THE BASIC TOPOLOGY MODEL OF SPHERICAL SURFACE DIGITAL SPACE

HOU Miao-le[1]

[1] Department of Surveying & Land Science, China University of Mining and Technology (Beijing), D11 Xueyuan Road, Beijing, China, 100083
e-mail: hml@cumtb.edu.cn

Keywords: SGDM, manifold, spherical surface digital space, spherical surface digital topology, spherical surface digital image, spherical surface spatial relationship

Abstract:

SGDM (Sphere Grid Data Model) is an efficient method to deal with the global data because of the advantages of multi-resolution and hierarchy. However, SGDM has no distinct descriptions and lack of round mathematical basis for various applications. What's more, most of mathematical model about global application have been based on the continuous methods. Although several researchers have considered the digital topology in 2-dimension and 3-dimension Euclidean Plane, complete theoretical foundation of proper theory for spherical surface digital space is still missing. In fact, it's more convenient and efficient to compute spatial relationship based on spherical surface digital space.

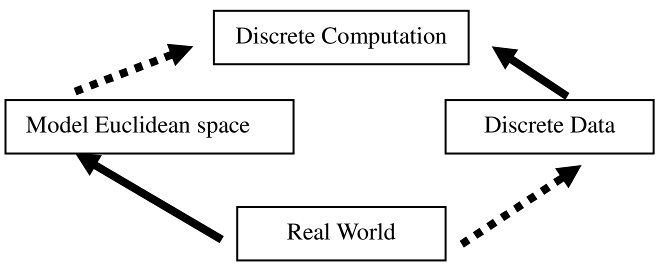
Firstly, this paper constructs spherical surface digital space based on manifold, *i.e.* the digitization of the spherical surface as a common spatial framework just as planar. Secondly, the concept of spherical surface digital topology will be given in the context of adjacency. We define a spherical surface digital image P as a triplet (T^2, R, H), where H is a finite subset of T^2 and R represents the adjacency relation in the whole lattice in a specific way. Then topological properties and paradox of discrete objects will be discussed in spherical surface digital space. In the end, we give some potential applications about spherical surface digital topology.

1. INTRODUCTION

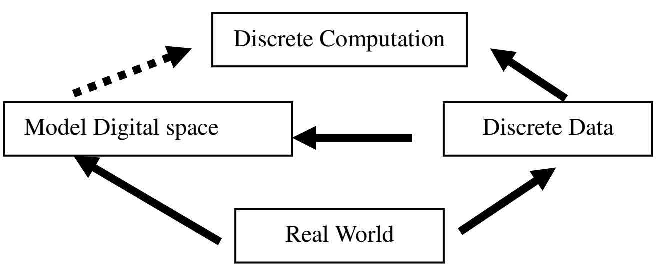
The surface of the earth is an important spatial domain, which is of course topologically equivalent to the surface of a spherical surface, ellipse and geoid. In fact, it's not topologically equivalent to any subset of the Cartesian plane (Sahr 1996 White 1992, 1998). So it's unpractical to analyze the spherical surface with the planar methods. Three typical conceptual modeling approaches are used to apply and adapt spatial analysis techniques to the spherical surface as follows (Raskin 1994).

- Map projection: The spherical surface is projected onto a plane using a conventional map projection. The projection approach is used implicitly in conventional studies that ignore the curvature of the earth, however, no one projection can keep both distance and area. What's more, the map projection transforms the spherical surface manifold to planar Euclidean space, therefore, the distance, orientation and area in large field are not accurate at all.
- Embeddings: The spherical surface is considered a constrained subset of the three-dimensional space R^3. Tree-dimensional spatial analysis is performed, with a constraint imposed to limit solutions to the spherical surface. The most typical one is direction cosine, which avoids the singularity of the pole. Although direction cosine has a perfect mathematics base, so it belongs to the vector method and does not accord with the discrete properties of real word in essential.
- Intrinsic: The spherical surface is considered an intrinsic space in its own right, with analysis performed in non-Euclidean space S^2. Longitude and latitude coordinate and SGDM are most typical two. SGDM (Sphere Grid Data Model) is research topic of this paper just because of the advantages of multi-resolution and hierarchy.

