GRAPHICAL SOLUTIONS OF ORIENTATION PROBLEMS IN PHOTOGRAMMETRY

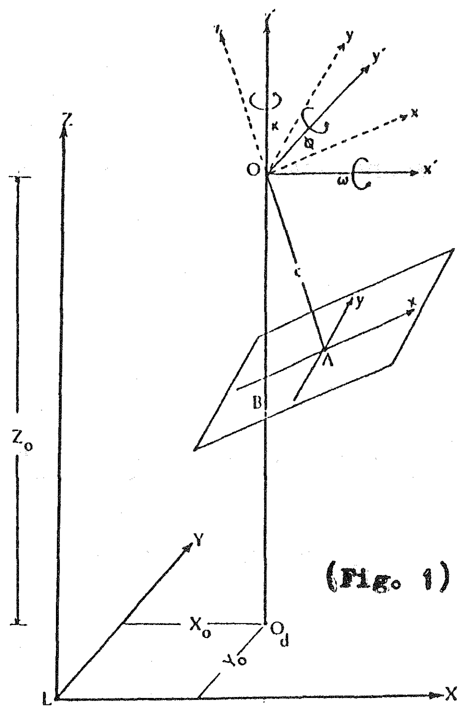
by
PROFESSOR WAGIH N. HANNA
Faculty of Engineering, Ain Shams University,
Abbasia, Cairo, Egypt

ABSTRACT

The main advantage of solving orientation problems is deriving formulae for determination the coordinates of the image point P_1 of space point P on the terrain, if the elements of orientation are known. The surface of the terrain is replaced by an inclined plane, which is a suitable approximation for erecting maps of small part of earth by using a single photo.
In this article two orientation problems are solved analytically and the corresponding equations of transformation are derived. These solutions are then represented graphically by assuming numerical values for the given data and using the principals of Descriptive Geometry. The measured results are compared with those obtained by the numerical substitution in the analytical solutions. The technique used here can be applied to solve more problems which may arise in practice.

1. Introduction

A Photograph taken with a metric camera permits the reconstruction of the bundle of rays with which the object was projected on the image plane. Orientation is the reconstruction of the photograph and optical model in space relative to a system of reference.
Interior orientation is the recovering of the projected cone of rays geometrically identical with the bundle of rays entered the camera lens to make the original exposure. It can be obtained if the camera constant c and the principal point A are known. Exterior orientation is placing the bundle of rays in the right position w.r.t. the object photographed. This happens by the location of the centre of projection O and by the angular orientation of the bundle of rays w.r.t. the space coordinate system, i.e. by determination of X_o, Y_o, Z_o, ω, ϕ, κ. (Fig. 1)



(Fig. 1)

In this article, new coordinate system of reference is established so as to suit the problems dealt with in this work. The surface of the terrain is replaced by an inclined plane π. For solving orientation problems, equations of transformation from this plane to the image plane are derived on the assumption that the photographic process is a central projection, and if some elements of orientation are known.
Two orientation problems are solved analytically, and by suitable assumptions of the data given, the unknowns are obtained numerically. The graphical solutions of these two problems, which depend on the principals of Descriptive Geometry, are then drawn and performed. The measured values of the unknowns are compared with its calculated corresponding values.

