Surface Models on the Basis of a Triangular Mesh – Surface Reconstruction

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ABSTRACT

Given is a triangulation of points on a surface which may be of abitrary topology. The aim is to reconstruct the surface using triangular patches. We present a solution that combines approaches from photogrammetry and from computer aided geometric design. The surface model built is capable of handling all kinds of surface data like normal vectors and break lines. The faces of the triangulation are replaced by triangular Bézier patches that meet smoothly along their boundary curves. The smoothness condition is weakened a bit to the concept of εG1-continuity, which allows us to use polynomial patches of low degree, in most cases without splitting.

1 INTRODUCTION

The problem of fitting a surface to a set of data points has been investigated in several scientific disciplines. On one hand there are the photogrammetetrists who are interested in building a Digital Terrain Model (DTM) from measured points on the earth's surface. In recent years the field of interest spread into medicine, modelling of buildings and many areas more. On the other hand there is the Computer Aided Geometric Design (CAGD) community, which has been investigating the same task from a different point of view and being motivated mainly by applications in Computer Aided Design. This paper aims at presenting a surface model, founded on the knowledge of both sides. It has the following characteristics:

- The model has no shape restrictions and it is independent of the coordinate system.
- The surface model is built on a triangulation of the data points, so the original measurements may be preserved. Filtering of measurement errors may be incorporated.
- The surface is smooth, except for so-called break lines where a difference in the first derivatives is desired. In this context 'smooth' means tangent plane continuous.
- It is possible to master the set of data for a whole country.
- The used methods are local. This means that the insertion of a new face in the triangulation influences the shape of the surface only in a restricted neighbourhood of this face.

Current digital terrain models do not satisfy all the characteristics described above. Many surface models are only 2.5 dimensional, which means that they can be described as graphs of bivariate functions. This approach does not allow one to represent bridges, overhangs, caves or similar effects. Surface models built of flat triangles (triangular irregular networks – TINs) either lack smoothness or require a large number of points to establish the desired smoothness. It stands to reason to utilize a spatial triangular mesh (a TIN) as a basis for triangular patches which establish the smoothness. Therefore the following steps are to be taken:

1. Estimation of surface normals at the vertices of the triangulation and filtering of measurement errors.
2. Constructing a curve mesh that interpolates the vertices of the triangulation as well as the surface normals at these vertices. The curves of the mesh are the boundary curves of the triangular patches.
3. Constructing triangular surface patches that interpolate these vertex data and these boundary curves of the network. Furthermore two adjacent patches must have the same tangent planes (with small deviations allowed) along their common boundary curve.

This is a common approach to the stated problem. Our contributions are as follows. In step 2, we also determine the field of surface normals along the boundary curves. This field is discretized by computing some of its representatives at various positions along the edge. The three fields of surface normals along boundary curves, which are the delimiters of one triangular face, are then approximated or interpolated in step 3. Hereby the interpolation conditions regarding the boundary curves are still fulfilled. Because discretized fields are used instead of interpolating the continous fields, two adjacent patches do not exactly join smoothly, but possess some deviation angle ε. We take care that ε is sufficiently small and thus it will have no disturbing effects when shading the surface, computing contour lines, calculating volumes, etc. To regularize the solution, the εG^1 algorithm is combined with the minimization of an appropriate fairness measure.

Let us briefly outline our presentation. In section 2, some preliminaries are described: the theory of Bézier triangles, which are a geometric representation of polynomial surfaces, some aspects of variational surface design and prediction. In section 3, related work from CAGD as well as photogrammetry is presented. Section 4 deals with the estimation of surface normals and the computation of boundary curves along with their normal vector fields. Section 5 shows how to insert the patches into the mesh of curves. In the concluding remarks we list the possibilities obtained with our approach.

2 PRELIMINARIES

2.1 Bézier triangles

Since Bézier triangles are rarely used in photogrammetry, we give a short introduction (for more details, see [Hoschek,1993]). A Bézier triangle (triangular Bézier patch) represents a polynomial surface of degree n in R^3 with help of a control net $\mathbf{P}_{ijk}$, with $i, j, k \geq 0$ and $i+j+k=n$ (see Figure 1). The points $\mathbf{P}_{ijk}$ are called control points, as the shape of the surface can be controlled by their position. The parameter domain is a triangle $\triangle(\mathbf{R}, \mathbf{S}, \mathbf{T})$. A Point $\mathbf{U}$ in this triangle can be described by its barycentric coordinates, the triple (r, s, t) with $r+s+t=1$ and $\mathbf{U}(r, s, t) = r\mathbf{R}+s\mathbf{S}+t\mathbf{T}$.

