A Stable Solution to Cell degeneracy in Grid Contouring

Nira Dyn
School of Mathematical Sciences, Tel-Aviv University,
Ramat Aviv 69978, Tel Aviv, Israel.

Samuel Rippa
Xi Information Processing Systems LTD.
P.O.Box 758, Tel-Aviv, Israel.

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Abstract Contour maps are a useful tool for describing surfaces of the form $z = f(x,y)$. Among the many algorithms for computing contour maps, an important class of methods create contours from gridded data. The input to these methods is a data matrix containing the values of f at the nodes of a rectangular grid. When implementing this type of contouring algorithm a problem of *degenerate cells* arise. A degenerate cell is a cell of the grid in which four intersections of a contour line with the cell's boundaries occur. The basic problem is the lack of information : there seems to be three different ways of linking together the intersection points to form the contour lines. Different schemes for choosing one of the three alternatives are suggested in the literature. Most of them are *unstable*, namely a small change in the values of f at the nodes of a degenerate cell may cause a significant change in the shape of the contour lines , which is undesirable. In this paper we propose a stable solution to cell degeneracy, resulting in a very simple scheme which compares favorably to other solutions.

1 Introduction

Contour maps are a useful, two dimensional , representation of three dimensional surfaces of the form $z = f(x,y)$. There exit various algorithms for computing contour maps and many of the schemes and ideas involved are summarized in the survey papers of Sutcliffe [3] and Sabin [2].

In this paper we are concerned with schemes for contouring *gridded data* It is assumed that f is known only at the nodes of a regular grid located over the rectangular domain $R = [a, b] \times [c, d]$. The input for the contouring scheme is a data matrix $Z = \{z_{ij}\}$ such that :

$$z_{ij} = f(x_i, y_j) \ , \ i = 1, \dots, n \ , j = 1, \dots, m$$

where

$$\begin{aligned} x_i &= a + \delta_x(i-1) \ , \ \delta_x = (b-a)/(n-1) \\ y_j &= c + \delta_y(j-1) \ , \ \delta_y = (d-c)/(m-1) \end{aligned}$$

Each sub-rectangle $R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ of the grid is called a *grid cell.*

The process of contouring gridded data is composed of two basic steps :

1. Find all intersections of the contour h (i.e. the contour defined by the equation $f(x, y) = h$) with edges of the cells of the grid. Intersection between the contour h and an edge of a grid cell (with end points A and B) occur when

   $$\min(f(A), f(B)) \leq h \leq \max(f(A), f(B))$$

   The usual assumption (see e.g. [3]) is that δ_x and δ_y are sufficiently small so that f behaves linearly on all edges of a grid cell. This means that no contour h crosses any edge of a grid cell more than once and that the actual intersection point may be approximated using inverse linear interpolation between points A and B.

2. Trace the intersection points through the grid cells and link them together to form contour lines.

When implementing a contouring algorithm by the above approach one soon discovers two types of ambiguous situations. The first involves a ***degenerate point*** which is a point (x_i, y_j) for which $z_{ij} = h$. This type of degeneracy is easily handled, e.g. by "virtually" changing z_{ij} by a small amount when processing Step 1. above. This virtual change also avoids the occurrence of a degenerate edge - an edge connecting two degenerate points. The second type of degeneracy ***a degenerate cell*** occurs when four intersections are recorded on the boundaries of one grid cell (Figures 1).

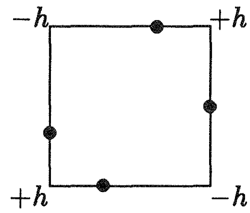

Figure 1: A degenerate cell

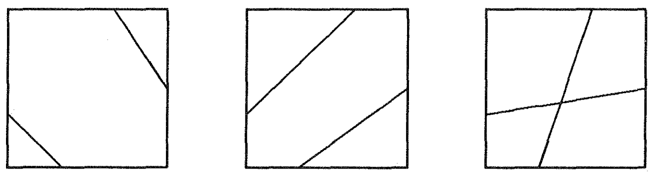

Figure 2: Three ways to trace contours in a degenerate cell.

This type of cell is called degenerate since, due to lack of information, there seems to be three possible ways for connecting the intersection points to form the contour lines, as shown in Figure 2.

We refer to any scheme for connecting the intersection points in a degenerate cell as a solution to the cell degeneracy problem (SCDP). We also call a SCDP *stable* if the shape of the connected contour lines changes only slightly when altering the values z_{ij} at the corners of the cell by a small amount. Clearly stability is a desired property of any SCDP. As far as we know this important feature has never been discussed before as a measure for the quality of a contouring method.

