VERTICAL PERSPECTIVE PROJECTION OF THE ROTATIONAL ELLIPSOID

Prof. Wagih N. Hanna
Faculty of Engineering Ain Shams University , Abbassia, Cairo, EGYPT
Comission IV , Working Group 2

KEY WORDS: Cartography, Space, Mapping, Analytical, Geometric.

ABSTRACT

The part of the earth visible from a certain outer point, which is the perspective center is illustrated in the photo from the space camera as a perspective projection. We consider the camera axis to be truly vertical through the center of projection and normal to the surface of the ellipsoid. In fact there are two real normals through any point in the plane of the ellipse. One normal belongs to the nearest part of latitude ϕ_o , another normal to the other part of latitude ϕ'_o. These latitudes are of different values and signs. The location of any point is given by the geographic coordinates ϕ, λ as well as its altitude h above the reference ellipsoid. In this paper we deduce analytically the corresponding map coordinates of the perspective projection of the points on the surface of the rotational ellipsoid onto the tangent plane through the two points of intersection of these normals with the ellipsoid. The coordinates are deduced in terms of the geodetic and geocentric coordinates.

1 INTRODUCTION

1.1 Notations Fig.(1), Fig.(2)

L	: The perspective center.
H	: The altitude of L.
LO′, LO″	: The perpendiculars on the ellipsoid from L.
ϕ	: Latitude.
λ	: Longitude
h	: Altitude above the reference ellipsoid.
$P(\phi, \lambda, h)$	: Any Point on the earth= $P(X_p,Y_p,Z_p)$.
$P_o(\phi_o,\lambda_o,h_o)$	: Center of map (minimal distance).
$P'_o(\phi'_o,\lambda'_o,h'_o)$	: Center of map (maximal distance).
T	: Tangent plane at P_o.
T_1	: Tangent plane at P'_o.
ϕ_g	: The geocentric latitude w.r.t the center of the earth O.
$P_1(x_1,y_1)$	: The map point corresponding to P
X_L-X_P	: ΔX.
Y_L-Y_P	: ΔY
Z_L-Z_P	: ΔZ
LP'_o	: D
$\phi_o-\phi'_o$	: $\Delta\phi$
$\lambda - \lambda_o$	: $d\lambda$

1.2 The Geometry of the Rotational Ellipsoid

Consider the earth as a rotational ellipsoid (Reference Ellipsoid) with semi-major axis a and semi minor axis c (axis of rotation). The equation of the rotational ellipsoid in the system shown in (Fig.2) is given by:

$$\frac{X^2+Y^2}{a^2}+\frac{Z^2}{c^2}=1 \tag{1}$$

The following elliptical parameters (Fig.3) are frequently used in the analytical calculations:

Eccentricity :
$$e^2=\frac{(a^2-c^2)}{a^2} \tag{2}$$

Radius of curvature: at a point on the ellipsoid:

$$N=\frac{a}{\sqrt{(1-e^2\sin\phi)}} \tag{3}$$

1.3 Coordinates

The position of a point on the surface of the rotational ellipsoid is expressed by either the geocentric cartesian coordinates (X,Y,Z) or by the two dimensional geodetic coordinates (ϕ,λ), where :

$$X=\frac{a}{\sqrt{1-e^2\sin\phi}}\cos\phi\cos\lambda \tag{4}$$

$$Y=\frac{a}{\sqrt{1-e^2\sin\phi}}\cos\phi\sin\lambda \tag{5}$$

$$Z=\frac{a(1-e^2)}{\sqrt{1-e^2\sin\phi}}\sin\phi \tag{6}$$

$$\phi=\arctan\left[\frac{Z}{(1-e^2)\sqrt{X^2+Y^2}}\right] \tag{7}$$

$$\lambda=\arctan\left(\frac{Y}{X}\right)=\arccos\left[\frac{X}{\sqrt{X^2+Y^2}}\right] \tag{8}$$

The equation of the tangent plane at any point (ϕ,λ) is given by:

$$x\cos\phi\cos\lambda+y\cos\phi\sin\lambda+z\sin\phi=\frac{a^2}{N} \tag{9}$$

Sometimes the points are expressed by the geocentric latitude ϕg, which is the angle subtended by the geocentric line OP and the equator. The relation between the geocentric latitude ϕg and the geodetic latitude ϕ is