Computers are digital, and most image acquisition and communication efforts at present are toward digital approaches. However, vector model including direction cosine and longitude and latitude coordination are continuous in essential. There must be some deficiency between continuous model and discrete computation just as figure 1. SGDM is a promising method to deal with the spherical surface. Therefore, it's conveniently to analyse the global data and make decision in spherical surface digital space.



(a) The relation between vector model and real world



(b) The relation between raster model and real world

Figure 1. Relation between real world and discrete computation

This paper is concerned with spherical surface digital topology. Digital topology provides a sound mathematical basis for various image-processing applications including surface detection, border tracking, and thinning in 2D Euclidean space. We often use voxel representation to describe objects on a computer. Specifically, spherical surface digital space is partitioned into unit triangles. In this representation, an object in spherical surface is described by an array of bits. In this way, a spherical surface digital object can be defined as an array augmented by a neighborhood structure. The emphasis of this paper is on the differences between planar and spherical surface digital topology. It is specific to the basic topology model on the surface of an earth, and thus, the ellipsoidal nature of the earth and its vertical dimension are not considered.

The paper is organized as follows. Next section presents the definitions of spherical surface digital space based on manifold. In Section 3, the basic topology model of spherical surface digital space is discussed. In the end, the discussions and the future works are given.

2. THE DEFINITION OF SPHERICAL SURFACE DIGITAL SPACE BASED ON MANIFOLD

Regular grid sampling structures in the plane are a common spatial framework for many applications. Constructing grids with desirable properties such as equality of area and shape is more difficult on a sphere (White et al. 1998). To deal with the problems on the Earth conveniently, it is necessary to construct a similar regular mesh structure as a common spatial framework for spherical surface just as planar. Such similar regular mesh system is named as spherical surface digital space, which is the digitization of the spherical surface. That is, spherical surface can be described with discrete point sample in spherical surface digital space. Therefore, it is necessary to subdivide the spherical surface according to its characteristics. There are three steps to get the sphere digital space just as follows.

2.1 Initial partition of the spherical surface

The Platonic solids are reasonable starting points for a spherical subdivision (shown in Figure 2). Three of the five polyhedrons have triangular faces, such as the tetrahedron (four faces), the octahedron (eight faces), and the icosahedron (20 faces). The other Platonic solids are the cube (six faces) and the pentagonal dodecahedron (12 faces). The icosahedron has the greatest number of initial faces, and would therefore show the least distortion in the subdivision. However, the larger number of faces makes it somewhat harder to deal with the problems through the borders of the initial faces. In a word, the sphere is more easily covered by triangles and the triangles of the initial partition need not be equilateral. Distortion could be decreased considerably by dividing each equilateral triangular side of an initial Platonic figure into equivalent scalene triangles (White et al. 1998).

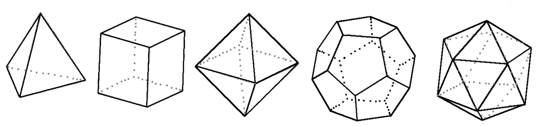


Figure 2. Platonic solids and it's spherical subdivision (White et al. 1992)

The octahedron has more distortion, but it has the advantage that its faces and vertices map to the important global features: meridians, the equator, and the poles (Goodchild and Shiren 1992). Therefore, in this paper octahedron is selected as common initial partition in which eight base triangles are produced.

2.2 Subdivision of triangular cells

There are several ways to hierarchically subdivide an equilateral triangle such as quaternary subdivision and binary subdivision (shown in Figure 3). All of these are subject to distortion when transferred to the spherical surface. Different decisions will have different effects on the uniformity of shape and size of cells within a given level of the hierarchy, as well as on the ease of calculation. Here, the quaternary subdivision is selected, in which a triangle is subdivided by joining the midpoints of each side with a new edge, to create four sub-triangles.

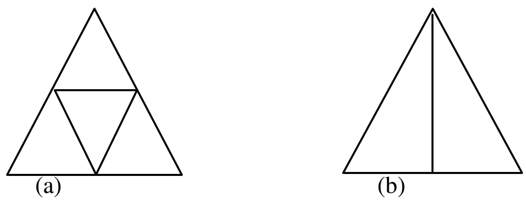


Figure 3. Quaternary subdivision (a) and binary subdivision (b)

The quaternary subdivision is a good compromise. It is relatively easy to work with, and non-distorting on the plane: a planar equilateral triangle is divided into four equilateral triangles. But a spherical base triangle may be divided into four equivalent triangles. The result of subdivision based on octahedron with quaternary subdivision is as follows in Figure 4 (Dutton 1996).

