TIME-SPACE MAPPING BASED ON FREE NET - TRILATERATION

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ABSTRACT:

Time-space mapping gives the distortion to a physical map such that the distances between any two points on the map are as consistent as possible with given time distances. If the time-space mapping procedure is integrated into GIS, it will be an attractive presentation tool for regional and transportation analyses. Time-space mapping procedure has been in general divided into the following two steps: i) multi-dimensional scaling (MDS) step which configures the points given the time distances between their points; ii) interpolation step which gives a mapping from the physical map to the time-space map based on the configured points by MDS. In conventional studies, the above two steps have been independently implemented. That is, the errors of points' configuration by MDS which are generally inevitable have been ignored in the interpolation step. This paper provides a possible time-space mapping procedure which is theoretically consistent with the least squares method by introducing the free network adjustment of trilateration.

1. INTRODUCTION

Time-space mapping gives the distortion to a physical map such that the distances between any two points on the map are as consistent as possible with given time distances. Time-space map shows visually an outline of the transportation level of service. Two time-space maps before and after a certain transportation improvement visualize impressively its impact.

Time-space mapping procedure is basically a technique of the transformation of map coordinates. If the time-space mapping procedure is integrated into GIS, it will be an attractive presentation tool for regional and transportation analyses.

There have been so far a few studies associated with the time-space mapping (Ewing and Wolfe, 1977; Shimizu, 1993; Spiekermann and Wegener, 1933). According to these studies, the time-space mapping procedure is in general divided into the following two steps:

- Given the time distances between some points, provide the configuration of the points on the time-space map. This procedure is interpreted as multi-dimensional scaling (MDS) into two-dimensional space.

- Interpolate (or extrapolate) the other map elements which are portrayed on the physical map onto the time-space coordinate system by using the points configured by the MDS as control points.

In conventional studies the above two steps have been independently implemented. That is, in the interpolation step, the points configured by the MDS have been assumed to have independent and equal positional error. However, we cannot necessarily assert positively that such an assumption holds for the configured points. It would be reasonable that we regard the coordinates of the points as the random variables which are mutually dependent and unequally distributed. Variance and covariance of the coordinates of configured points should be dealt with in the interpolation procedure.

This paper provides a possible time-space mapping procedure which is theoretically consistent with the least squares method by employing the free network adjustment of trilateration based on Moore-Penrose generalized inverse. The next chapter shows Torgerson's MDS which has been most frequently used in MDS applications and discusses its the problems as MDS procedure for time-space mapping. Chapter 3 gives the basic formulation of the least squares MDS. In Chapter 4, the least squares MDS based on the free network concept (Free Network MDS) is formulated. The following chapter shows the interpolation procedure using the points configured by Free Network MDS.

2. TORGERSON'S MDS

Let i $(i=1,2,..,n)$ be the points between which the time-distances, t_{ij}, are given. Let the coordinates of the time-space map be denoted by (u_i, v_i). Assume that the time-distances are given to all pairs of the points, that is, the number of the observations is $m=n(n-1)/2$.

Torgerson's MDS (Torgerson, 1952) begins from the calculation of the inner product matrix $R = (r_{ij})$ constituted by the position vectors taking the gravity point as the origin (Young and Householder's transformation). The inner product of points i and j is given by

$$r_{ij} = \frac{1}{2}\left(\frac{1}{n}\sum_i t_{ij} + \frac{1}{n}\sum_j t_{ij} - \frac{1}{n^2}\sum_i \sum_j t_{ij} - t_{ij}^2\right). \tag{1}$$

Assume that the points can be distributed in Euclidean space and the rank of the inner product matrix R is r.

From the Young and Householder's theorem, R can be decomposed as

$$R = DD^t \tag{2}$$

and each row vector of D shows the coordinates of the point concerned in r-dimensional space. Since D is a real symmetric matrix and the rank is r, R is diagonalized by the orthogonal matrix X as follows;

$$X^t R X = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{bmatrix} \tag{3}$$

where $\lambda_1,..,\lambda_r$ are the positive eigen values of R and X is the matrix in which the column vectors are composed by the normalized eigen vectors $x_1,..,x_r$, i.e., $|x_i| = 1$, corresponding to $\lambda_1,..,\lambda_r$. Since the diagonal components of Λ is all positive,

$$\begin{aligned} R &= X\Lambda X^t \\ &= \left(X\Lambda^{1/2}\right)\left(X\Lambda^{1/2}\right)^t. \end{aligned} \tag{4}$$