2. The Coordinate Systems

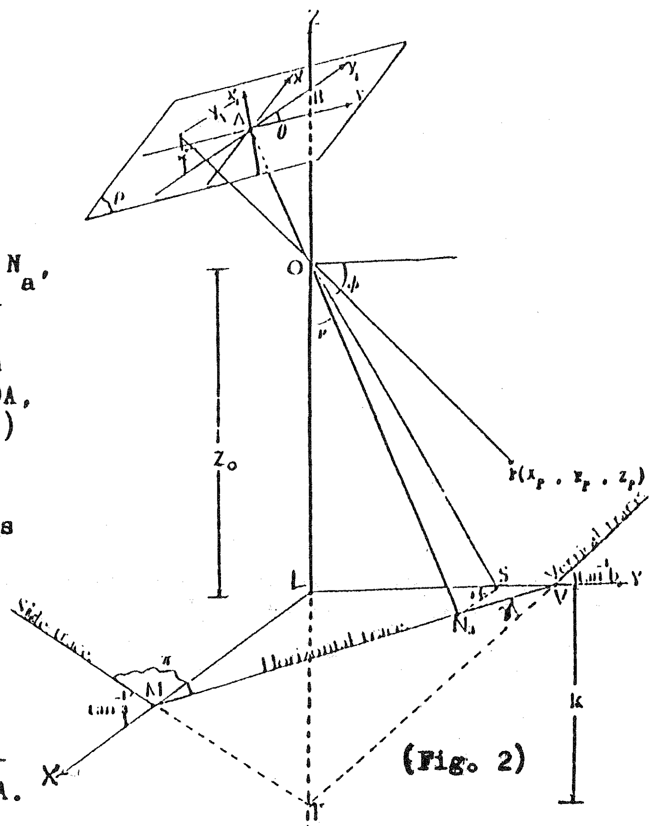
We deal in this item with the space coordinate system, the image coordinate systems and the equation of the surface of the terrain.

2.1 Space coordinate system of reference

Considering the terrain as an inclined plane ,then its representation on the image plane by central projection will depend only on its shape,the position of the centre of projection and the direction of the camera axis at the moment of the exposure,but not on the chosen coordinate system. We use the coordinate system shown in (Fig. 2) in which :

1. O is the centre of projection.
2. π is the plane of terrain.
3. ρ is the image plane.
4. A is the principal point.
5. ν is the tilt angle.
6. θ is the swing angle.
7. c is the camera constant.

The camera axis OA meets π in the point N_a, through which we let pass the horizontal plane of reference π_1 (X,Y). The Y-axis passes through the origin L and makes an arbitrary angle ϕ with the camera axis OA, where $\phi + \nu \geq 90$ . . . . (1)



(Fig. 2)

2.2. Image coordinate systems

The image plane ρ passes through A and is perpendicular to the camera axis OA. We consider two coordinate systems :

(a) The ordinary image coordinate system (x,y) with A as origin,and the lines joining the fudicial marks as axes.
(b) The image coordinate system (x_1, y_1) with A as origin, y_1-axis = AB and x_1-axis is perpendicular to it through A.

2.3. Representation of the surface of the terrain

Since we deal with small domains of earth,we consider the surface of terrain as a plane π,which is represented by the equation :

$$Z = aX + bY - K \qquad . \quad . \quad . \qquad (2)$$

The choice of an inclined plane instead of the usually taken horizontal plane as an approximating surface,although rendering the calculations more difficult and the solutions of the problems of transformation more tedious,nevertheless is a better approximation to the real surface of the terrain,and also leads to more accurate results.

In(Fig. 2), let MVT be the trace-triangle of π with the planes of reference, the angle $T\hat{V}M = \gamma$, then from equation (2) we get :

$$\overline{VM} = \frac{K\sqrt{a^2+b^2}}{ab}, \quad \overline{VT} = \frac{K\sqrt{1+b^2}}{b}, \quad \overline{MT} = \frac{K\sqrt{1+a^2}}{a} \qquad . \ . \ . \ . \qquad (3)$$

$$\overline{MT}^2 = \overline{VT}^2 + \overline{VM}^2 - 2\,\overline{VT}\,.\,\overline{VM}\cos\gamma$$

$$\gamma = \cos^{-1}\frac{a}{\sqrt{(1+a^2)(1+b^2)}} \qquad . \ . \ . \ . \qquad (4)$$

$$\overline{VN}_a^{\,2} = \frac{K^2}{b^2} - \frac{2K Z_0 \cos\phi}{b\cos\nu} + Z_0^2\tan^2\nu \qquad . \ . \ . \qquad (5)$$

$$\overline{TN}_a^{\,2} = Z_0^2\tan^2\nu + K^2 = \overline{VN}_a^{\,2} + \overline{VT}^2 - 2\,\overline{VN}_a\,.\,\overline{VT}\cos\nu \qquad . \ . \ . \qquad (6)$$