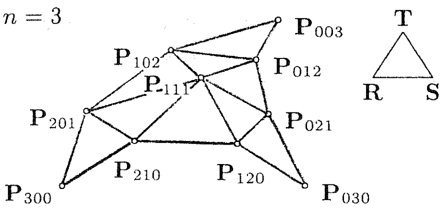

Figure 1: Control net and parameter triangle

The point $\mathbf{P}(r, s, t)$ on the surface to the point $\mathbf{U}$ in the parameter triangle is computed with the following recursive de Casteljau algorithm (see Figure 2):

1. $\mathbf{P}^0_{ijk} := \mathbf{P}_{ijk}$
2. $\mathbf{P}^l_{opq} := r\mathbf{P}^{l-1}_{o+1,p,q} + s\mathbf{P}^{l-1}_{o,p+1,q} + t\mathbf{P}^{l-1}_{o,p,q+1}$ with $o+p+q+l=n$. $\mathbf{P}^l_{opq}$ is therefore the image of $\mathbf{U}$ under the affine map from $\triangle(\mathbf{R}, \mathbf{S}, \mathbf{T})$ onto $\triangle(\mathbf{P}^{l-1}_{o+1,p,q}, \mathbf{P}^{l-1}_{o,p+1,q}, \mathbf{P}^{l-1}_{o,p,q+1})$.
3. $\mathbf{P}^n_{000} := \mathbf{P}(r, s, t)$ is the desired point on the surface.

Bézier triangles have the following properties:

- The surface has a polynomial parametric representation of degree n, expressible with Bernstein polynomials as

$$\mathbf{P}(r, s, t) = \sum_{i+j+k=n} \mathbf{P}_{ijk} B^n_{ijk}(r, s, t) \qquad (1)$$

$$\text{with } B^n_{ijk}(r, s, t) = \frac{n!}{i!j!k!} r^i s^j t^k.$$

- End point interpolation: $\mathbf{P}_{n00}$ is the point on the surface corresponding to the point $\mathbf{R}$ in parameter space. The tangent plane at $\mathbf{P}_{n00}$ contains the points $\mathbf{P}_{n-1,1,0}$ and $\mathbf{P}_{n-1,0,1}$. Analogy applies for $\mathbf{P}_{0n0}$ and $\mathbf{P}_{00n}$.
- The boundary curves of a Bézier triangle are Bézier curves. Their control polygons are the boundary polygons of the control net. The other control points of the surface are called 'inner points'.

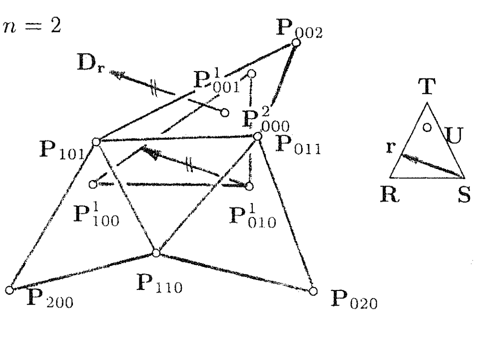

Figure 2: Constructing a point, subdivision and derivatives

- The tangent plane at $\mathbf{P}^n_{000}$ is defined by the three points $\mathbf{P}^{n-1}_{100}, \mathbf{P}^{n-1}_{010}, \mathbf{P}^{n-1}_{001}$.
- The directional derivative $\mathbf{D_r}$ to the direction $\mathbf{r}$ in parameter space is given by the affine image of $\mathbf{r}$ in the triangle that is obtained in the last but one step of the de Casteljau algorithm.
- The de Casteljau algorithm has the following subdivision property: During the computation of $\mathbf{P}(r, s, t)$ the control nets of the three subpatches to the parameter triangles $\triangle(\mathbf{R}, \mathbf{S}, \mathbf{U})$, $\triangle(\mathbf{U}, \mathbf{S}, \mathbf{T})$ and $\triangle(\mathbf{R}, \mathbf{U}, \mathbf{T})$ are obtained. The control points are $\mathbf{P}^l_{ij0}$ with $i+j+l=n$ and analogously for the other two control nets. As an example, in Figure 2 one of the new nets is $\mathbf{P}_{200}, \mathbf{P}_{110}, \mathbf{P}_{020}, \mathbf{P}^1_{100}, \mathbf{P}^1_{010}, \mathbf{P}^2_{000}$. The new control nets lie closer to the surface than the original one.