Many suggestions for SCDPs are summarized in [3], most of them address the problem in the second step of the contouring algorithm. Starting with an entry point to the degenerate cell, the goal is to find the exit point. One solution uses a table from which the exit edge is selected according to the entry edge. Another solution uses the principle "keep the high ground on the right". The edges are scanned in the order right, top, left (relative to the starting edge) until a suitable exit point is found. Another possibility is to

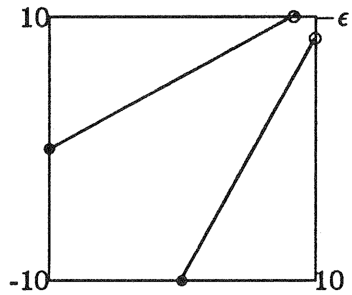

Figure 3: Unstable contour lines connection in a degenerate cell.

choose the exit point so that the change in direction from the previous step will be minimal. All these solutions share one thing in common : they are unstable. To see the instability consider the example in Figure 3 in which a contour line $h = 0$ is traced and ϵ is a small positive number. Suppose that the starting edge was the bottom edge. The high ground to the right principle will connect the solid points with the circled points and so will do the tabular solution. Let us assume that the lower-left corner of the cell below our degenerate cell has a value of ϵ and the lower-right corner has value of 10. In this case, the minimal change principle also connects the solid and the circled points since it result in zero change in direction. By changing the value on the top-right corner of the cell in Figure 3 from $-\epsilon$ to ϵ we eliminate the degeneracy of the cell. The two remaining (solid) intersection points are then connected to each other, indicating the instability of all SCDP's , presented above (a small change in the value of the top-right corner changed completely the shape of the contour lines).
Another SDCP ([4]) connects the intersection points so that the resulting contour lines separate the higher ground points. This solution is also unstable as can be verified on the above example.
We note that in ([3]) one stable SCDP is presented. We discuss this solution in in light of our solution.

In this paper we propose to use a piecewise bilinear model for the degenerate cells. This model provides a natural and stable solution to cell degeneracy, resulting in a very simple scheme which compares favorably to other solutions. In the above example of Figure 3 the solution with the bilinear model connects the solid points to each other and the circled points to each other.

2 A stable solution for cell degeneracy

A bilinear model for the surface in a degenerate cell R_{ij} with lower-left corner (x_i, y_j) is :

$$P_{ij} = a_{ij} + b_{ij}\frac{(x - x_i)}{\delta_x} + c_{ij}\frac{(y - y_j)}{\delta_y} + d_{ij}\frac{(x - x_i)(y - y_j)}{\delta_x \delta_y}, \ (x, y) \in R_{ij} \quad (1)$$

The interpolation conditions at the four corners of R_{ij} :

$$P_{ij}(x_r, y_s) = z_{rs}, r = i, i + 1 \ , \ s = j, j + 1$$

uniquely determine P_{ij}. The coefficients of P_{ij} are :

$$\begin{aligned} a_{ij} &= f_{ij} \\ b_{ij} &= f_{i+1,j} - f_{ij} \\ c_{ij} &= f_{i,j+1} - f_{ij} \\ d_{ij} &= f_{i+1,j+1} - f_{i+1,j} - f_{i.j+1} + f_{ij} \end{aligned}$$

Using this model,the problem of contouring within a degenerate cell can be replaced by the problem of calculating exactly the contour curves of P_{ij}. This approach provides a stable solution for cell degeneracy since P_{ij} is a continuous function and thus, small perturbations of the values at the four corners change the locations of the contour curves of P_{ij} only slightly.

To trace the contours of P_{ij}, we observe that P_{ij} is linear on each edge of the degenerate cell and thus no more than one intersection point exists between a contour and such an edge. Moreover the intersection point can be calculated exactly using inverse linear interpolation. We note, also, that P_{ij} is linear on each horizontal or vertical line inside the cell. Thus *there can be at most one intersection point between the contours of height h and any horizontal or vertical line inside a degenerate grid cell* (except when the vertical or horizontal line is a segment of the contour line). This observation provides the rule for connecting the intersection points within a degenerate cell.

In words : a point on the left (right) edge is connected to the point with minimal (maximal) x coordinate or equivalently, a point on the top (bottom) edge is connected to the point with maximal (minimal) y coordinate. By

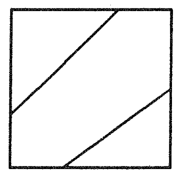

Figure 4: A connection incompatible with the bilinear model.

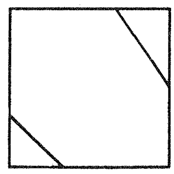

Figure 5: A connection compatible with the bilinear model.

this rule Figure 4 is excluded and the only possible way to connect the intersection points is as in Figure 5.