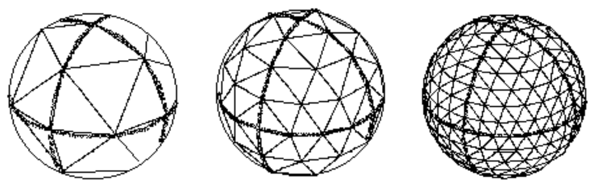


Figure 4. The result of subdivision based on octahedron (Dutton 1996)

2.3 The definition of sphere digital space based on manifold

Manifold is the extension of Euclidean just because every point in manifold has a homeomorphism of an open set in Euclidean. So local coordinates system can be set up for every point in manifold. It seems that manifold is a result plastered with many Euclidean spaces. It can be proved that sphere is a 2-dimension smooth manifold (Evidence omitted).

If the spherical surface is divided by quaternary subdivision based on octahedron, the spherical surface digital space is $8\times 4^N (N=\{0,1,...,n-1\})$ regular mesh based on finite discrete space, expressed as T^2. In the first level, spherical surface has the 8 base triangles, which are local coordinates systems of manifold. The relationship between 8 local coordinate systems can be described by spherical surface spacefilling curves (shown as Figure 5), which is a continuous mapping from a one-dimensional interval, to the points on the spherical surface. Continuous ordering based on spacefilling curves have been proven useful in heuristics related to a number of spatial, combinatorial, and logistical problems (Bartholdi and Goldsman 2001)

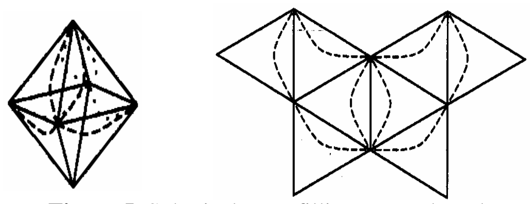
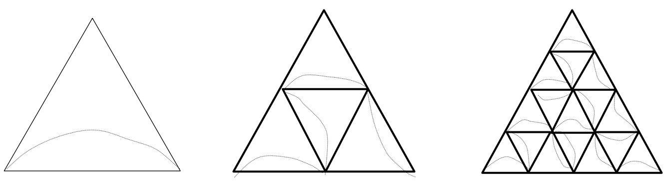


Figure 5. Spherical spacefilling curves based on octahedron with quaternary subdivision

To every base triangle, quaternary spherical surface spacefilling curve still can be used to express the relationship between every sub-triangle. In quaternary subdivision, the relationship between sub-triangles can be depicted with quaternary spherical surface spacefilling curve (shown as Figure 6). In given resolution, spherical surface digital space can be continuously indexed by quaternary spherical spacefilling curve (Details in Bartholdi 2001). Comparing with the other modal (Dutton 1991), spherical surface digital space has the advantage of continuous ordering. It makes us to index the sphere digital space continuously to allow quick and efficient search at multi-scale. At the same time, spherical surface digital space has the intrinsic disadvantage that the triangle is equivalent but not equal with each other.



(a) level 1 (b) level2 (c) level 3

Figure 6. The quaternary spherical surface spacefilling curve

Planar digital space is a simple Euclidean space, but spherical surface digital space is a more complex manifold. So spherical surface digital space is not the simple copy of planar digital space. It has some special properties just as follows. Spherical surface digital space is not a Euclidean space, that is to say, it is no homomorphous to planar and no single coordinates system can be set up to express every point in spherical surface. Although cells of spherical surface digital space are approximately equivalent, it still has a multi-scale and continuous ordering advantages (Bartholdi and Goodsman 2001).

