THE DEFINITION OF REFERENCE SURFACES FOR ARCHITECTURAL PHOTOGRAMMETRY

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ABSTRACT

In this contribution the author presents a complete approach in order to calculate surfaces of second degree fitted to 3D point clouds for further applications mainly in architectural photogrammetry. A least square adjustment is used for this purpose. The process consists of two distinctive parts. The first part involves the calculation of the parameters of the general equation that describes a surface of the second degree. From these parameters the surface type is classified, and approximate parameters for shape, size and orientation are calculated. These values are necessary for the second part in which individual programs for each surface are described. As an example this process is applied to a conical tower on the island of Andros in Greece.

1 INTRODUCTION

Architectural applications of photogrammetry require very often the determination of the shape of a building or of a specific structure such as a dome, a vaulted ceiling, or a tower. In particular surfaces of second degree come frequently upon in those applications. These are mainly the ellipsoid, sphere, cone, paraboloid, hyperboloid and cylinder. They can fully or partially describe the object. single image photogrammetry techniques very often profit from the knowledge of the analytical equation describing the surface of the object. (Theodoropoulou, 1999). Since stereo techniques mainly complicate tasks in terms of equipment and algorithms, monoscopic methods are advised due to their simplicity and effectiveness. Image rectification remains the most widespread tool for architectural and archeological applications and in this case only one photo is adequate for the representation of the 3D space. The necessary information provided by the second image could be replaced by a geometric property of the object. For example, if the object's surface is smooth and can be described by analytical expressions the second image is no longer indispensable. In this case the knowledge of the mathematical equation describing the surface is essential to the application therefore the problem of surface fitting to measured points arises. Such problems have been tackled repeatedly in the past where mathematically expressed surfaces were developed or projected (flattened) successfully.

Ellipsoids are very often encountered in domes. An interesting example of fitting a set of photogrammetrically determined redundant points on an ellipsoid was carried out at the Dome of the Rocks in Jerusalem. (Restle and Stephani, 1988). Ellipsoidal domes were restituted by means of a technique based on projecting the surface orthogonally on rectangular polyhedrons. The knowledge of the shape of the triaxial ellipsoid was essential. (Vozikis and Jansa, 1980). As a subcase of the ellipsoid, the sphere is frequently used to describe architectural buildings. The restitution of old globes entails the investigation of their shape and condition and therefore requests the adjustment of a best fitting spherical surface (Kager et al., 1992; Hemmleb and Suthau, 1999). Redundant points determined photogrammetrically were fitted to an elliptical paraboloid (Fotiou et al.,1991). The aim was to detect deformations at the Dome of the House of Sports in Teramo. The shape was determined as well as the origin.

Cones and cylinders are moreover developable surfaces, i.e. they can be unwrapped or flattened on the plane without any distortions (Karras et al.,1996; Hemmleb and Wiedemann, 1997; Karras et al., 1997). Conical surfaces have been treated in the past , (Petsa,1993), whereas cylindrical surfaces have been subject to a more frequent research (Chandler and Cooper,1991; Robson et al.,1992).

In all the cases mentioned above the question of determining the best fitting mathematical equation arises. For this purpose points could be sampled in many ways, for example by surveying methods, or photogrammetry. Finding and defining the analytical equation is the first step for a further parametrical rectification of the objects. There is an increase of architectural and archeological applications which demand a raster representation or even a 3D visualization of the object.

2 SURFACE FITTING

2.1 The Equation of Second Degree

A surface of second order is defined as a set of points, which satisfy the following equation:

$$a_1x^2 + a_2y^2 + a_3z^2 + b_1xy + b_2yz + b_3xz + c_1x + c_2y + c_3z + d = 0 \quad (1)$$

The above equation contains 10 coefficients. By dividing the equation by any constant there are 9 effective coefficients. Thus, in general one unique quadric passes through nine given points. In other words, if nine independent conditions are satisfied by the 9 effective constants, the surface can be determined uniquely (Jain and Ahmed, 1991). These 9 coefficients represent the three parameters of the canonical forms defining shape (surface size and type) and the 6 parameters for surface position and orientation. The intersection of such a surface with a plane, is a second order curve. If all six coefficients of second order are all zero, then the surface will represent a plane.