Substituting by (3) , (4) , (5) in (6) we get :

$$K = \frac{b\,Z_o \cos\phi}{\cos\nu} \pm \frac{a\,Z_o}{\cos\nu}\sqrt{\sin^2\nu - \cos^2\phi} \quad . \quad . \quad . \qquad (7)$$

Equation (2) of reduces to :

$$Z = a\,X + b\,Y - \frac{Z_o}{\cos\nu}\left[\, b\cos\phi \pm a\sqrt{\sin^2\nu - \cos^2\phi}\,\right] \quad . \quad . \qquad (8)$$

3. Equations of transformation

The equations of transformation express the following coordinates in terms of the space coordinares (X_P, Y_P, Z_P) of any point P in :

(a) The space coordinates P_1 (X_{P1}, Y_{P1}, Z_{P1}) of the image P_1 of P .

(b) The image coordinates $(x_1\ , y_1)$ of P_1 .

(c) The ordinary image coordinares $(x\ , y)$ of P_1 .

3. 1 Problem (1)

Given : $P(X_P\ , Y_P\ , Z_P)\ , Z_o\ , c\ , \nu , \phi\ , \theta$.

Required : a. Space coordinares P_1 $(X_{P1}\ , Y_{P1}\ , Z_{P1})$

b. Image coordinates P_1 $(x_1\ , y_1)$

c. Image ordinary coordinates P_1 $(x\ , y)$

3. 11 Determination of $P_1(X_{P1}, Y_{P1}, Z_{P1})$

The coordinates of the centre of projection O $(0\ , 0\ , Z_o)$, the coordinates of the principal point A $(c\sqrt{\sin^2\nu - \cos^2\phi}\ , -c\cos\phi\ , Z_o + c\cos\nu)$.

Equation of image plane ρ is :

$$\alpha_1(X - X_A) + \beta_1(Y - Y_A) + \gamma_1(Z - Z_A) = 0 \quad . \quad . \qquad (9)$$

where the direction ratios of the camera axis OA are :

$$(\alpha_1\ , \beta_1\ , \gamma_1) = (\sqrt{\sin^2\nu - \cos^2\phi},\ \cos\phi\ , -\cos\nu) \qquad (10)$$

The equation (9) will be :

$$X\sqrt{\sin^2\nu - \cos^2\phi} + Y\cos\phi - Z\cos\nu + Z_o\cos\nu + c = 0 \quad . \qquad (11)$$

Parametric equations of OP are :

$$X = t\,X_P\ , \qquad \infty \leqslant t \leqslant \infty$$
$$Y = t\,Y_P\ ,$$
$$Z = t\,(Z_P - Z_o) + Z_o \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Solving (11) and (12) we get :

$$X_{P1} = \frac{-c\,X_P}{D}\ , \quad Y_{P1} = \frac{-c\,Y_P}{D}\ , \quad Z_{P1} = \frac{c\,(Z_o - Z_P)}{D} + Z_o \qquad (13)$$

3. 12 Determination of image coordinates $P_1(x_1\ , y_1)$

Since the y_1-axis is the line joining AB, where B $(0\ , 0\ , Z_o + c\sec\nu)$, then its direction ratios are :

$$(\alpha_2\ , \beta_2\ , \gamma_2) = (\sqrt{\sin^2\nu - \cos^2\phi},\ \cos\phi,\ \sin\nu\tan\nu) \quad . \quad . \qquad (14)$$

$$y_1^2 = \frac{1}{\alpha_2^2 + \beta_2^2 + \gamma_2}\left[\alpha_2(X_A - X_{P1}) + \beta_2(Y_A - Y_{P1}) + \gamma_2(Z_A - Z_{P1})\right]^2 \qquad (15)$$

