VARIANCE DECOMPOSITION AND ITS APPLICATION IN PHOTOGRAMMETRY

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ABSTRACT:

The mathematical modeling of dynamic and time-dependant perturbations affecting space sensors is a common practice in photogrammetry. This modeling is usually done through the fitting of parametric or interpolative models. However the extension of the model and the addition of parameters may lead to unstable solution due to high correlations between parameters. The identification of correlated parameters to take corrective measures is often based on the analysis of the correlation matrix. The correlation matrix shows however, only correlations pairewise and does not give any indication on functional groupings.
In this paper, the variance decomposition based on singular value decomposition is presented. In this method the number of small singular values indicates the number of near depndencies and parameters involved in these are identified as those that have more than 50% of their variances associated with the same small singular value.
A sase studey based on in-flight camera calibration was conducted with simulated and real data, and showed the efficiency of the method in dealing with fuctional groupings of the paramameters.

1. INTRODUCTION

The effects of external conditions and errors affecting the system constitute a limiting factor on the attainable accuracy in computational photogrammetry. The mathematical modeling of such phenomena are a common practice so as to take into accounts these effects. This modeling is usually done by the fitting of:

- a parameteric model based on the geometrical or physical characteristics of the phenomenon.
- an interpolative model represented by a polynomial.

Modification of existing models through their extension and addition of parameters to account for these perturbation may lead to an unstable solution due to the correlation between parameters.
For almost all least square users, the identification of the correlated parameters is based on the analysis of the correlation matrix. Hence, in the case of additional parameters, the decision of rejecting and deleting parameters is essentially based on the magnitude of the correlation coefficient. In this respect, some authors recommended 0.90 as a rejection standard (Grun, 1980), while others suggested 0.85 (Faig and Shih, 1988).
However, the alternative of rejecting and deleting parameters on the grounds of their significance and stability is not alwys justified. In fact, in some applications the physical significance of the parameter may be of great importance to the modeling; besides this, the rejection decision may not be fully reliable due to the fact that hypothesis testing may be rendered inconclusive because of the high variances inducued by the ill-conditioning.
On the other hand, the correlation matrix shows only correlations between parameter pairs and does not give any indication on fuctional groupings where more than two parameters are simultaneously invloved in a correlation. In this respect, experience has shown that, it is possible for three or more parameters to be correlated when taken together, but no two of these taken in pairs are highly correlated. Moreover, when the system is ill-conditioned, high correlation coefficients may be indicative of correlated parameters, but the absence of high correlation coefficients cannot be considered as evidence of no problem.
To overcome the drawbacks of the correlation matrix mentioned above, we present in this paper a method based on the singular value decomposition and that deals efficiently with multiple correlations or functinal groupings of parameters.

2. BACKGROUND ON VARIANCE DECOMPOSITION

2.1 Singular Value Decomposition

The singular value decomposition is a concept closely related to the eigensystem , but that applies directly to the design matrix **A** insted of the normal matrix (A^TA) .

Hence, if **A** is an **(mxn)** rectangular matrix, the singular values λ_i of **A** are the positive square roots of the eigenvalues of the square matrix $(\mathbf{A}^T\mathbf{A})$ of order **n** (Lascaux and Theodor, 1986).

In fact, for any arbitrary **(mxn)** matrix **A**, there exists an unitary **(mxn)** matrix **U** and an unitary **(nxn)** matrix **V** such that:

$$\mathbf{A} = \mathbf{UDV}^T \tag{2.1}$$

with **D** a diagonal matrix of the form:

$$\mathbf{D} = \begin{bmatrix} \mathbf{D_{11}} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.2}$$

and :

$$\mathbf{D_{11}} = \mathbf{diagonal}(\lambda_1, \lambda_2, \ldots, \lambda_r) \tag{2.2a}$$

$\lambda_1, \lambda_2, \ldots, \lambda_r$ are the nonvanishing singular values of **A**, and **r** is its rank.

It is known that, when some singular values are exactly zeros, the matrix **A** is not full rank; in this case we say that **A** contains exact linear dependencies whose number is exactly equal to the number of null singular values.
Therefore, since a null singular value is an indication of exact linear dependency, Kendall and Silvey (Belsley et al, 1980) have extended this idea to say that, the existence of a small singular value is indicative of a near dependency; which means that, there will be as many near dependencies as there are small singular values.