There are cases where the surface type is not obvious or there are doubts concerning the surface type, for example between an ellipsoid and a sphere. By fitting this 9 parameter equation to the set of points, it is possible to classify the surface uniquely by calculating the so called 4 invariants and 2 semi invariants of the quadric (Bronstein et al.,1997), or by performing a 3D transformation to another cartesian coordinate system where the quadric will be in its simplest form (canonical form). Quadrics could be classified in 17 different surface types. Among these, the ones most often encountered in photogrammetry are ellipsoids, spheres, cylinders, paraboloids and cones. It is possible, from the general 9-parameter equation by performing a 3D transformation (translation and rotation) to get the parameters of the canonical form. Nevertheless, once the surface type is determined, it is advised to process each surface with its own canonical equations. The information that the photogrammetrist should determine for each surface is shape, size and orientation (position and orientation).

2.2 The Least Squares Model

For the solution a Least Squares Adjustment based on conditions equations is selected (Dermanis and Fotiou, 1997). Other possible methods have been already mentioned in the literature (Robson et al.,1992) The coordinates of the observed points are treated as observations. This allows different accuracies $\sigma x, \sigma y, \sigma z$ to be included in the model. Given the observations (x,y,z) then the sum of the squares of the observation errors are to be minimized. The functional model of the condition equations has the form:

$$u(x^a, y^a) = 0 \quad (2)$$

x^a: vector of unknown parameters

y^a: vector of observed quantities

In matrix form the linearised equation has the form :

$$Ax - Bv + w = 0 \quad (3)$$

w: vector of misclosures

A,B : design matrices

v : vector of observations

x: vector of unknowns

The best linear unbiased estimates $\hat{x}$ are computed:

$$\hat{x} = -N^{-1}u = -(A^T(BP^{-1}B^T)A)^{-1}(A^T(BP^{-1}B^T)^{-1}w) \quad (4)$$

P : weight matrix

also

$$\hat{x}^a = x^o + \hat{x} \quad (5)$$

x^o : vector of approximate values of parameters

$$\hat{v} = P^{-1}B^T(BP^{-1}B^T)^{-1}(w + A\hat{x}) \quad (6)$$

$$\hat{\sigma}^2 = \frac{\hat{v}^T P \hat{v}}{s-m}, \quad (7)$$

$\hat{\sigma}^2$: a posteriori variance

s : number of equations

m : number of unknowns

The functional model for the estimation of the unknown parameters in the least squares adjustment is:

$$u = a_1x^2 + a_2y^2 + a_3z^2 + b_1xy + b_2xz + b_1xz + c_1x + c_2y + c_3z + d = 0 \quad (8)$$

Where

$a_{1,}a_2,...d$: are the unknown parameters

x,y,z : the observations (coordinates)

These 10 coefficients $a_{11}...d$ are parameters without any physical meaning. They tell us nothing about the shape or the orientation of the surface in question. It is however possible to extract useful information from them and to answer the following questions :

- which is the type of surface ? The 4 invariants can classify uniquely the surface.

- the required parameters for shape, translation and orientation can be extracted from the 10 coefficients.

Only 9 of the 10 parameters $a_1...d_4$ are independent. Therefore in order to get a solution from the least squares adjustment a constraint has to be introduced. This is a problem that has been already subject to research in the literature before. It is common in photogrammetric applications to use the constraint d = constant for example d = 1. However this constraint can produce several problems as it is not invariant to translation and rotation for example in the classification of the surface. Other constraints have been proposed in the literature. The author selected the constraint (Bookstein, 1979):

$$a_1+a_2+a_3+0.5(b_1+b_2+b_3) = \text{constant} \quad (9)$$

2.3 The Process

The nine independent parameters of the equation of second degree are calculated based on three dimensional coordinates for a sufficient number of points. For the functional model (8) approximate values must be provided. For the observations the approximate values are the measured coordinates. However approximate values must be calculated for the unknowns. Hence a special process for the calculation is applied. The approximate values for the parameters a_1, a_2...d are computed by setting temporarily the parameter d = constant and creating another linear model.

$$f(x^{a)}) = \text{constant} \tag{10}$$

The parameters are recalculated by using the constraint (9) and the final approximate values are inserted to the non linear functional model (8).