The subdivision algorithm mentioned above can be performed for more points in parameter space simultaneously. Choosing three points in the parameter space at the midpoints of each edge of the parameter triangle yields four new control nets describing the surface. Each of these control nets can be subdivided with the same points in parameter space again. This leads to a sequence of control nets that are fastly converging towards the surface (see [Hoschek,1993]). If subdivision is mentioned in the following, it has always to be understood as subdivision based on the edge midpoints of the parameter triangle.

2.2 Variational surface design

Pleasing surface shapes or surfaces with specific physical properties are often obtained as solutions of variational problems.

A frequently used fairness measure for a surface $\mathbf{S}(u, v)$ is the linearized thin plate energy

$$E = \int (\mathbf{S}^2_{uu} + 2\mathbf{S}^2_{uv} + \mathbf{S}^2_{vv}) du dv. \qquad (2)$$

Although it is dependent on the parametrization, it is often sufficient to minimize this quadratic functional in a linear space of surface candidates. This clearly amounts to the solution of a linear system. Other functionals, partially parametrization invariant, are discussed in [Greiner,1994], along with a technique to solve the nonlinear optimization iteratively by linear problems. These are obtained when the solution surface in step N is used as parameter domain for the improved solution in step $N+1$.

In our approach, variational design is basically used for regularization. Therefore, it is sufficient to use even simpler measures that may be applied directly to a piecewise linear mesh, such as the control net of a Bézier triangle. The net shall consist of points $\mathbf{P}_i, i = 1, \ldots, n$, and of edges $\mathbf{e}_j, j = 1, \ldots, m$. Then, an energy for the network may be formulated as the energy of a configuration of springs with one spring placed along each edge $\mathbf{e}_j$,

$$E_S = \sum_{j=1}^{m} \kappa_j \|\mathbf{e}_j\|^2. \qquad (3)$$

The spring constants κ_j introduce additional flexibility in choosing the functional. We often set all constants equal to 1. Another choice discussed in [Eck,1995] yields approximations of harmonic maps.

A similar functional that may be applied to linear networks is based on so-called 'umbrella vectors' which are associated with each vertex $\mathbf{P}_i$ of the net. If $\mathbf{P}_i$ has k_i neighbours $\mathbf{Q}_{ij}, j = 1, \dots, k_i$, the umbrella vector at $\mathbf{P}_i$ is defined as [Kobbelt,1995]:

$$\triangle \mathbf{P}_i = \mathbf{P}_i - \frac{1}{k_i} \sum_{j=1}^{k_i} \mathbf{Q}_{ij}. \tag{4}$$

The euclidean length of $\triangle \mathbf{P}_i$ can be seen as a discrete measure for the mean curvature at $\mathbf{P}_i$. This results in the fairness functional

$$U = \sum_{i=1}^{n} \|\triangle \mathbf{P}_i\|^2. \tag{5}$$

An application is the refinement of a network. New vertices are inserted such that U is minimized.

2.3 Predicting surface data

Prediction is a surface interpolation and approximation method ([Wild,1983]). It uses radial basis functions which are well studied in approximation theory, numerical analysis and CAGD (e.g. [Hoschek,1993]). The surface is considered to be a bivariate function which shall be approximated by the prediction function. The prediction function is sufficiently often differentiable, therefore surface normals and curvature can be calculated. It is also possible to force the prediction function to have certain derivatives at given points. Thus tangents to the surface in a given direction or surface normals can be prescribed.

3 RELATED WORK

There is a rich literature in CAGD dealing with the construction of surfaces based on polynomial patches. Most methods are covered in the monograph [Hoschek,1993]. We therefore just point to a few more recent developments.