Solving (13) , (14) , (15) we get :

$$y_1 = \pm\left[c \cot\nu - \frac{c\,(Z_o - Z_P)\,\operatorname{cosec}\nu}{D}\right] \quad . \quad . \qquad (16)$$

Since $x_1^2 = \overline{AP}_1^2 - y_1^2$, then substituting in (16) we get :

$$x_1 = \pm\, c\left[\frac{X_P \operatorname{cosec}\nu \cos\phi - Y_P \operatorname{cosec}\nu\sqrt{\sin^2\nu - \cos^2\phi}}{D}\right] \quad . \quad . \qquad (17)$$

where

$$D = X_P\sqrt{\sin^2\nu - \cos^2\phi} + Y_P \cos\phi + (Z_o - Z_P)\cos\nu$$

3. 13 Determination of the ordinary image coordinates $P_1(x,y)$

Since, $x = x_1 \cos\theta + y_1 \sin\theta$

$$y = y_1 \cos\theta - x_1 \sin\theta \quad . \quad . \quad . \quad . \qquad (18)$$

then substituting from (16) , (17) in (18) we get $P_1(x,y)$.

3. 2 Problem (2)

Given : $P(X_P , Y_P , Z_P)$, $P_1(x,y)$, Z_o , c , ν .

Required : a. Space coordinates $P_1(X_{P1} , Y_{P1} , Z_{P1})$

b. Image coordinates $P_1(x_1 , y_1)$

c. ϕ

d. θ

3. 21 Determination of $P_1(X_{P1} , Y_{P1} , Z_{P1})$

From(Fig. 3) we have :

$$r^2 = x^2 + y^2 = x_1^2 + y_1^2 ,$$

$$\overline{OP}_1 = \sqrt{c^2 + r^2}$$

$$\overline{PP}_1 = \sqrt{X_P^2 + Y_P^2 + (Z_o - Z_P)^2} + \sqrt{c^2 + r^2}$$

$$X_{P1} = \frac{-\overline{OP}_1 \cdot X_P}{PP_1 - OP_1} = \frac{-X_P\sqrt{c^2 + r^2}}{L}$$

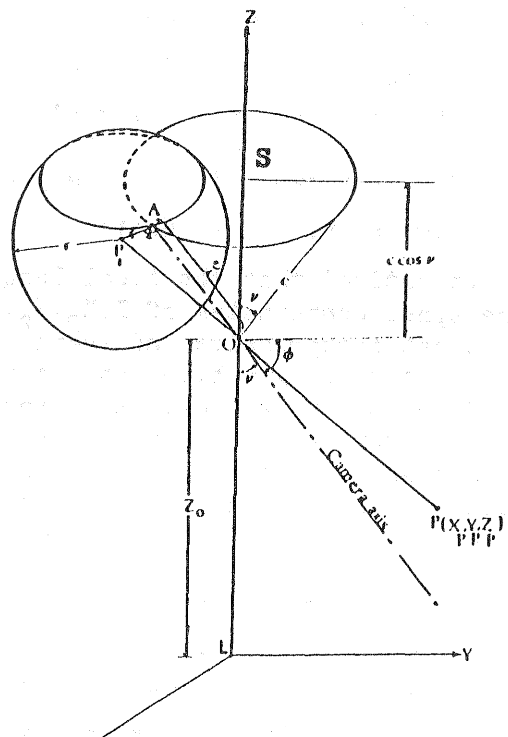
$$Y_{P1} = \frac{-\overline{OP}_1 \cdot Y_P}{PP_1 - OP_1} = \frac{-Y_P\sqrt{c^2 + r^2}}{L}$$

$$Z_{P1} = \frac{\overline{PP}_1 \cdot Z - \overline{OP}_1 \cdot Z_P}{PP_1 - OP_1}$$

$$= Z_o + \frac{(Z_o - Z_P)\sqrt{c^2 + r^2}}{L} \qquad (19)$$

where

$$L = \sqrt{X_P^2 + Y_P^2 + (Z_o - Z_P)^2}$$



(Fig. 3)

