$$f_s^2(d_i, d_k, \alpha_{ik}) - f_s^2(d_i', d_k', \alpha_{ik}') = 0, \quad i = 1,2 \qquad (17)$$

Note: $(17)_{i=1} - (17)_{i=2}$ and $17_{i=1}$ (or $(17)_{i=2}$) give

$$d_k = a_k + \varepsilon_k\sqrt{g} \ , \quad d_k' = a_k' + \varepsilon_k\sqrt{g_k'} \ ; \quad \varepsilon_k = \pm 1$$

where $g_k = 0$ is introduced if $g_k < 0$ and $|g_k| < g_{tol}$. So also for g_k'.

- Compute some figure S^0 on the accuracy of $(\mathbf{d},\mathbf{d}')^0$, e.g.:

$$S^0 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (f_s(d_i,d_j,\alpha_{ij}) - f_s(d'_i,d'_j,\alpha_{ij}))^2$$

The we can try the iterative least squares method, first with the set $(\mathbf{d},\mathbf{d}')^0$ that has given the least S, etc.

ESTIMATING APPROXIMATIONS BEFORE BLOCK ADJUSTMENT

The general case with no restrictions as regards the size of the orientation parameters of bundles is assumed.

The following algorithm for estimating initial approximations involves subroutines for: relative orientation, transformation [12] intersection and resection [6], any of which need no or only coarse approximations of their respective unknowns.

1. Performance of individual relative orientations

   In block of n photos covering a common area, $\frac{1}{2}n(n-1)$ models can theoretically be formed. However, it is only the models that have a sufficient number of relative orientation points properly distributed in space to ensure reliability and little trouble with ambiguity, which are in fact formed.

2. Formation of sub-blocks and block

   Models are transformed in a proper sequence to form multiple sub-blocks or one block, in local coordinate systems. When determinating transformation parameters, common object-points (X,Y,Z) and common photo(s) $(X_0,Y_0,Z_0,\varphi,\omega,\kappa)$ being transformed, are to be considered. Sub-blocks are transformed like models, to form larger sub-blocks etc.

3. Absolute orientation of block units

   Models, or sub-blocks being formed can be transformed to the object coordinate system, as soon as sufficient control is contained, or the whole block can be absolutely orientated.

4. Intersections and resections

   There might be cases when operations 1-3 do not determine all object-coordinates and photo orientations. Therefore, the algorithm should also include the following operations:

   a. Intersections of points which have not been determined so far.

   b. Resections of photos which have not been orientated so far. (These resections might use intersected points as well).

   c. If necessary, repetition from a, etc.

   Operations 4 are performed during or after operations 2-3, in a local coordinate system or in the object-system.

The algorithm can be conveniently used on-line with error detection included.

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