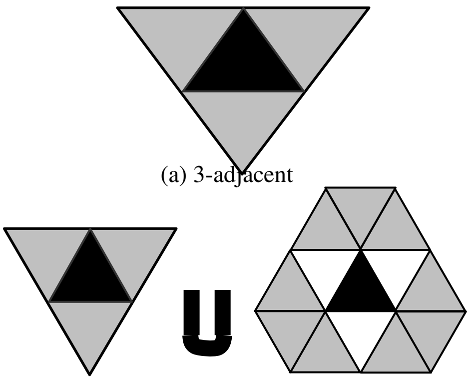
3. THE BASIC TOPOLOGY MODEL OF SPHERICAL SURFACE DIGITAL SPACE

From the definition of spherical surface digital space, T^2 is the result of partitioning the connected spherical surface into small triangular pieces that cover the whole spherical surface space. Each triangle is viewed as an element, called “spel” (short for spatial element). All the spels in the spherical surface can form a new set, which can be named as grid set T^2. The set T^2 can then be regarded as the hardware of the spherical surface digital space. The transitive closure δ of the adjacency relation between the two spels in T^2 can be considered as software. This system can be expressed as $\langle T^2,\delta\rangle$, where δ is the binary relations. This binary relation determines the connectedness between the spels in T^2. $\langle T^2,\delta\rangle$ is also referred to as “spherical surface digital topology”. S^2 is a connected space, but the T^2 is not connected space. In T^2, this implicit assumption of connectedness in S^2 no longer works.

3.1 General definitions and notations

Points of T^2 associated with triangles that have value 1 are called black points, and those associated with triangles with value 0 are called white points. The set of black points normally corresponds to an object in the digital image. First, we consider objects as subsets of the spherical surface digital space T^2. Elements of T^2 are called “spels” (short for spatial element). The set of spels which do not belong to an object O is included in T^2 constitute the complement of the object and is denoted by $\overline{O}$. Any spel can be seen as a unit triangle centered at a point with integer coordinates. Now, we can define some binary symmetric antireflexive relations between spels. Two spels are considered as 3-adjacency if they share an edge and 12-adjacent if they share a vertex. For topological considerations, we must always use two different adjacency relations for an object and its complement (shown as Figure 7). We sum this up by the use of a couple (n,n') with $(n,n')=\{3;12\}$, the n - adjacency being used for the object and the n' - adjacency for its complement. By transitive closure of these adjacency relations, we can define another one: connectivity between spels. We define an n - path π with a length k from spel a to spel b in O included in T^2 as a sequence of voxels $(\mathrm{i})\ \mathrm{i}=0;\ldots,\mathrm{k}$, such that for $0\le \mathrm{i}\le \mathrm{k}$, the spel v_i is n - adjacent or equal to $\mathrm{v}_{\mathrm{i}+1}$, with $\mathrm{v}_0=\mathrm{a}$ and $\mathrm{v}_\mathrm{k}=\mathrm{b}$. Now we define connectivity: two voxels a and b are called n - connected in an object O if there exists an n - path π from a to b in O. This is an equivalence relation between spels of O, and the

n - connected components of an object O are equivalence classes of spels according to this relation. Using this equivalence relation on the complement of an object we can define a background component of O as an n' - connected component of O'.



(a) 3-adjacent

(b) 12-adjacent

Figure 7. The definition of 3-adjacent and 12-adjacent

In 2D spherical surface digital space, we consider spherical surface triangle mesh to express spherical surface digital image. In this paper, points refer to grid points in spherical surface digital space unless stated otherwise. Two nonempty sets of points S_1 and S_2 are said to be 3 - adjacent or 12 - adjacent if at least one point of S_1 is 3 - adjacent or 12 - adjacent to at least one point of S_2. The adjacency definition is important not only in the computation of raster distance between two spels but also in topological analysis (LI et al. 2000). Let S be a nonempty set of points. An 3 - path between two points p, q in S means a sequence of distinct points $p = p_0, p_1, ..., p_n = q$ of S such that p_i is 3-adjacent to p_{i+1}, $0 \le i < n$. Two points $p, q \in S$ are 3-connected in S if there exists an 3 - path from p to q in S. An 3 - component of S is a maximal subset of S where each pair of points is 3 - connected.