3. 22 Determination of ϕ

$$\phi = \cos^{-1}\frac{-Y_A}{c} \quad . \quad . \quad . \qquad (20)$$

The coordinates of the principal point A are determined from the two following locii :
(i) A sphere with P_1 as centre and its radius equals r :

$$(X - X_{P1})^2 + (Y - Y_{P1})^2 + (Z - Z_{P1})^2 = r^2 \quad . \quad . \quad . \quad (21)$$

(ii) A horizontal circle whose centre S(0 , 0 , Z_o + c cos ν) and its radius r_1 = c sin ν :

$$Z = Z_o + c \cos \nu$$
$$X^2 + Y^2 = c^2 \sin^2 \nu \quad . \quad . \quad . \quad (22)$$

3. 23 Determination of image coordinates $P_1(x_1,y_1)$

By substituting from (19),(20),(21),(22) in (16) and (17) we get $P_1(x_1,y_1)$.

3. 24 Determination of swing angle θ

We get by applying the following relation :

$$\theta = \sin^{-1} \frac{1}{r^2} (x y_1 - y x_1) \quad . \quad . \quad . \quad . \quad (23)$$

4. Grphical solutions and numerical results

The graphical solutions of the previous two problems, that have been solved analytically, are established. These solutions depend mainly on the principals of Descriptive Geometry. They are performed and drawn to satisfy the following numerical assumptions :