2.2 Condition Number and Condition Indices

The condition number is one of the most popular stability indicators. The condition number is defined as:

$$\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \quad (2.3)$$

or in terms of the singular value decomposition of **A** as the ratio of the largest to the smallest singular values as:

$$\kappa = \frac{\lambda_{max}}{\lambda_{min}} \quad (2.4)$$

In a similar fashion, the i^{th} condition index can be defined as the ratio of the largest singular value to the i^{th} singular value :

$$\kappa_i = \frac{\lambda_{max}}{\lambda_i} \quad (2.5)$$

Hence there will be as many condition indices as there are nonzero singular values.

2.3 Variance-Decomposition Proportions

It is well known to all least square users that the variance-covariance matrix of the adjusted parameters (if we assume a unit weight matrix $\mathbf{P} = \mathbf{I}$, and unit a priori variance of unit weight $\sigma_0^2 = 1$) is given by:

$$\Sigma_X = (\mathbf{A}^T\mathbf{A})^{-1} \quad (2.6)$$

Expressing this in terms of the singular values of the design matrix **A** as in Eq.(2.1) we get:

$$\Sigma_X = (\mathbf{VDU^TUDV^T})^{-1} \quad (2.7)$$
$$= \mathbf{VD^{-2}V^T}$$

Therefore, the variance of the i^{th} parameter x_i may be written as:

$$\sigma_{x_i}^2 = \sum_{j=1}^{r} \left(\frac{v_{ij}^2}{\lambda_j^2}\right) \quad (2.8)$$

which means that, the variance of any parameter decomposes into a sum of components, each of which is associated with one of the singular values λ_j. For instance, the component of the variance of the i^{th} parameter associated with the j^{th} singular value is given by:

$$(\sigma_{x_i}^2)_j = \frac{v_{ij}^2}{\lambda_j^2} \quad (2.9)$$

The proportion of the variance the parameter x_i associated with the j^{th} singular value is given as:

$$(\rho_i)_j = \frac{(\sigma_{x_i}^2)_j}{\sigma_{x_i}^2} \quad (2.10)$$

The decomposition of the variances of all parameters with respect to all singular values gives what is called the variance-decomposition proportions matrix.

2.4 Identification of Correlated Parameters

Since a null singular value is an indication of an exact linear dependency, a small singular value is indicative of a near dependency. Therefore, there will be as many near dependencies as there are small singular values.

On the other hand, since in Equation(2.9) λ_j's appear in the denominator of the expression of the components of the variance, components associated with small singular values will be large compared to the other components; which will lead to high proportions.
It follows that , two or more parameters can be said to be involved in a near dependeny when a high proportion of their variances is associated with the same small singular value (or same high condition index).

Hence, the method of variance decomposition will enable identify:

- The number of near dependencies (multiple correlation) affecting the system as the number of high condition indices (small singular values).

- The parameters involved in these multiple correlations as those that have a large proportion of their variances associated with the same high condition indiex.

It remains however to decide on what should be considered as a large proportion of the variance and what should be considered as high condition index.
In this matter no standard exists on which to base this decision. Concerning the threshold for the proportion of the variance, Belsley et al (1980) considered a proportion to be large when it accounts for more than 50% of the variance of a parameter. The threshold for the condition index is however more complicated, because what can be considered a high condition index inducing ill-conditioning for a particular application , may not be a source of ill-conditioning for other type of applications.
Hence in the testing that follows, all condition indices will be considered until a conclusion can be reached during the testing on threshold to consider as harmfull.

3. CASE STUDY: IN-FLIGHT CAMERA CALIBRATION

In-flight camera calibration is a typical and an oldfashioned problem where the model is extended to account for perturbations.
The model used is the well-known collinearity condition equations extended to include variations in the interior orientation parameters expressed as:

$$F(x) = (x - x_p) + (x - x_p)(K_1\bar{r}^2 + K_2\bar{r}^4 + K_3\bar{r}^6 + ...) + \left\{P_1(\bar{r}^2 + 2\bar{x}^2) + 2P_2\overline{xy}\right\}\left\{1 + P_3\bar{r}^2 + ...\right\} - c\, X'/Z' = 0 \quad (2.11)$$

$$F(y) = (y - y_p) + (y - y_p)(K_1\bar{r}^2 + K_2\bar{r}^4 + K_3\bar{r}^6 + ...) + \left\{2P_1\overline{xy} + P_2(\bar{r}^2 + 2\bar{y}^2)\right\}\left\{1 + P_3\bar{r}^2 + ...\right\} - c\, Y'/Z' = 0$$

where:

$$\begin{aligned} X' &= a_{11}(X - X_o) + a_{12}(Y - Y_o) + a_{13}(Z - Z_o) \\ Y' &= a_{21}(X - X_o) + a_{22}(Y - Y_o) + a_{23}(Z - Z_o) \\ Z' &= a_{31}(X - X_o) + a_{32}(Y - Y_o) + a_{33}(Z - Z_o) \end{aligned} \quad (2.11a)$$

$$\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2};\ \bar{x} = x - x_p;\ \bar{y} = y - y_p \quad (2.11b)$$

with:

$a_{11}, a_{12}, ... a_{33}$: elements of the rotation matrix of the gimbal angles defining the orientation between the survey and photo coordinate systems.