The second part consists of individual programs that treat every surface type separately. The adjustment is performed once more. This is necessary in order to get precision estimations of the parameters having a physical significance. These are the ellipsoid (with the sphere as a subcase), the elliptical paraboloid, the one and two sheet hyperboloid and the elliptical cone. Degenerate cases of surfaces of second order like a pair of planes or a pair of intersecting lines are not included in this approach. The minimum number of independent parameters required to define the size and shape is used. Additional unknowns for the position (translation vector) and orientation (rotation matrix) are included. For these parameters (which in this case have a physical significance) approximate values are required. From the results derived in the first part of the program these approximate values can be extracted. The program finally provides :

- the necessary parameters defining shape and orientation
- the corrected (adjusted) coordinates of all the 3D points.
- precision estimation of all the quantities that have been estimated in the adjustment

2.4 Individual Surfaces

In this paragraph the individual surfaces of second degree are briefly presented. For each surface observation equations are derived and the particularities are shortly discussed. For the adjustment of the individual surfaces the same model is used, i.e. the observations are the measured coordinates of the points. This contribution does not exhaust all the cases of second degree surfaces. The hyperbolical paraboloid is excluded because of its rareness in architectural applications. The cylinder is on the other hand not mentioned since the derivation of the model and the observation equations are repeatedly treated in the literature.

2.4.1 Ellipsoid The equation of the triaxial ellipsoid has the form**:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{11}$$

For the full knowledge of the parameters of the triaxial ellipsoid one should determine the semiaxes a,b,c and the orientation: position (three unknowns for the origin x_c,y_c,z_c) and a rotation matrix (three unknowns). The observation equation is:

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{12}$$

Usually the coordinates of the points are referred to an arbitrary geodetic coordinate system, therefore orientation parameters should be introduced.: Translation: $x' = x-x_c$, $y' = y-y_c$, $z' = z-z_c$. The rotations ω, φ, κ are introduced in the form of a rotation matrix (Moffitt and Mikhail, 1980) as follows:

$$x'' = r_{11}(x-x_c)+r_{21}(y-y_c)+r_{31}(z-z_c) \tag{13}$$

$y'' = r_{12}(x-x_c)+r_{22}(y-y_c)+r_{32}(z-z_c)$

$z'' = r_{13}(x-x_c)+r_{23}(y-y_c)+r_{33}(z-z_c)$

Let x'' = u1, y'' = u2, z''= u3.

The observation equation can be rewritten:

$$u = \frac{1}{a^2}u1^2 + \frac{1}{b^2}u2^2 + \frac{1}{c^2}u3^2 - 1 = 0 \qquad (14)$$

Therefore the partial derivatives of the unknowns (a,b,c,xc,yc,zc,ω,φ,κ) must be calculated for the matrix A, and partial derivatives for the observed quantities (x'',y'',z'') for the matrix B.

A rotational ellipsoid occurs when two axes are equal, say a=b, whereas when a=b=c, the ellipsoid becomes a sphere. For the rotational ellipsoid the unknown parameters are seven since a=b and the rotation over the axis z is irrelevant. For the sake of completion the derivation of the observation equation for the sphere is also given here. In order to determine a sphere, 4 parameters are demanded, its origin x_c, y_c, z_c and its radius R. Thus, in some cases, and within a certain tolerance limit, the triaxial ellipsoid could be treated as a rotationary surface or even as a sphere. The observation equation of the sphere is:

$$u = (x-x_c)^2+(y-y_c)^2+(z-z_c)^2-R^2 = 0 \qquad (15)$$

where (x_c,y_c,z_c) the center of the sphere and R the radius.