$c = 25$ cms , $\nu = 60^\circ$, $\phi = 40^\circ$, $\theta = 20^\circ$

$P(X_P , Y_P , Z_P) = (40 , 30 , 42.5)$, $Z_o = 45$ cms

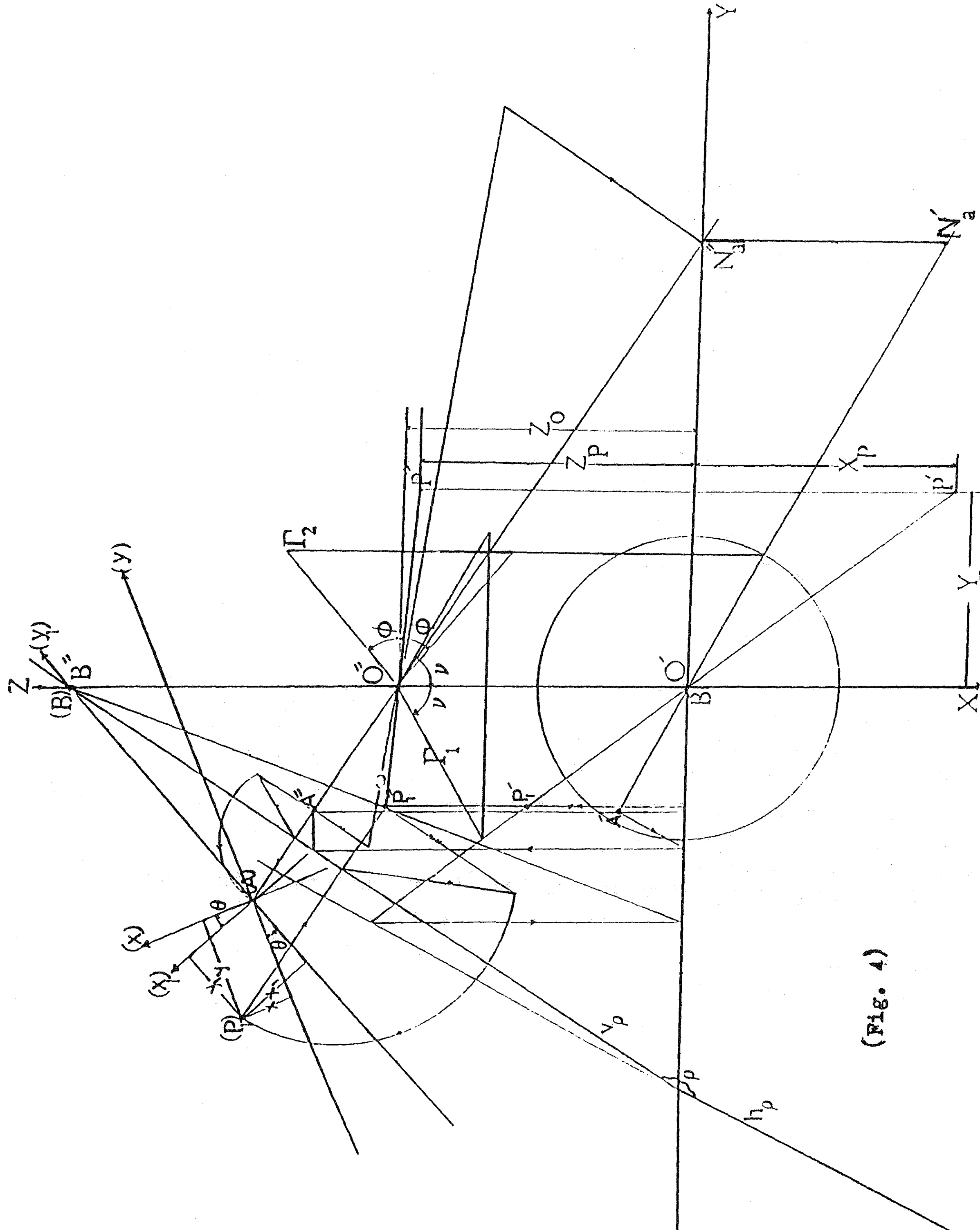
For the equation of the terrain plane :
$a = -1$, $b = - 1.185$
If P_1 is known , it will be given by the following coordinates in cms. :
$P_1\ (x,y) = (8.11 , - 16.41)$, $P_1\ (x_1,y_1) = (13.24 , - 12.65)$
It is noticed that the focal length c , the image coordinates (x,y),(x_1,y_1), the space coordinates of $P(X_P,Y_P,Z_P)$, and the height of flight Z_o are somewhat inexpressive and inapplicable values, that is to obtain a suitable and acceptable graphical configuration.
As described hereafter, the camera axis OA is represented by intersecting two right circular cones having the same vertex O, the axis of the first is the Z-axis and its semi vertex angle is ν , the axis of the second is the line passing through O and parallel to the Y-axis and its semi vertex angle is ϕ . These two cones intersect in real generators if : $\nu + \phi \geqslant 90$.

4.1 Graphical solution of problem (1) (Fig. 4)

Given : $P(X_P , Y_P , Z_P)$, Z_o , c , ν , θ , ϕ .
Required : a. Space coordinates $P_1\ (X_{P1} , Y_{P1} , Z_{P1})$
b. Image coordinates $P_1\ (x_1 , y_1)$
c. Ordinary image coordinates $P_1\ (x , y)$

Proceedure

1. Represent O by its horizontal and vertical projections O' , O'' respectively.
2. Determine the camera axis OA by the intersection of two cones having the same vertex O , the axis of the first is the Z-axis and its semi vertex angle ν, the axis of the second is parallel to the Y-axis and its semi vertex



(Fig. 4)

angle $= \phi$.

3. Represent the principal point A , where OA = c .
4. Represent the image plane ρ , which passes through A and is perpendicular to OA .
5. Represent the point P and join OP , which intersects ρ in P_1 . Hence the space coordinates P_1 (X_{P1} , Y_{P1} , Z_{P1}) are obtained .
6. Rabat the image plane ρ about its vertical trace on the (Y,Z) plane and determine (A) , (P_1) , the axis (y_1) = (A)(B) . Hence we get the image coordinates P_1(x_1 , y_1) .
7. Draw in rabatment the ordinary axes using the swing angle θ, and measure P(x , y) .