Generalizations of the classical minimum norm networks by G. Nielson to parametric surface design have been discussed in [Kolb,1995]. The methods are global and therefore not applicable for huge data sets.

An elegant approach for dealing with smooth surfaces composed of triangular or rectangular polynomial patches are the surface splines in [Peters,1995]. For the present application, the method would generate too many patches.

Approximately smooth surfaces have been investigated in [Mann,1992]. The patches require curvature information at the vertices and yield very pleasing results if enough data are available for producing good curvature estimates. This does not really correspond to the present scenario.

A very promising approach to surface modeling are the hierarchical techniques in [Eck,1995]. They work very well on TINs and more generally on so-called subdivision surfaces. Some ingredients of these algorithms, such as parametrization based on harmonic maps, may also be useful for our application.

In view of the lack of a single method that would satisfy all our requirements we decided to implement our own version, which is an appropriate combination of known techniques taylored towards the applications we have in mind.

4 ESTIMATION OF NORMALS

4.1 Estimating the surface normals

The process of estimating the surface normals must be independent of the coordinate system and local. For each point $\mathbf{P}$, its surface normal $\mathbf{n}$, its tangent plane τ respectively, is calculated.

To make it a local process, only a neighbourhood of a point $\mathbf{P}$ is used to calculate the surface normal at $\mathbf{P}$. The neigbourhood $\{\mathbf{Q}_1, \mathbf{Q}_2, \dots\}$ of a point can either be defined via generations of points (topological norm) in the triangulation around this point or via distances (euclidean norm). The set of points $\{\mathbf{Q}_1, \mathbf{Q}_2, \dots\}$ having a distance to the centerpoint $\mathbf{P}$ less than or equal to s can be described as a subset of the neighbourhood of $\mathbf{P}$ with n generations, where the points in the n-th generation have a distance to $\mathbf{P}$ larger than or equal to s. This definition prevents including points which are close to $\mathbf{P}$ in the sense of distance but lie completely elsewhere on the surface. This situation can arise for example when interpolating the surface data of a cave. Still there may exist one or more points $\mathbf{Q}_j$ in the neighbourhood of $\mathbf{P}$ which will have to be excluded from the estimation process. This results from an estimation technique, where we need surface regions that are definable as graphs of bivariate functions over an appropriate parameter plane (using the approach in [Opitz, 1994], this restriction could be avoided however). To find these points, the direct way from $\mathbf{P}$ to $\mathbf{Q}_j$ via the edges of the triangulation has to be examined. If one of the slopes of the planes against an approximate tangent plane at $\mathbf{P}$ exceeds a threshold value (e.g. 80°), then $\mathbf{Q}_j$ has to be excluded (see Figure 3).

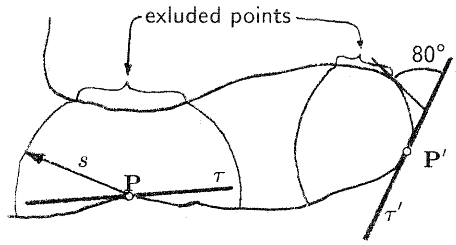

Figure 3: Interpolation in a cave

To guarantee that the estimation of $\mathbf{n}$ is independent of the coordinate system, the interpolation is done in a local coordinate system whose (x, y)-plane is the actual approximate tangent plane τ of the point $\mathbf{P}$. The plane τ is updated in an iterative process. The starting value τ_0 for the iteration can either be a result of the triangulation process, or can be computed by averaging the normal vectors of the triangles meeting at $\mathbf{P}$. The iteration runs as follows:

1. The points $\mathbf{Q}_i$ and $\mathbf{P}$ are transformed into a coordinate system based on an approximation for the tangent plane (τ_i).
2. Prediction yields a surface, whose tangent plane at $\mathbf{P}$ delivers the next approximation (τ_{i+1}). If desired, filtering can be performed in this step as well.
3. If the angle between τ_i and τ_{i+1} is smaller than a certain threshold value, the process is stopped. Otherwise τ_{i+1} serves as a new entry for step 1.

The curvature can also be computed from the prediction function. This could be used for constructing the curves between two vertices of the triangulation. If filtering is applied, the vertices of the triangulation obtain new coordinates.