Thus,

$$\begin{aligned} D &= X\Lambda^{1/2} \\ &= \left[\sqrt{\lambda_1}x_1, \ldots, \sqrt{\lambda_r}x_r\right] \\ &= \left[\sqrt{\lambda_1}\begin{bmatrix} x_{11} \\ \vdots \\ x_{1n}\end{bmatrix}, \ldots, \sqrt{\lambda_r}\begin{bmatrix} x_{r1} \\ \vdots \\ x_{rn}\end{bmatrix}\right]. \end{aligned} \tag{5}$$

This is the outline of Torgerson' MDS. Next, we discuss the relationship between Torgerson's MDS and the least squares MDS. Consider the configuration of points on two-dimensional space from the point of view of the application into time-space mapping. Define the (u,v) coordinates of points in the time-space by the first and second column of D, that is, the coordinates of points i and j are given by

$$\begin{aligned} u_i &= \sqrt{\lambda_1}x_{1i}, \quad v_i = \sqrt{\lambda_2}x_{2i} \\ u_j &= \sqrt{\lambda_1}x_{1j}, \quad v_j = \sqrt{\lambda_2}x_{2j}. \end{aligned} \tag{6}$$

The inner product of points i and j, $\hat{r}_{ij}$, is

$$\hat{r}_{ij} = \lambda_1 x_{1i} x_{1j} + \lambda_2 x_{2i} x_{2j}. \tag{7}$$

Hence the mean squares error for all components of the inner product matrix, m^2, is

$$\begin{aligned} m^2 &= \sum_i \sum_j \left(r_{ij} - \hat{r}_{ij}\right)^2 \\ &= \sum_i \sum_j \left(\lambda_3 x_{3i} x_{3j} + \lambda_4 x_{4i} x_{4j} \ldots + \lambda_r x_{ri} x_{rj}\right)^2 \\ &= \sum_i \sum_j \left(\lambda_3^2 x_{3i}^2 x_{3j}^2 + \lambda_4^2 x_{4i}^2 x_{4j}^2 \ldots + \lambda_r^2 x_{ri}^2 x_{rj}^2\right) + \\ &\quad \sum_i \sum_j \left(2\lambda_3\lambda_4 x_{3i} x_{3j} x_{4i} x_{4j} + \ldots\right) \\ &= \lambda_3^2\left(\sum_i x_{3i}^2\right)\left(\sum_j x_{3j}^2\right) + \lambda_4^2\left(\sum_i x_{4i}^2\right)\left(\sum_j x_{4j}^2\right) + \\ &\quad \ldots 2\lambda_3\lambda_4\left(\sum_i x_{3i}x_{4i}\right)\left(\sum_j x_{3j}x_{4j}\right) + \ldots \\ &= \lambda_3^2 + \lambda_4^2 + \ldots + \lambda_r^2. \end{aligned} \tag{8}$$

Therefore, if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, the configuration of points by Torgerson's MDS is said to be optimal in the sense of approximating the components of the inner product matrix in the least squares criterion. However, we should not forget that Torgerson's MDS approximates not the given time-distances but the inner product matrix. If the given time-distances have errors, these two criteria have the following differences according to the error propagation law;

- If two time-distances are same both in magnitude and in direction, the precision of the time-distance which is far from the gravity point is higher than another.

- If two time-distances are same both in magnitude and in the distance between the center point of two points and the gravity point, the precision of the time-distance which is in the radial direction from the gravity point is higher than another.

Accordingly the variance-covariance matrix of the estimated coordinates is affected not only by the time-distance itself but also by its direction and distance from the gravity point. Thus it is concluded that the configuration of points based on Torgerson's MDS is not optimal in the restrictive sense of consistence with the given time-distances. It is requested to employ the MDS that approximates directly the given time-distances in the least squares criterion.

3. BASIC FORMULATION OF LEAST SQUARES MDS

The least squares MDS is basically equivalent to the error adjustment problem of the trilateration. The physical distance on the time-space map, d_{ij}, is represented by

$$d_{ij} = \sqrt{(u_i-u_j)^2 + (v_i-v_j)^2}. \tag{9}$$

Define $(u_i^0, v_i^0)\,(i=1,2..,n)$ to be the approximate values for the coordinates. Then, the linearized form of (9) by the Taylor series expansion is given by

$$d_{ij} = -\frac{u_j^0-u_i^0}{d_{ij}^0}\Delta u_i + \frac{u_j^0-u_i^0}{d_{ij}^0}\Delta u_j - \frac{v_j^0-v_i^0}{d_{ij}^0}\Delta v_i + \frac{v_j^0-v_i^0}{d_{ij}^0}\Delta v_j + d_{ij}^0 \tag{10}$$

where $\Delta u_i = u_i - u_i^0$, $\Delta v_i = v_i - v_i^0$ and $d_{ij}^0 = \sqrt{(u_i^0-u_j^0)^2 + (v_i^0-v_j^0)^2}$. Denoting