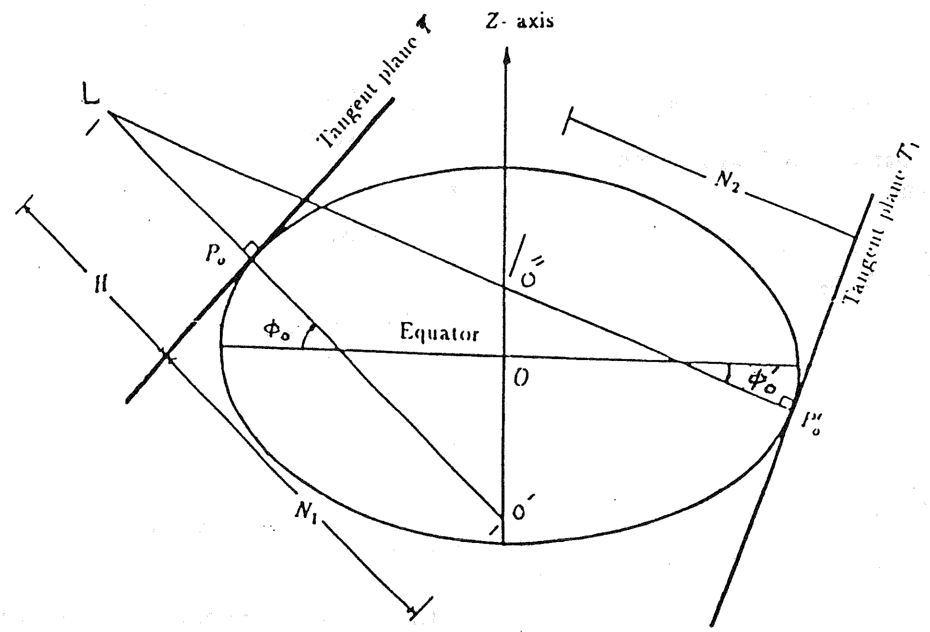

Fig. (1) Two real normals from one point in the plane of the ellipse

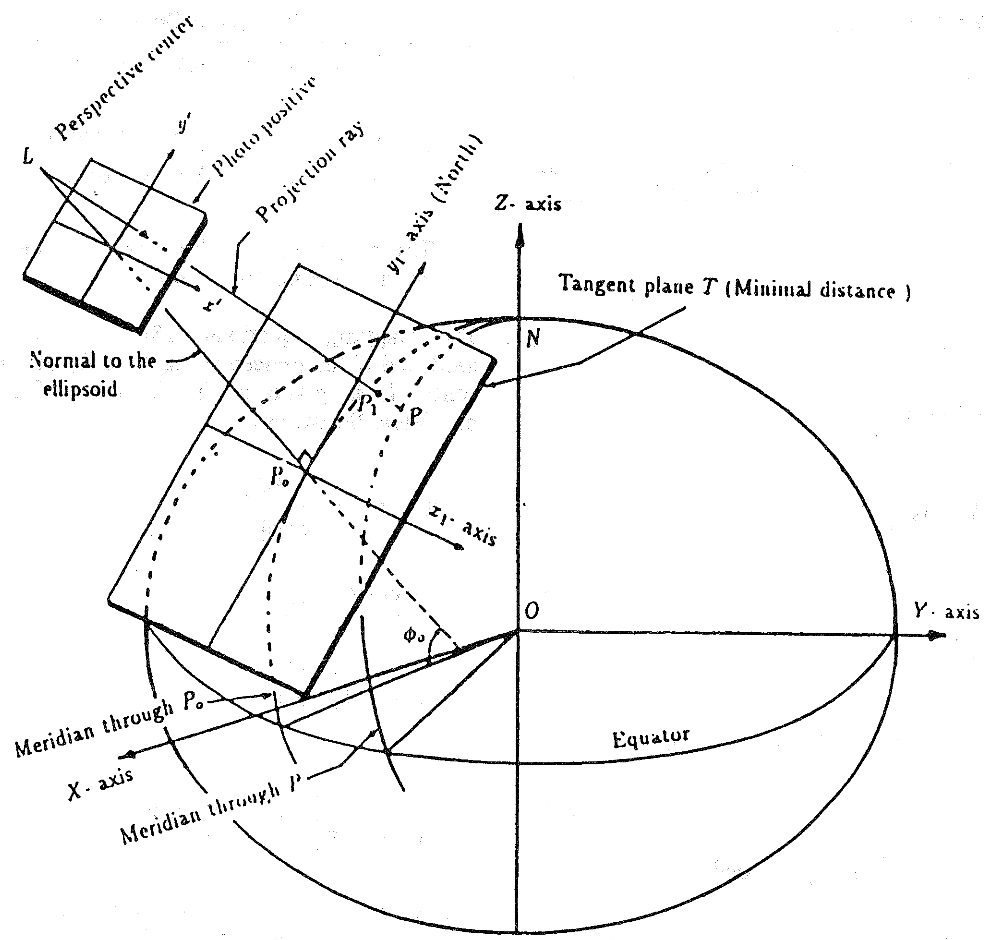

Fig (2) Geometry of the perspective projection (minimal distance) given by:

$$\phi_g = \arctan[(1-e^2)\tan\phi] \quad (10)$$

2 CENTRAL PROJECTION

2.1 Vertical Projection onto a Tangent Plane T (Minimal Distance)

From Fig.(4), we have for the point P:

$$y_1 = LP_o \tan\Delta_1 = H\tan(\Delta - \Delta_2)$$

$$\tan\Delta = \frac{\Delta X}{\Delta Z}$$

$$\Delta_2 = \frac{\pi}{2} - \phi_o$$

So we get:

$$y_1 = H\frac{\Delta X\sin\phi_o - \Delta Z\sin\phi_o}{\Delta Z\sin\phi_o + \Delta X\cos\phi_o} \quad (11)$$

Also

$$x_i = \frac{y_1 . Y_P}{PK} = \frac{y_1 . Y_P}{LP\sin\Delta_1}$$

where LP=ΔX/sinΔ. So we get

$$x_1 = H\frac{Y_P}{\Delta Z\sin\phi_o + \Delta X\cos\phi_o} \quad (12)$$

To derive the mapping equation in terms of the geocentric coordinates (ϕ,λ,h) of the point P, P_o and L we have:

For the point P:

$$X_p = PJ = (N+h)\cos\phi\cos d\lambda$$

$$Y_p = PS = (N+h)\cos\phi\sin d\lambda \quad (13)$$

$$Z_p = JO = [N(1-e^2)+h]\sin\phi$$

For the point P_o:

$$X_{p_o} = MO = (N_o + h_o)\cos\phi_o$$

$$Y_{p_o} = 0 \quad (14)$$

$$Z_{p_o} = P_oM = [N_o(1-e^2)+h_o]\sin\phi_o$$

$$X_L = RO = RM + MO = [H + h_o + N_o]\cos\phi_o = R_o\cos\phi_g$$

$$Y_L = 0 \quad (15)$$

$$Z_L = LR = [H + h_o + N_o(1-e^2)]\sin\phi_o = R_o\cos\phi_g$$

Substituting from the above geocentric coordinates of the points P, P_o, and L into equations (11) and (12), we obtain after some trignometrical and algebraic reductions the mapping equations:

$$x_1 = H\frac{C\sin d\lambda}{G - S\sin\phi_o - C\cos\phi_o\cos d\lambda}$$

$$y_1 = H\frac{F + S\cos\phi_o - C\sin\phi_o\cos d\lambda}{G - S\sin\phi_o - C\cos\phi_o\cos d\lambda} \quad (16)$$

where

$$C = \frac{(N+h)}{a}\cos\phi$$

$$S = \frac{N(1-e^2)+h}{a}\sin\phi$$

$$F = \frac{e^2 N_o\sin\phi_o\cos\phi_o}{a} \quad (17)$$

$$G = \frac{H + h_o + N_o(1 - e^2\sin^2\phi_o)}{a}$$

After expanding the forms of sine and cosine of (λ-λ_o) and substitiuting into equations (16), we get the general mapping formulas of the vertical perspective projection of the rotational ellipsoid onto the tangent plane of minimal distance:

$$x_1 = H\frac{C\cos\lambda_o\sin\lambda - C\sin\lambda_o\cos\lambda}{G - S\sin\phi_o - C\cos\phi_o\cos\lambda_o\cos\lambda - C\cos\phi_o\sin\lambda_o\sin\lambda} \quad (18)$$

$$y_1 = H\frac{F + S\cos\phi_o - C\sin\phi_o\cos\lambda_o\cos\lambda - C\sin\phi_o\sin\lambda_o\sin\lambda}{G - S\sin\phi_o - C\cos\phi_o\cos\lambda_o\cos\lambda - C\cos\phi_o\sin\lambda_o\sin\lambda} \quad (19)$$

2.2 The mapping equations using geocentric latitude of the perspective center

The mapping equations (18) and (19) can also be modified if the geocentric latitude ϕ_g of the perspective center L is given as in the case of some satellite problems. So we have:

$$R_o = (H + h_o + N_o)\frac{\cos\phi_o}{\cos\phi_g}$$

$$G = \frac{R_o\cos(\phi_o - \phi_g)}{a}$$

$$F = \frac{R_o\cos(\phi_o - \phi_g)}{a} \quad (20)$$

$$\phi_o = \phi_g + \arcsin\left[\frac{e^2 N_o\sin\phi_o\cos\phi_o}{R_o}\right]$$

The value of ϕ_o may be found by a rapidly converging iteration, with an initial value of ϕ_o=ϕ_g and using equation (20) to obtain Ro. The new value R_o is used to obtain the next approximation of ϕ_o . The values of R_o and ϕ_o are iterated until the change in ϕ is considered negleigible.