4.2 Calculating the boundary curves and their normal vector fields

Boundary curves Each boundary curve replaces an edge of the triangulation. The end points of the curves are the vertices of the triangulation. All curves meeting in one point must have tangents at this point that lie in a common tangent plane. In general this tangent plane is obtained by calculating normal vectors for each point of the triangulation.

The boundary curves shall be of degree three. This guarantees compatibility to various systems based on spatial curves. For example in [Halmer,1996] a way to adjust spatial curve nets composed of curves of degree three is described. In cases where the curves are given, only the normal vector fields along these curves have to be computed. Otherwise the task is to determine a curve $\mathbf{c}(t), t \in [t_a, t_b]$, that interpolates two given points $\mathbf{P}_0 = \mathbf{c}(t_a)$ and $\mathbf{P}_1 = \mathbf{c}(t_b)$. $\mathbf{P}_0$ and $\mathbf{P}_1$ are vertices of the triangulation, their surface normals are $\mathbf{n}_0$ and $\mathbf{n}_1$. The curve $\mathbf{c}(t)$ must also fullfil the 'tangent plane conditions' $\dot{\mathbf{c}}(t_a) \cdot \mathbf{n}_0 = 0$ and $\dot{\mathbf{c}}(t_b) \cdot \mathbf{n}_1 = 0$. As the curves are of degree three, we just need to compute $\dot{\mathbf{c}}(t_a)$ and $\dot{\mathbf{c}}(t_b)$.

The edges of the triangulation emanating from one point prescribe an order: Looking at that point against the direction of its normal vector each edge has one edge to its left and one to its right. This order must be maintained by the curves [Kobbelt,1995]. Laying the tangent $\dot{\mathbf{c}}(t_a)$ in $\mathbf{P}_0$ in the plane through $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{n}_0$ guarantees that the ordering prescribed by the edges is maintained. Together with $\dot{\mathbf{c}}(t_a) \cdot \mathbf{n}_0 = 0$ the direction of the tangent is determined completely. Analogy applies for $\dot{\mathbf{c}}(t_b)$. The length of the first derivative vectors can be chosen in a way that the curvature of the surface computed in a previous step is approximated (interpolation is possible only for planar curves, which needs another choice of the tangent directions; e.g. [Mann,1992]).

Normal vector field Given is a curve $\mathbf{c}(t), t \in [t_a, t_b]$, on a surface. The surface is represented by points and their surface normals. The goal is to compute the surface normal vector field along the curve. The field is discretized. For that, parameter values t_i, $t_a < t_i < t_b$, $i = 1, \ldots, k$, to points on this curve are given. The notation for the endpoints remains as explained before. The points in the neighbourhood of $\mathbf{P}_0$ and $\mathbf{P}_1$ are $\mathbf{Q}_1, \mathbf{Q}_2, \ldots$ and their normals $\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2, \ldots$.

To calculate the normal vector field the points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{Q}_1, \mathbf{Q}_2, \ldots$ and their normals are transformed into a coordinate system which is solely dependent on these points and possibly also on their normals. However, the triangulation of these points has to be describable as a graph of a bivariate function. For determining the normals in the points $\mathbf{c}(t_i)$ prediction is used. The data to be interpolated are: the points on the curve $\mathbf{c}(t_i)$ and the tangents $\dot{\mathbf{c}}(t_i)$, the points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{Q}_1, \mathbf{Q}_2, \ldots$ and their normals. The discretized normal vector field is formed by the normals of the prediction function at $\mathbf{c}(t_i)$.

5 COMPUTING THE BÉZIER TRIANGLES

The previous section described how to interpolate surface normals, boundary curves and normal vector fields along these boundary curves. Now triangular patches have to be constructed for each face of the triangulation interpolating the data. The examples given in this section are based on triangulations of analytical surfaces.

5.1 εG1 continuity

Two adjacent patches are joined smoothly if they have the same tangent planes along their common boundary curve. In this case they are geometrically continuous of first order (G1).