$$a_{ij} = -\frac{u_j^0-u_i^0}{d_{ij}^0},\ b_{ij} = -\frac{v_j^0-v_i^0}{d_{ij}^0} \tag{11}$$

and lettering $l_{ij} = -d_{ij}^0 + t_{ij}$, we obtain the observation equations as follows;

$$\begin{aligned} e_{ij} &= d_{ij} - t_{ij} \\ &= a_{ij}\Delta u_i - a_{ij}\Delta u_j + b_{ij}\Delta v_i - b_{ij}\Delta v_j - l_{ij} \end{aligned} \tag{12}$$

where e_{ij} is the residual. Although it is not necessarily requested to give the time-distances to all pairs of points in the least squares MDS, we assume that only for descriptions. Denoting the following matrix notations;

$$e = \begin{bmatrix} e_{12} \\ e_{13} \\ \vdots \\ e_{n-1,n} \end{bmatrix},\quad T = \begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_n \\ \Delta v_1 \\ \vdots \\ \Delta v_n \end{bmatrix},\quad L = \begin{bmatrix} l_{12} \\ l_{13} \\ \vdots \\ l_{n-1,n} \end{bmatrix} \tag{13}$$

and

$$A = \begin{bmatrix} a_{12} & -a_{12} & 0 & 0\dots0 & b_{12} & -b_{12} & 0 & 0\dots \\ a_{13} & 0 & -a_{13} & 0\dots0 & b_{13} & 0 & -b_{13} & 0\dots \\ \dots & & & & \dots & & & \end{bmatrix}, \tag{14}$$

then we get

$$e = AT - L. \tag{15}$$

Since, in general, the number of the observations, m, is larger than the number of unknown coordinates, $2n$, the least squares adjustment becomes necessary. Minimizing

$$min.\quad e^tPe \tag{16}$$

where P is a weight matrix, it is usually possible to get the solution as follows:

$$T = (A^tPA)^{-1}A^tPL. \tag{17}$$

Final values of the adjusted coordinates, (u_i, v_i), can be obtained by the iteration process from (10) to (17) until the norm of T becomes nearly equal to zero.

Although the above least squares solution is seemingly reasonable and possible, it has actually an important problem. It is because the rank of the matrix A^tPA is smaller than $2n$, that is, A^tPA is singular and $det(A^tPA)=0$. Thus there are infinite number of solutions and we cannot fix the point coordinates. It is obvious that observations only of the distances between points give only the shape of the trilateration network if there are no conditions for the coordinates. To obtain an unique optimal solution by (17), at least three coordinates should be fixed. That is, the defect of the rank (A^tPA) is 3.

Also in the case of time-space mapping, the configuration of points can be obtained by fixing at least three coordinates. If the configuration of points is only an objective and it is not necessary to interpolate the map elements, the above-mentioned MDS with three fixed coordinates is regarded as the optimal MDS in the sense of least squares criterion to given time-distances. It is concluded that the trilateration adjustment by least squares can be functioned as a metric MDS.

However, what does it mean by fixing three coordinates ? This means that the fixed coordinates are assumed to be without error and the errors of the other coordinates are estimated in dependence on the fixed coordinates. In order to calibrate the interpolation functions compatibly with the least squares theory, the errors of the coordinates, i.e., the variance and covariance matrix of the coordinates should not be dependent on which coordinates are fixed. The least squares MDS for time-space mapping is required to be able to fairly evaluate the position errors without fixing any coordinates.

4. LEAST SQUARES MDS BASED ON FREE NETWORK CONCEPT

The geodesy or surveying network without the fixed coordinates is called a free network. The MDS for time-space mapping is analogous to the trilateration free network. To adjust such a free network and obtain the coordinates, the problem of $det(A^tPA)=0$ needs to be resolved. For this purpose, so-called Moore-Penrose general inverse (MP inverse) is applied to (17) (Rao and Mitra, 1971).

The representation of (17) using MP inverse is

$$T = (A^tPA)^- A^tPL. \tag{18}$$

where $(A^tPA)^-$ is MP inverse matrix of A^tPA. It is known that the MP inverse realizes both $min.\ e^tPe$ and

$$min. \quad T^tT. \tag{19}$$

It is obvious from (19) that the MP inverse enables the free network to be adjusted by adding the condition of minimizing the norm of unknown variables. This condition is equivalent to the restriction of the rotation and translation of the free network. Therefore, MP inverse gives the adjusted coordinates without any fixed coordinate.

In addition, Mittermayer (1972) proved that the optimization of (16) and (19) is equivalent to the following optimization;

$$min. \qquad trace\,(\Sigma_T). \tag{20}$$

where Σ_T is the variance-covariance matrix of the adjusted coordinates. This means that the errors of the coordinates are fairly minimized. The least squares MDS based on MP inverse is regarded as the most appropriate one to MDS for time-space mapping, in which the interpolation procedure is requested as the next task.