2.4.2 One Sheet Hyperboloid : The locus of the equation**:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (16)$$

is a hyperboloid of one sheet. If a = b the hyperboloid is a rotationary hyperboloid or a hyperboloid of revolution.

For the full knowledge of the parameters of the hyperboloid of one sheet one should determine the semiaxes a,b,c and the orientation :position (the origin x_c,y_c,z_c) and a rotation matrix. The observation equation and the orientation parameters are introduced as in paragraph 2.4.1, so the observation equation can be written:

$$u = \frac{1}{a^2}u1^2 + \frac{1}{b^2}u2^2 - \frac{1}{c^2}u3^2 - 1 = 0 \qquad (17)$$

2.4.3 Two Sheet Hyperboloid : The equation :

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \tag{18}$$

represents a hyperboloid of two sheets. The surface is separated in two portions, one above the plane z = c and the other below the plane z = -c. For the full knowledge of the parameters of the hyperboloid of two sheet one should determine the semiaxes a,b,c and the orientation :position and a rotation matrix. The orientation parameters are introduced in the same way as in paragraph 2.4.1 so the observation equation can be written:

$$u=\frac{1}{a^2}u1^2-\frac{1}{b^2}u2^2-\frac{1}{c^2}u3^2-1=0 \tag{19}$$

2.4.4 Elliptical paraboloid : An elliptical paraboloid is a set of points with the following equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0 \tag{20}$$

Except for the origin, the surface lies above the xy plane. From the mathematical expression of the surface it is clear that the parameter c should be arbitrarily set in order to compute the parameters a and b. So for an arbitrary value of c, a pair of a and b will exist and vice versa. Six more unknowns are also in this case necessary for the orientation. Therefore for the full knowledge of the parameters of the elliptical paraboloid one should determine the semiaxes a,b, and the orientation :position and a rotation matrix. The orientation parameters are introduced as in paragraph 2.4.1, so the observation equation can be rewritten:

$$u=\frac{1}{a^2}u1^2+\frac{1}{b^2}u2^2-\frac{1}{c}u3=0 \tag{21}$$

2.4.5 Elliptical Cone : A conical surface is the surface which is generated by a straight line, the generatrix, which always intersects a given curve the directrix, and always passes through a given point, the vertex. The quadric cone is an elliptical cone whose canonical equation is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \tag{22}$$

The cone is a central non-bounded surface. It extends to infinity. It is a simple curved surface, that is the tangent plane intersects the cone on a line. If a=b then the surface is a circular cone or a cone of revolution. The parameter c should be arbitrarily fixed for the elliptical cone as for the elliptical paraboloid. For the full knowledge of the parameters of the elliptical cone one should determine the semiaxes a, b, and the orientation :position and a rotation matrix. The orientation parameters are introduced as in paragraph 2.4.1, so the observation equation can be rewritten :

$$u=\frac{1}{a^2}u1^2+\frac{1}{b^2}u2^2-\frac{1}{c^2}u3^2=0 \tag{23}$$

3 PRACTICAL EXAMPLE

The documentation of a hellenistic tower on the island of Andros, in Greece was requested by the Hellenic Ministry of Culture. The tower has an overall right circular conical form and is approximately 20 meters high (Figure 1). Points on the external surface were geodetically measured in order to perform the adjustment as well as the orientation of the images for the documentation of the building.

At first the 208 measured points were introduced to the program "Adjust" fitting the general equation of second degree. The constraint mentioned above was set to the constant value 0.5. Values for the 10 unknown coefficients were estimated and the invariants were calculated (Table 1). Actually the invariant Δ should be equal to zero for the conical surface. In this case it looks as if the surface is identified as one sheet hyperboloid. This is probably due to the poor point distribution. The vertex of the cone is at 95 meters whereas the points are distributed to the lower 20 meters of the surface. The double curvature indicated by this invariant can be attributed to the roughness of the stones especially on the lower part of the tower.