X, Y, Z : Object point coordinates in the survey system.

X_o, Y_o, Z_o : Exposure station coordinates in the survey system.

x, y : Observed image coordinates in the fiducial system.

x_p, y_p : Principal point coordinates in the fiducial system.

c : Camera constant.

K_1, K_2, K_3 : Polynomial coefficients of symmetric radial distortion.

P_1, P_2, P_3 : Polynomial coefficients of decentring distortion.

In this model , parameters of interior orientation are to be recovered in a simultaneous least squares adjustment a long with exterior orientation parameters and survey coordinates. the problem of highly correlated parameters has been demonstrated by many authors (Pogorelov and Popova, 1975; Mrchant, 1974; Salmanpera, 1974; Kupfer, 1985; Brown 1969). Most often, the identification of correlated parameters has been based on the correlation matrix or the analysis of partial derivatives of the function with respect to each of the parameters.
In this paper the procedure of variance decomposition is applied to data pertaining to in-flight camera calibration.

3.1 Description of the Data

To allow for extensive testing of the method a synthetic data was generated mathematically. Since in camera calibrartion the resulting matrix is large, to keep the computations within reasonable limits, only four (4) photos were considered. Object space coordinates were generated for 25 ground control points. Exposure station coordinates and attitudes were assumed, allowing for the computation of the image coordinates with a focal length of 152.25 mm (Ettarid, 1992). Several factors such as elevation differences on ground control, use of convergent photos and a priori information on parameters were considered in the testing.

To confirm the validity of the procedure and the conclusion drawn from simulations, the testing of the method was conducted also with real data. The configuration consisted of 4 photos, two of which were vertical and the two others were convergent. The convergent photos were taken by modification of the flight scheme, using "the standard coordinted turn at 45 degrees" as described in Tudhope (1988).

3.2 Discussion of the Results

The variance decomposition method was applied to the resulting design matrix. Different cases were considered such as the use of convergent photos, the influence of elevation differences on ground control and a priori information on parameters. As the resulting variance decomposition proportions matrices are very large, only extracts of the condition indices and parameters associated with are presented here; the reader interested in the complete set of results may refer to Ettarid (1992). Correlated parameters are identified as those having more than 50% of their variances associated with the same high condition index.

* The variance decomposition applied to the design matrix resulting from **calibration over a flat terrain** showed that the high condition index is $5.8x10^{10}$ (Table 3.1), induced by a multiple correlation involving the parameters P_2 and ω rotations, with 100% of the variance of P_2 associated with this index. Parameters K_1, K_2 and K_3 are involved in a second corrlation associated with a condition index of magnitude $1.1x10^7$. The other correlated parameters identified are respectively the camera constant c and Z_o's, x_p and Xo 's, and y_p and Y_o's. The involvement of P_2 in a stronger correlation with Omega rotations has masked its involvement with y_p in a weaker correlation as we will see later. Similarly the involvement of x_p with X_o's has masked the correlation between x_p and P_1.

Table 3.1. Correlated Parameters in the Case of Calibration Over Flat Terrain

Condition Indices	Correlated Prameters (Variance-Decomposition Proportions)
$5.8x10^{10}$	P_2 (1.00) ω_1 (.74) ω_2 (.78) ω_3 (.72) ω_4 (.60)
$1.1x10^7$	K_1(.87) K_2 (.96) K_3 (.99)
$8.8x10^5$	C (.91) Z_1(.91) Z_2(.91) Z_3(.91) Z_4(.91)
$2.8x10^5$	x_p(.96) X_{o_1}(.96) X_{o_2}(.95) X_{o_3}(.97) X_{o_4}(.94)
$2.3x10^5$	y_p(.76) Y_{o_1}(.71) Y_{o_2}(.72) Y_{o_3}(.70) Y_{o_4}(.69)

The introduction of orthogonal Kappa rotations on successive exposures has no effect on correlated parameters in the case of calibration over flat terrain.

On the contrary, the use of convergent photos (photos 1 and 2) resulted in the elimination of the correlation between y_p and Y_o's and alleviated the correlation between camera constant c and Z_o's of the convergent photos (Table 3.2); but involved on the other hand κ rotations in the near dependency involving P_2, y_p and ω rotations.