4.11 Numerical results of problem (1)

	Graphical	Analytical
(X_{P1} , Y_{P1} , Z_{P1})	(- 24.75 , - 18.56 , 46.55)	(- 24.759,-18.569,46.547)
P_1 (x_1 , y_1)	(13.24 , -12.65)	(13.239 , -12.647)
P (x , y)	(8.12 , -16.41)	(8.11 , 16.412)

4.2 Graphical solution of problem (2) (Fig. 5 , 6)

Given : P(X_p , Y_p , Z_p) , P_1(x , y) , Z_o , c , ν .

Required : a. P(X_{P1} , Y_{P1} , Z_{P1})

b. P_1(x_1 , y_1)

c. ϕ

d. θ

Procedure

1. Determine the space coordinates P_1(X_{P1} , Y_{P1} , Z_{P1}) as in problem (1).
2. Represent the sphere with point P_1 as centre and radius $r = \sqrt{x^2 + y^2}$.
3. Represent a horizontal circle whose centre S(0 , 0 , Z_o + f) lies on the Z-axis , and its radius r_1 , where f and r_1 are obtained from (Fig. 5) .
4. Determine the principal point A by intersecting the sphere (P_1 , r) and the horizontal circle (S , r_1) .
5. Join OA and measure ϕ .
6. Rabat the image plane ρ as in problem (1) and determine the axes (x_1) , (y_1) and the point (P_1) , then measure x_1 and y_1 .
7. By using the coordinate x of P_1 determine the axes (x) and (y) and then determine the swing angle θ .

4.21 Numerical results of problem (2)

	Graphical	Analytical
P_1(X_{P1} , Y_{P1} , Z_{P1})	(-24.7 , -18.6 . 46.5) $\phi = 40°$	(-24.759,-18.569,46.547) $\phi = 40°$
P_1(x_1 , y_1)	(13.2 , -12.65) $\theta = 20°$	(13.239 , -12.647) $\theta = 20°$

References

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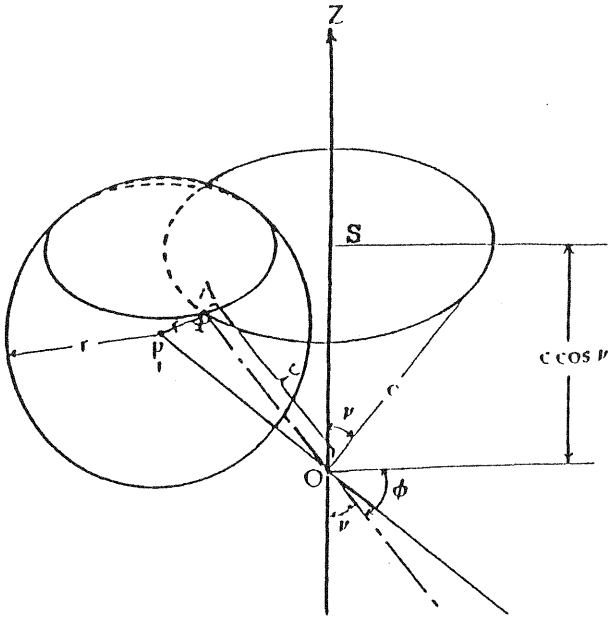
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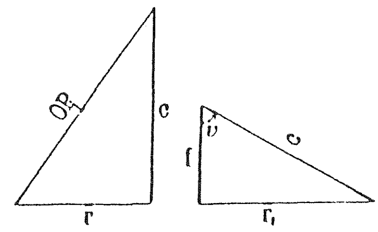
$c \cos \nu = f$