To maintain locality, the determination of the inner control points of adjacent patches shall be independent from each other, and just based on the boundary data (curves and normals). Continuous normal vector fields can in general not be interpolated with Bézier triangles. However, if the field is interpolated or approximated at a number of positions, the tangent planes of the adjacent patches will not match exactly, but it can be expected, that they deviate from each other by a small angle only. Therefore this kind of "continuity" is called εG1. The deviation of tangent plane fields can be judged in two ways:

1. Comparing two adjacent patches directly: The maximum deviation angle shall be γ.
2. Comparing the normal vector field of a patch with the given normal vector field: The maximum deviation angle shall be $\frac{\gamma}{2}$; then the first criterion is satisfied as well.

The angle γ is a user defined quality measure. What needs to be done if the γ–criterion can not be satisfied, is described later on.

To ensure fast reconstruction and to reduce the amount of data, we restricted ourselves to use patches of degree three and four. Patches of higher degree also lead to larger systems of equations and their shape is more likely to have unpleasing areas of high curvature or oscillation.

5.2 Inner control points and tangent planes

It is well known and follows immediately from the tangent plane construction with the algorithm of de Casteljau that the tangent planes of a Bézier triangle along a boundary curve depend only on the corresponding boundary control points and on the points of the neighbouring row. In the case of a Bézier triangle of degree three, there is only one inner point. All the tangent planes along the boundary are dependent on the choice of this point. In the case of degree four, always a pair of two inner control points have influence on the tangent plane along a boundary curve.

The functional dependency between the tangent plane at $\mathbf{P}$ and the inner points is easily derived. Let $\mathbf{P}$ be a point of the Bézier curve whose control points are $\mathbf{P}_{300}$, $\mathbf{P}_{210}$, $\mathbf{P}_{120}$, $\mathbf{P}_{030}$; its curve parameter shall be t. Further, let $\mathbf{Q}$ be the point with parameter t on the Bézier curve with control points $\mathbf{P}_{201}$, $\mathbf{P}_{111}$, $\mathbf{P}_{021}$. Then $\mathbf{q} = \mathbf{Q} - \mathbf{P}$ is a tangent vector at $\mathbf{P}$. With $\mathbf{n}$ as normal vector at $\mathbf{P}$, we therefore have

$$\mathbf{n} \cdot \mathbf{q} = 0. \tag{6}$$

In terms of the unknown inner point $\mathbf{P}_{111}$ and using the Bernstein polynomials for Bézier curves this can be written as

$$\begin{aligned} \mathbf{n} \cdot \mathbf{q} = \mathbf{n} \cdot (\mathbf{Q} - \mathbf{P}) = 0 = & \qquad (7) \\ \mathbf{n} \cdot ((1-t)^2\mathbf{P}_{201} + 2t(1-t)\mathbf{P}_{111} + t^2\mathbf{P}_{021} - \mathbf{P}), & \\ 2t(1-t)\mathbf{n} \cdot \mathbf{P}_{111} = \mathbf{n} \cdot (\mathbf{P} - (1-t)^2\mathbf{P}_{201} - t^2\mathbf{P}_{021}). & \end{aligned}$$

Equation 8 is a scalar product and therefore each boundary normal to be interpolated introduces one linear scalar equation

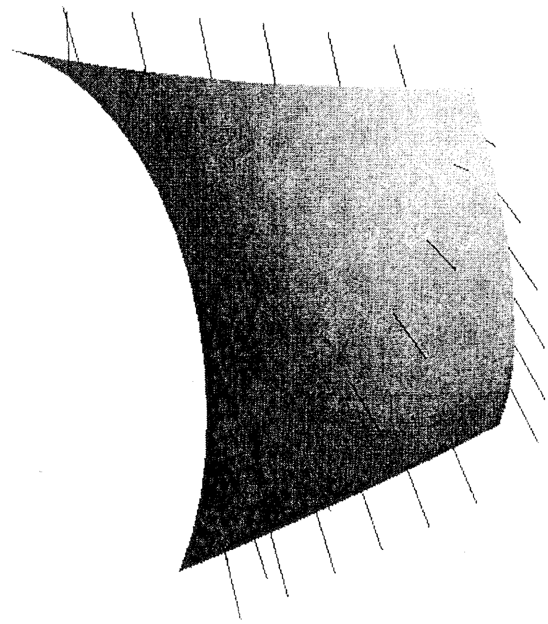

Figure 4: Approximation of normals

for $\mathbf{P}_{111}$. Hence, with a cubic Bézier triangle three normal vectors can be interpolated. The way to extend equation 8 for Bézier triangles of degree four is obvious. A Bézier triangle of degree four has three inner points and thus nine normals at the edges can be interpolated.