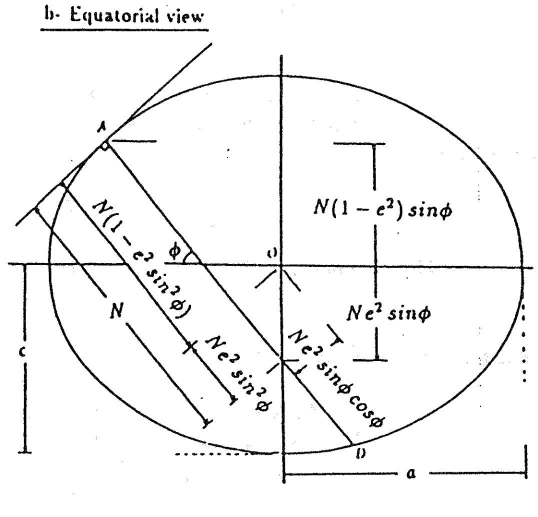

Fig (3) Dimensions on the rotational ellipsoid

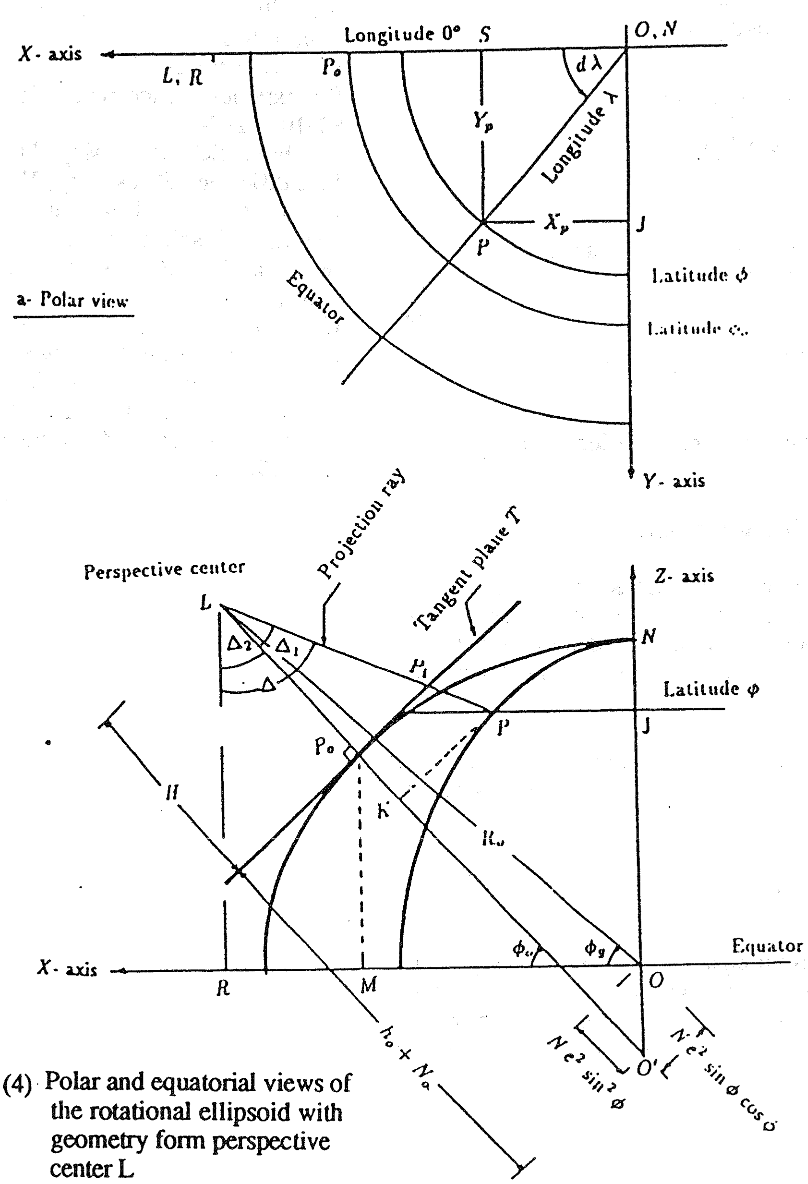

Fig (4) Polar and equatorial views of the rotational ellipsoid with geometry form perspective center L