A 2D spherical surface digital image Ç can be treated as a subset of Z2 together with some fixed neighborhood structure. It is defined as a quadruple $(I, 3, 12, E)$. Here I is the image space, which is a set of all grid points. E is defined as the set of black points that is spatial entity in I and $I - E$ is the set of white points. 3-adjacency or 12-adjacency are the adjacencies used for finding 3 - components and 12 - components in E and $I - E$, respectively. Note that $I - E$ denotes the set of white points in E. In this paper, we use 12 - adjacency for black points and 3 - adjacency for white points and call 12 - components of E black components and 3 - components of $I - E$ white component. The basic topological components of a spatial entity in spherical surface digital space are still interior, boundary and exterior. A point $p \in E$ is called an interior point of E if $N(p) \subset E$, otherwise p is called a border point of E. The set of all interior points of E is called the interior of E and is denoted as E°. The set of all border points of E is called the border of E and is denoted as ∂E. The closure of E is denoted as $\bar{E}$. The relationship between interior, closure and boundary is as follows:

$$E^o \cap \partial E = \Phi$$

$$E^o \cup \partial E = \bar{E}$$

$$\partial E = \bar{E} \cap (\bar{E})^-$$

3.2 Spherical surface digital Jordan Therom

3.3 Topological paradox associated with definition of adjacency in T^2

The classical Jordan curve theorem says that the complement of a Jordan curve in the Euclidean in the Euclidean plan R^2 consists of exactly two connectivity components. This theorem is the basic topological property in vector space and it would be preferable to keep it in the Z^2 raster space. So a topological paradox in Z^2 has arisen (figure 8). Kong and Rosenfeld has solved this problem in Z^2 if solved if the white spels are defined as being 4-connected and black spels 8 - connected, or vice versa.

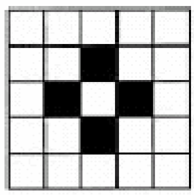


Figure 8. Topological paradox in raster space (from LI et al. 2000)

However, no one has discussed the topological paradox in T^2. In Figure 9, there are six black spels, one gray spel and some white spels. The gray spel is surrounded by the six black spels. If 12-adjacency is defined, the black spels are connected and should form a closed line; however, this black line cannot separate the central gray spel from the white spels. If 3-adjacency is defined, the black spels do separate the central gray spel from the white spels; however, these black spels are totally disconnected and thus no closed line has been formed by the black spels in this case. So this leads to the topological paradox in raster space T^2. To deal with this paradox, the white spels are defined as being 3 - connected and black spels 12 - connected, vice versa. In spherical surface digital space, background and object have the different connectedness. That is to say, the spatial entity in spherical surface is defined as being 12-connected, but the background is defined as being 3-connected. So, the six black spels defined as 12-connected should be connected. However, gray spel and black spels just as background should be not connected if the background is defined as being 3-connected. So the continuous curve (connected path in T^2) separate the spherical surface two parts.

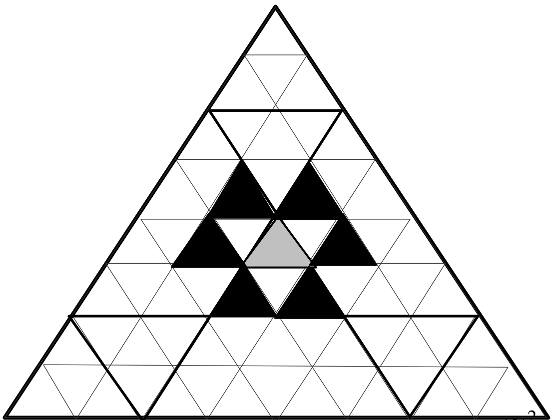


Figure 9. Topological paradox in raster space T^2

But why this topological paradox happens? We use the six spels in figure 10 to explain it. In reality, when one considers spels 1 ,3 and 5 to be connected, one has already implicitly assumed that P belongs to the black line. On the other hand, when one considers spels 2, 4 and 6 to be connected, one has already implicitly assumed that P belongs to the white spels. That is, the point P belongs to two different things. If the black spels represent spatial entities and the white spels represent the background, then point P belongs to both the background and the entity at the same time, thus having dual meanings. This of course leads to paradox–a kind of ambiguity. To solve the problem, one must eliminate the dual meanings of point P. One should only allow P to belong to either the entity or the background but not both. In this paper, the spels belonging to background are defined as 3-connected, however, the spels belonging to the object are defined as 12-connected.