OP_1
c
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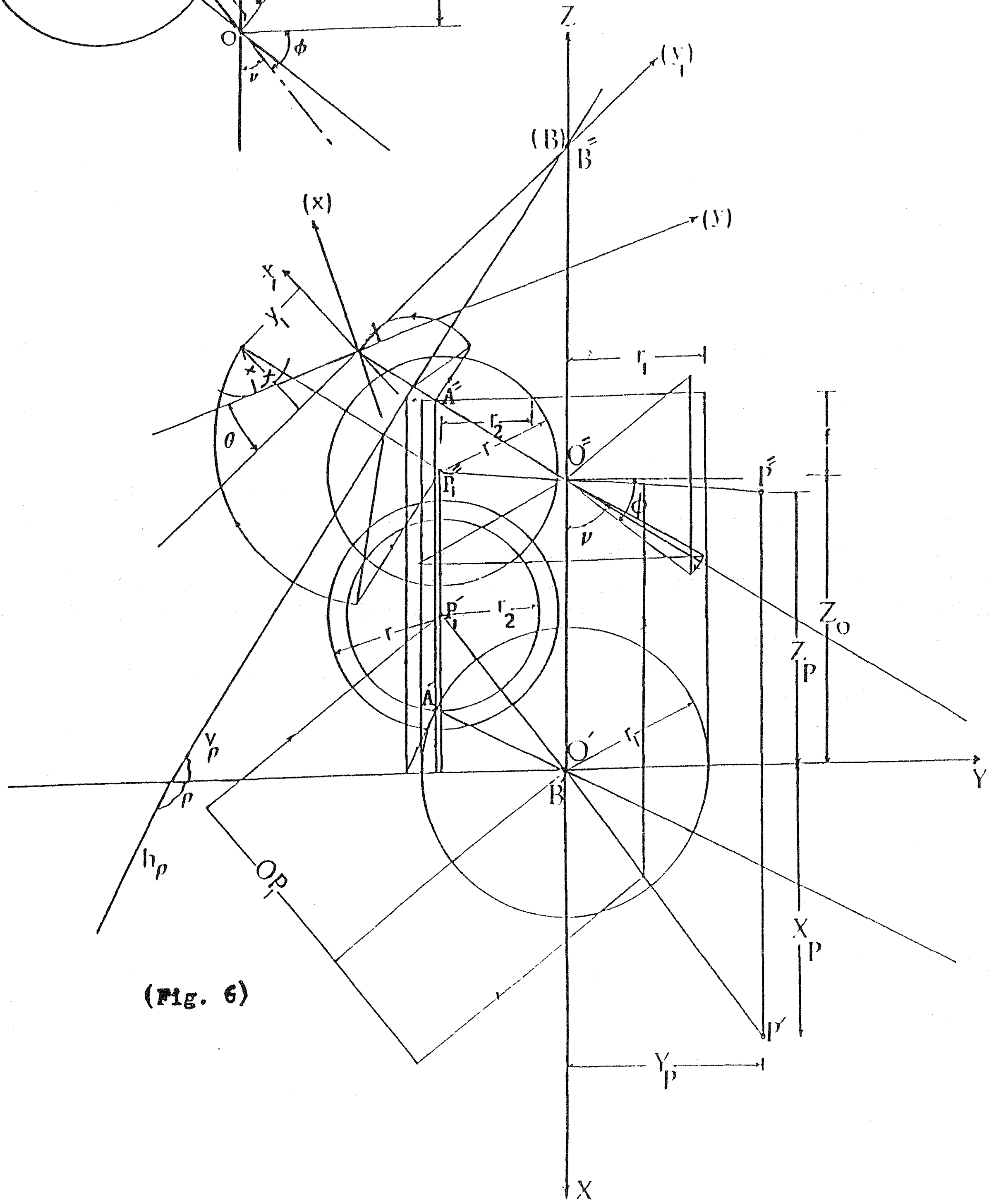
(Fig. 5)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(Fig. 6)