Table 3.2. Correlated Parameters in the Case of Calibration over Flat Terrain with Convergent Photos.

Condition Indices	Correlated Prameters (Variance-Decomposition Proportions)
$4.9x10^{10}$	P_2 (1.00) y_p (.63) κ_1 (.61) κ_2 (.64) ω_3 (.72) ω_4 (.60)
$1.2x10^7$	K_1(.87) K_2(.97) K_3(.99)
$3.6x10^4$	C (.85) Zo_1(.60) Zo_2(.64) Zo_3(.85) Zo_4(.85)
$2.5x10^4$	x_p(.65) Xo_3(.90) Xo_4(.90) φ_1(.65) φ_2(.60)

* In the case of camera **calibration with elevation differences on ground control** (Dh/H = 30%), the variance decomposition showed that the highest condition index ($3.7x10^{10}$) is still induced by the correlation involving P_2, y_p and Omega rotations (Table 3.3). This also has alleviated the correlation between camera constant c and Z_o's; but as c is freed it becomes part of the correlation involving K_1, K_2 and K_3.

Table 3.3. Correlated Parameters in the Case of Calibration with Elevation Differences on Ground Control.

Condition Indices	Correlated Prameters (Variance-Decomposition Proportions)
$3.7x10^{10}$	P_2 (1.00) y_p (.69) ω_1 (.88) ω_2 (.87) ω_3 (.90) ω_4 (.90)
$1.1x10^7$	C(.51) K_1(.86) K_2(.95) K_3(.97)
$3.0x10^4$	P_1 (.85) φ_1(.71) φ_2(.52) φ_3(.71) φ_4(.64)
$1.1x10^4$	C(.30) Zo_1(.87) Zo_2(.88) Zo_3(.86) Zo_4(.84)

The introduction of orthogonal Kappa rotations on exposures and the use of convergent photos has eliminated or alleviated all the correlations except those involving respectively K_1, K_2 and K_3, and P_2 and y_p (Table 3.4).

Table 3.4. Correlated Parameters in the Case of Calibration with Elevation Differences on Ground Control, Orthogonal Kappa and Convergent Photos..

Condition Indices	Correlated Prameters (Variance-Decomposition Proportions)
$1.1x10^{10}$	P_2 (1.00) y_p (.71) ω_1 (.62) ω_2 (.79) ω_3 (.82) ω_4 (.83)
$1.0x10^7$	K_1(.83) K_2(.96) K_3(.99)
$8.3x10^3$	C(.20) Zo_1(.50) Zo_2(.36) Zo_3(.68) Zo_4(.72)

* In the case of calibration with elevation differences on ground control, **the introduction of prior information** on exterior orientation parameters did not bring any change to the existing pattern of correlations. It seems that elevation differnces on control encompasses the information brought by constraints on exterior orientation parameters. Prior information here is introduced as appropriate weighting of the unknown parameters leading to what is generally called indirect observations or quasi-observations.

Introduction however of constraints on the interior orientation parameters (c, x_p and y_p) succeded in isolating y_p from the near dependency involving P_2 and Omega rotations; it remains only two near dependencies as in Table 3.5.

Table 3.5. Correlated Parameters in the Case of Calibration with Elevation Differences on Ground Control, Orthogonal Kappa, Convergent Photos and Constraints on Interior Orientation Parameters.

Condition Indices	Correlated Prameters (Variance-Decomposition Proportions)
$8.0x10^9$	P_2 (1.00) ω_1 (.36) ω_2 (.61) ω_3 (.68) ω_4 (.68)
$8.3x10^6$	K_1(.76) K_2(.94) K_3(.98)

* The application of variance decomposition to the design matrix resulting from the calibration based on real data showed the same pattern of near dependencies and the same correlated parameters as those found with the simulated data.

4. CONCLUSION

The testing done with simulated and real data for different geometric configurations indicated that the variance decomposition proportions method is a valuable analytical tool for the identification of parameters involved in functional groupings.

In the case of non linear models, which is usually the case in photogrammetry, the design matrix is changing because it has to be updated after each iteration. In order to reduce the computational burden, the variance decomposition needs to be applied only to the initial design matrix if realistic first approximations are introduced as initial values.

In the testing all condition indices were considered; but the results of the calibration showed that estimates of parameters involved in near dependencies associated with condition indices smaller than $\mathbf{10^6}$ are not degraded.

On the other hand, Belsley et al (1980) advocated the scaling of the design matrix to a unit column norm before applying the variance decomposition. In this paper the variance decomposition was applied to an unscaled design matrix because the scaling can undo ill-conditioning associated with features such as mixed units, which may mask the real correlations between parameters.

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