Since the calculation of MP inverse is somewhat complicated, the approximate calculation procedure is provided (Rao and Mitra, 1971). With this, the MP inverse of an arbitrary matrix M is calculated by

$$M^- = \lim_{\delta \to 0} (M^tM + \delta I)^{-1} M^t. \tag{21}$$

Applying the formula (21) to (18), we get the adjusted coordinates. The unbiased estimate of the variance of unit weight is given by

$$\sigma^2 = \frac{e^tPe}{m - rank\,(A)} = \frac{e^tPe}{m - (2n - 3)}. \tag{22}$$

Therefore, the variance-covariance matrix of the estimated coordinates is obtained by the error propagation law as follows;

$$\Sigma_T = \sigma^2 (A^tPA)^- \tag{23}$$

5. INTERPOLATION PROCEDURE BASED ON ADJUSTED COORDINATES

The configured points (u_i, v_i) $(i=1,2,..,n)$ by the MDS are used as the control points in the interpolation. Define the coordinates of the physical map by (x_i,y_i). The following interpolation functions are calibrated by the least squares method based on the control points.

$$u = f(x,y), \qquad v = g(x,y) \tag{24}$$

Only for convenience, assume the interpolation functions to be linear to the unknown parameters. Of course, the functions do not need to be linear to the (x,y) coordinates. Let $\boldsymbol{\varepsilon}$ represent the residual vector of the interpolation functions as follows;

$$\boldsymbol{\varepsilon} = (\varepsilon_{u1},..,\varepsilon_{un}, \varepsilon_{v1},..,\varepsilon_{vn})^t \tag{25}$$

Let the unknown parameter vector included in (24) be denoted by $\boldsymbol{W} = (w_1,w_2,..w_k)^t$ and the coefficient matrix for $\boldsymbol{W}$ be denoted by $\boldsymbol{B}$. Then the observation equations of the interpolation problem is given by

$$\boldsymbol{\varepsilon} = \boldsymbol{BW} - \boldsymbol{S}, \tag{26}$$

where S is the estimated coordinates when the adjusted terms, T , converges.

Note that $\boldsymbol{\varepsilon}$ depends on Σ_T $(=\Sigma_S)$: hence the least squares estimates of W are

$$W = \left(B^t \Sigma_T^{-1} B\right)^{-1} B^t \Sigma_T^{-1} S. \qquad (27)$$

In addition, the variance-covariance matrix of the estimated parameters is given by

$$\Sigma_W = \sigma_W^2 \left(B^t \Sigma_T^{-1} B\right)^{-1}. \qquad (28)$$

where σ_w^2 is the variance of unit weight parameter and can be get by

$$\sigma_W^2 = \frac{\varepsilon^t \Sigma_T^{-1} \varepsilon}{2n - k}. \qquad (29)$$

The function form of the interpolation functions needs to be in general derived after a trial and error process.

6. CONCLUSION

Torgerson's MDS, which has been most commonly used as a metric MDS technique, is not an optimal for time-space mapping. It is because Torgerson's method fits the point configuration not directly on given time-distances but on the inner products derived from the given time-distances, and brings the distortions of estimates of coordinates when there exist the errors in MDS process onto two-dimensional space. The least squares MDS should be applied into time-space mapping.

The least squares MDS is basically equivalent to the error adjustment problem of trilateration. There is need to fix at least three coordinates in order to get a set of coordinates. With this, the configuration of points can be obtained by the ordinary least squares method. The least squares MDS based on trilateration adjustment can be utilized as a metric MDS technique. However, such a method does not provide the accurate variance-covariance of the adjusted coordinates because it is dependent on which coordinates are fixed. This is a fatal problem as a MDS technique for time-space mapping, since it has need to employ the interpolation procedure based on the adjusted coordinates. In the interpolation process, the variance-covariance of coordinates should be dealt as weights in the least squares criterion. The least squares MDS is required to fairly evaluate the variance-covariance of the adjusted coordinates.

Free network adjustment of trilateration using Moore-Penrose generalized inverse enables us to obtain the points' configuration without fixing any coordinates. Furthermore, we are able to fairly evaluate the variance-covariance of the adjusted coordinates. Applying this free network MDS, time-space mapping procedure can be systematically represented on the theoretical framework of the least squares method.

It is needless to say the proposed free network MDS is significant as a metric MDS technique itself. There have been so far scarcely any studies which aim to discuss the accuracy of the coordinates of configured points. MDS is a general-purpose technique to visually present the complicated structure and analyze its basic structure. There have been a wide variety of applications in the fields including psychology, sociology, geography, and regional science. The free network MDS may be potentially a new powerful weapon in these fields.

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