Given the same data as in figure 4 — two adjacent patches of degree four — but using interpolation of three normals per boundary, the mean value of the deviation angle of the tangent planes would be 0.54°, the maximum value would be 1.21°. Furthermore, the shape of the Bézier triangles would be curved less regularly.

5.3 Approximation of normal vector fields

Instead of interpolating the field of surface normals at a certain number of positions it can also be approximated at more positions. As it might be expected, this leads to better results. To approximate the normal vector field in the sense of least squares,

$$F = \sum_{i=1}^{k} (\mathbf{n}_i \cdot \mathbf{q}_i)^2, \quad \text{with } \mathbf{q}_i = \mathbf{Q}_i - \mathbf{P}_i, \tag{8}$$

must be minimized. $\mathbf{P}_i$ is a point on the boundary of the patch, $\mathbf{Q}_i$ relates to it as explained before. $\mathbf{n}_i$ represents the estimated normal vector at $\mathbf{P}_i$ and k is the number of normal vectors for approximation. We chose $\|\mathbf{n}_i\| = 1$, otherwise equation 8 is a weighted adjustment. The weight for one equation would be the square of the length of $\mathbf{n}_i$. Function 8 is quadratic. Its minimum is the least squares solution of the overdetermined linear system $\mathbf{A}x = l$. $\mathbf{A}$ contains the normal vectors, multiplied with a Bernstein polynomial, x the coordinates of the inner points and l the known points of the control net, each multiplied with one of the Bernstein polynomials.

Figure 4 shows two adjacent patches of degree four. Along the boundary curves the surface normals for approximation are drawn. The mean value of the deviation of the tangent planes is 0.44°, the maximum value is 0.94°.

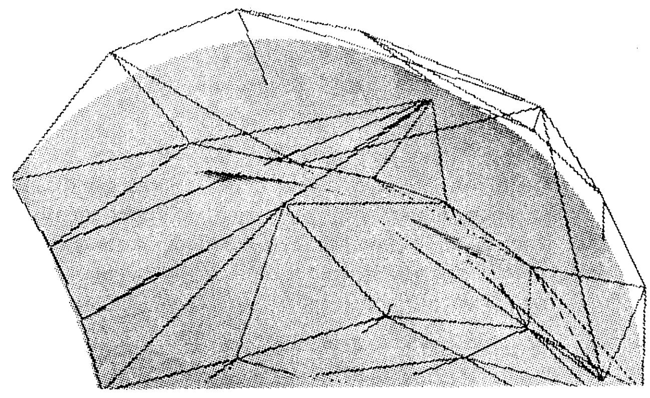

Figure 5: Approximation, Bézier triangles that tumble over

The question of how many normals per edge shall be approximated needs to be raised. For Bézier triangles of degree four a number of six normals for approximation per edge turned out to be sufficient in general.

5.4 Approximation of normal vector fields and minimizing the surface energy

In areas of very small curvature (the Bézier triangles are flat) the positions of the inner points are only weakly determined by equations 6 or 8. In the case of Bézier triangles of degree three the system of equations becomes singular, if there are no three normals that form a basis in R^3. By examining $(\mathbf{A}^t\mathbf{A})^{-1}$ it can be shown that the inner points are determined weakly in their position perpendicular to the normals but strongly in the direction of the normals. It can be said that the position of the inner points perpendicular to the normals does not have a strong influence on the approximation of the normals. Thus the inner points also could lie "outside" of the points along the edges. This leads to a Bézier triangle that overturns, tumbles over (see Figure 5). In such a case, the vectors $\mathbf{q}$ tend to be very small where the patch tumbles over.

Therefore, some kind of regularization must be implied. Instead of minimizing equation 8, the following combined functional will be minimized:

$$F_C = \alpha(\sum_{i=1}^{k} (\mathbf{n}_i \cdot \mathbf{q}_i)^2) + (1-\alpha)E, \quad \text{with } \alpha \in [0,1]. \tag{9}$$

E is a term for the surface energy. It can be any of the functionals discussed in section 2.2. We simply used the spring energy functional E_S with equal spring constants applied to the control net after one subdivision. More than two subdivisions do not make sense. Experiments showed that α-values should be in the range of 0.9 to 0.95.