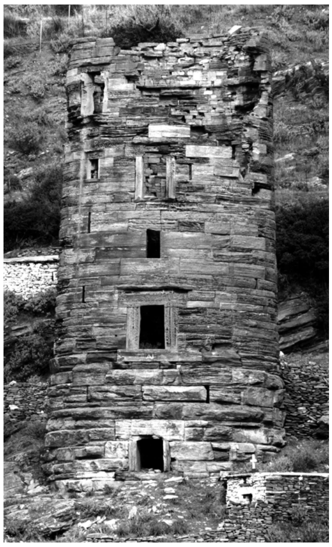

Figure 1 : The hellenistic conical tower on Andros

a_1 = 0.4991	Δ = 0.000945
a_2 = 0. 500	δ =-0.000348
a_3 =-0.00139	T = 0.2486
b_1 =-0.0022	S = 0.9986
b_2 =-0.000997	
b_3 = -0.00148	
c1 =-211.78121	
c2 =-214.263	
c3 = 0.753733	
d = 45267.4891	

Table 1 : Program Adjust : Results

Approximate	Results	
a = 2.32 (m)	a = 2,287	$\hat{\sigma}$ a = 4,95(cm)
b = 2.32 (m)	b = 2.274	$\hat{\sigma}$ b = 4,78(cm)
x_c= 211.791	x_c = 211.649	$\hat{\sigma}$ x_c= 10.69(cm)
y_c= 213.510	y_c = 213.445	$\hat{\sigma}$ y_c = 10.69(cm)
z_c= 81	z_c = 95.360	$\hat{\sigma}$ z_c = 1.8 (m)
ω = 0°	ω =-0.002°	$\hat{\sigma}$ ω = 0.01°
φ = 0°	φ =0.035°	$\hat{\sigma}$ φ = 0.01°
κ = 0°	κ =-1.538°	$\hat{\sigma}$ κ = 1.8°
c = 44 (constant)	c =44 (constant)	

Table 2 : Program Cone : Results

The overall a posteriori variance of the adjustment is :

$$\hat{\sigma}^2 = 6.01cm$$

Further on the program "Cone" for the adjustment of an elliptical cone needs approximate values for the unknowns a and b, for the vertex (x_c,y_c,z_c) and for the rotation matrix (Table 2). The approximate values for the semi axis are calculated from the eigenvalues of the equation. From the above results one can immediately conclude that the cone is circular since the values for a and b do not differ significantly. The translation vector is calculated together with the invariants. Moreover, the coefficients for the rotation are practically zero, and so are the approximate values for the rotation angles. The overall a posteriori variance of the adjustment is 6 centimeters. The distribution of the points explains the big uncertainty of the value z_c. It is clear from the results that the cone is circular and right i.e. the angles φ and ω are small. The values for the semi axes a and b are calculated for the constant value c = 44m. In fact the angles β_1 and β_2 resulting from the ratios a/c and b/c, indicate how "open" the cone is, and are 2.975° and 2.958° respectively. A program adjusting a right circular cone where the unknowns are the vertex and one angle β indicating the ratio a/c . This best fitting adjustment uses another mathematical model, giving slight different estimations for x_c, y_c but a better overall a posteriori variance (Theodoropoulou et al., 2000).

4 CONCLUDING REMARKS

It is evident that surfaces of second degree appear frequently in architectural buildings, momuments or other industrial constructions. The knowledge of the mathematical equation which best approximates the surface of the object is very often necessary in photogrammetric surveys and offers possibilities in particular in single image applications. The best fitting process in this paper uses the method of least squares and minimises the sum of the square errors of the observations. With this model adjusted coordinates for each point are calculated. Nevertheless other models where another expression to minimise is selected, for example the perpendicular distance to the surface, might in some cases provide more suitable results. Accordingly the choice of the mathematical model will influence the quality of the adjustment.

The final product of a photogrammetric restitution is very often 2D vector or raster representation of the object. The presented approach provides the user with a reference surface for the choice of the appropriate projection.

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