2.3 Vertical Projection onto a Tangent Plane T_1 (Maximal Distance)

In this case, the perspective center L is at a distance LP'_o=D from the tangent plane T_1, the relation between ϕ_o and ϕ'_o is given by the equation:

$$\phi'_o = -\phi_o + \arcsin\left[\frac{e^2 \sin(2\phi_o)}{1+\dfrac{H}{N_o}}\right] \tag{21}$$

The normal distance LP'_o=D between the center L and the plane T_1 is given by:

$$D = N'_o + [N_o + H]\frac{\cos\phi_o}{\cos\phi'_o} \tag{22}$$

The mapping equations can be obtained by the same method explained above. The analogy of equations (16) gives the mapping coordinates relating to the new system:

$$x'_1 = D\,\frac{C\sin d\lambda}{G' - S\sin\phi'_o - C\cos\phi'_o\cos d\lambda}$$

$$y'_1 = D\,\frac{F' + S\cos\phi'_o - C\sin\phi'_o\cos d\lambda}{G' - S\sin\phi'_o - C\cos\phi'_o\cos d\lambda} \tag{23}$$

where

$$F' = \frac{e^2 N_o \sin\phi_o \cos\phi'_o - (H + h_o + N_o)\sin\Delta\phi}{a}$$

$$G' = \frac{-e^2 N_o \sin\phi_o \sin'\phi + (H + h_o + N_o)\cos\Delta\phi}{a} \tag{24}$$

C and S are determined from equation (17).

2.4 Inverse Mapping Equations

The inverse mapping equations for the vertical prespective projection of the ellipsoid permit the calculation of geoditic latitude ϕ and longitude λ for a selected point which has rectangular coordinates x_1, and y_1.

2.4.1 Geoditic Latitude ϕ : Since the mapping equations (18),(19) are linear in $\sin\lambda$ and $\cos\lambda$, then they can be arranged as follows:

$$A_1 C \sin\lambda + A_2 C \cos\lambda = A_3 S + A_4$$
$$A_5 C \sin\lambda + A_6 C \cos\lambda = A_7 S + A_8 \tag{25}$$

By elliminating λ from equations (25) we get a quadratic equation in S, which has two solutions. One of them gives the latitude of the nearest point on the ellipsoid, the other of the hidden point on the other side. Since Z>0 then the positive root is considered. So we have :

$$\sin\phi = \frac{a\,S}{N(1 - e^2) + h} \tag{26}$$

Then by using an itteration method we get ϕ.

2.4.2 Geoditic Longitude λ: After some trignometric calculations equations (25) can be reduced to:

$$\lambda = \lambda_o + \arctan\left[\frac{x_1(T\sin\phi_o - S)}{S(y_1\sin\phi_o + H\cos\phi_o) - (y_1 G - HF)}\right] \tag{27}$$

where: $T=[H+h_o+N_o(1-e^2)]/a$

References

[1] Abbas, Y. Triple projection of a topographic surface from an external persepective center, Ph.D Thesis, Assuit University, Egypt (1993).

[2] Alfred, L. GPS Satellite Survying, John Wiley and Sons, Inc., New York (1990).

[3]Beate, M., Kartenprojektionen des dreiacbsigin Ellipsoid Geodatishes Institut, Stuttgart (1991).

[4] Dragomir, V. Theory of Earth Shape., Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York (1982).

[5] Dumetrescu T.V., Kosmographishe Persepective Projektionen. Allgemeine Vermeasugs Nachrichten pp. 91-104 (1974).

[6] Fredenick P. , Map Projection Methods, Sigma Scientific Inc., Blacksburg, Virginia (1984)

[7] Kahle H.G., Einführung in die Höhere Geodasie Verlag der Fachverein, Zürich, (1988).

[8] Lauf G.B. Geodesy and Map Projection , Tafe Pub. Unit (1983).

[9] Peno P., Transformation of Rectangular Coordinates into Geographical by Closed Formulas, Mapping and Photogrammetry Vol. 20, No. 3, pp 175-177, (1978).

[10] Snyder P.J., The perspective Map Projection of the Earth, The American Cartographer Vol. 8, No. 2, pp 149-160 (1981).