Figure 6 shows eleven Bézier triangles; $\alpha = 0.95$, the net is subdivided once. The control nets are drawn as well. The surface shown in figure 5 has been computed with the same data, but without applying regularisation.

5.5 Splitting the face for a better approximation

There may occur surface data that is not compatible with the γ-criterion, described in section 5.1. In such a case a face has to be split along the edge where the criterion could not be fulfilled. If the inner points shall be stored permanently, the faces have to be divided and the triangulation must be updated. If the inner points are calculated each time they are

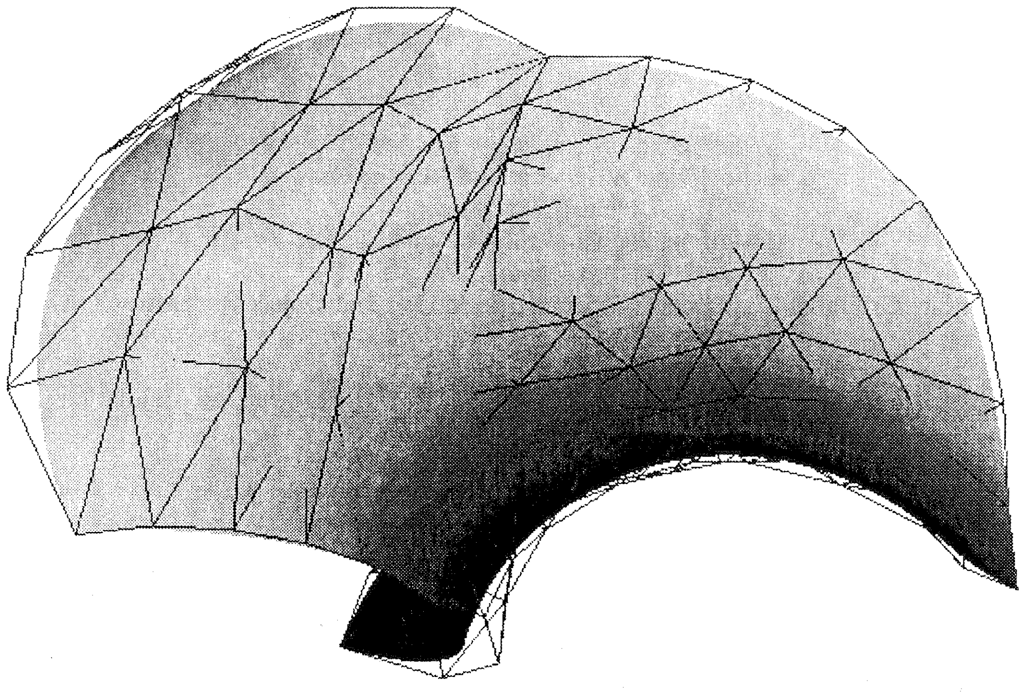

Figure 6: Approximation with surface energy minimization

used, it is sufficient to divide the patch that cannot fullfil the criterion.

6 CONCLUDING REMARKS

Due to the parametrization of the surface over a TIN, this approach obtains a universality which allows one to model surfaces of abitrary topology. Because of the use of the surface normals at the vertices of the triangles, it is furthermore very flexible in the adaption of the subsistent data:

- Breaklines can be taken into account by giving two normal vectors to those surface points that are situated along the breaklines, one for each side. Only points lying on the appropriate side of the breakline may influence the estimation process of the surface normal.
- Structure lines, e.g. mountain ridges or the bottom of a valley, may be taken into account by an appropriate triangulation and choice of the surface normals and the tangent plane field.
- Contour lines, e.g. digitized from a map, are perpendicular to the surface normals, and hilltops are points with known surface normals. This can be exploited in the estimation of the normals by introducing constraints for the derivatives.

This is a preliminary report on our progress in a larger research project. Further studies are necessary, such as data reduction by appropriate preprocessing, hierarchical modelling, refined estimation techniques and others.

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