Note that the two criterions above are equivalent only if the intersection points are computed by inverse linear interpolation. In any other case it might happen that the two criterions result in different connections of the intersection points as demonstrated in Figure 6. In this case the two connecting lines are intersected either by a horizontal or by a vertical line (e.g. the dashed lines in Figure 6).

The above SCDP does not lead to a unique choice of alternatives only in the very special case where the left and right intersection points have the same y coordinate (consequently the top and bottom intersection points must have the same x coordinate) as is illustrated in Figure 7. The only correct way to connect the intersection points , according to the bilinear model, is as in Figure 7a (think about the contour $h = 0$ of the function $z = xy$). This type of connection is unpleasant to the eye and represents an ideal mathematical model. To avoid such situations we suggest to use the same treatment as in the case of point degeneracy, i.e. to alter "virtually" the values of the data

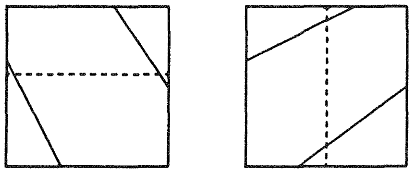

Figure 6: Intersection points *not obtained* by inverse linear interpolation : no connection is compatible with the bilinear model.

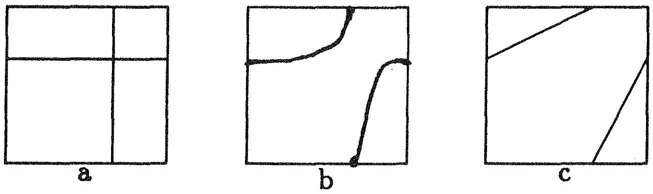

Figure 7: Special case of cell degeneracy : (a) contours of the bilinear model, (b) smooth contours produced by formulas (4) and (5) after virtual change, (c) connection by line segments after virtual change.

at one corner of the cell so that the different contour branches will not cross. To retain the stability of the solution one should not connect the intersection points by straight line segments as in Figure 7c but rather follow the contour lines of P_{ij} as in figure 7b. To do so consider the grid cell R_{ij} and define $t = (x - x_i)/\delta_x$, $s = (y - y_j)/\delta_y$. Equation (1) can be rewritten as :

$$P_{ij} = P_{ij}(t, s) = a_{ij} + b_{ij}t + (c_{ij} + d_{ij}t)s \ , \ s, t \in [0, 1] \tag{2}$$

The implicit formula for the curve of the contour h is given by :

$$h = P_{ij}(t, s) \tag{3}$$

Denote by (t_0, s_0) and (t_n, s_n) two intersection points of the contour h with the cell's edges, i.e. $P_{ij}(t_0, s_0) = P_{ij}(t_n, s_n) = h$. Suppose furthermore that these points are to be connected, by the bilinear model, to form a contour line. We can distinguish between two cases :

(a) $|t_n - t_0| \geq |s_n - s_0|$. In that case we can use (2) and (3) to write s as a function of t :

$$s = s(t) = \frac{h - a_{ij} - b_{ij}t}{c_{ij} + d_{ij}t} \tag{4}$$

and evaluate the function s(t) at the points :

$$t_i = t_0 + \frac{t_n - t_0}{n} i \ , \ i = 1, \ldots, n-1$$

Then we obtain the contour line by connecting the points $(t_i, s(t_i)), i = 0, \ldots, n$ with line segments. This curve is much smoother (depending on how large is n) than the line segment connecting (t_0, s_0) and (t_n, s_n).

(b) $|t_n - t_0| < |s_n - s_0|$. In that case we can use (2) and (3) to write t as a function of s :

$$t = t(s) = \frac{h - a_{ij} - c_{ij}s}{b_{ij} + d_{ij}s} \tag{5}$$

and attain the contour line as above, with the roles of t and s interchanged.

It should be remarked that formulas (4) and (5) can be used to obtain smooth contour lines within a grid cell.
Before concluding we mention the method of Heap for the solution of cell degemeracies and point out its close relation to the method proposed in this paper. Heap divides each grid cell into four triangles by the diagonals , taking the value at the center to be the mean of the four values at the corners of the cell. This approach gives a stable solution to degenerate cells (a property not observed up to now) and, also, smooth the contour lines since in each cell a contour curve is composed of two linear pieces instead of one (see [3] and [2] where this method is called "St. Andrew's cell").
Observing that the value of P_{ij} at the center of the cell R_{ij} is the average of the values at the four corners of R_{ij}, we conclude that Heap's method is based on a piecewise linear approximation on triangles to the bilinear model P_{ij}. Thus our method is an improvement over Heap's method yielding smoother contour lines.

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