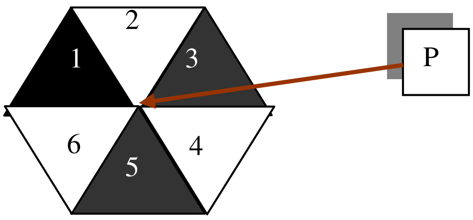
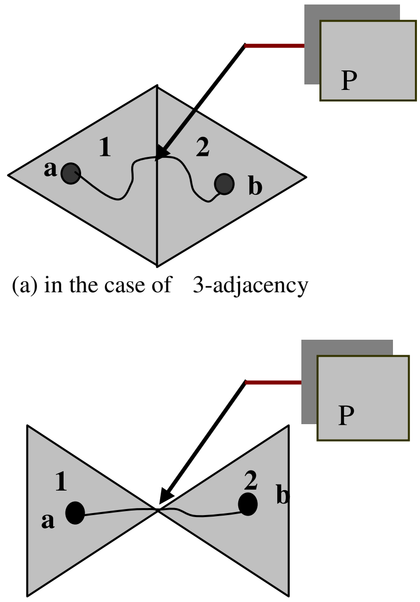


Figure 10. Topological paradox caused by the ambiguity at point P

3.4 Relationship between topologies on spherical surface digital space and spherical surface continuous space

The connectedness of raster space is based on the adjacency of two neighboring spels (LI et al. 2000). In spherical surface digital space, there is a common line (Figure 11a) between the two spels in the case of 3-connectedness. On the other hand, in the case of 12-connectedness, the common part could be either a line, a point, or both. In other words, there is at least a point in common if the two spels are to be connected. If an arbitrary (vector) point is selected from each spel, say ``a'' and ``b'', then the path from ``a'' to ``b'' intersects the common line at P. Points ``a'', ``b'' and P are points in vector space. Points ``a'' and P are connected in the left spel and points P and ``b'' are also connected in the right spel in vector space. As the connectedness is transitive, points ``a'' and ``b'' are therefore connected. As a result, any point in the left spel is connected to any point in the right spel. It means that the connectedness concept in vector space has been implicitly adopted when the connectedness concept in raster space is discussed.



(a) in the case of 3-adjacency

(b) in the case of 12-adjacency

Figure 11. Implicit dependency of topological connectedness in T^2

CONCLUSION

SGDM (Sphere Grid Data Model) is an efficient method to deal with the global data because of the advantages of multi-resolution and hierarchy. However, SGDM has no distinct descriptions and lack of round mathematical basis for various applications. This paper gave the definition of spherical surface digital space, which has the characters as follows:

- Similar regular grids based on spherical surface discrete space.
- Spherical spacefilling curves can be used to express the relationship between local coordination.
- No single coordination system can express every point in the spherical surface.
- Multi-scale and continuous ordering.

As an important part, this paper set up the basic topology model which include the topological structure of sphere digital space, the basic topological components of a spatial entity in T^2, topological paradox associated with definition of adjacency in T^2 and so on. This paper is just an introduction to studying the characterization of 2-digital sphere manifold and the Jordan–Brower separation theorem, which are all round mathematic basis of spherical spatial computing and reasoning.

Acknowledgements

The work presented in this paper was substantially supported by an Outstanding Youth Award from the Natural Science Foundation of China (under grant No.40025101). I also thank my two supervisors, Prof. Chen Jun and Dr. Zhao Xuesheng, whose comments allow me to improve this article.

REFERENCES:

[1] Bartholdi. III and Goldsman P., 2001, Continuous indexing of hierarchical subdivisions of the globe, *Int. J. Geographical Information Science*, 15(6): 489-522.
[2] Bieri H. and Nef W., 1984, Algorithms for the Euler characteristic and related additive functional of digital objects. *Computer Vision Graphic Image Process*. 28: 166–175.
[3] Dyer C. R., 1980, Computing the Euler number of an image from its quadtree. *Computer Graphics Image Process*. 13, 270–276.
[4] Dutton G., 1991,Polyhedral hierarchical tessellations: The shape of GIS to come.. *Geographical Information Systems*, 1(3): 49-55.
[5] Chen J., Zhao X., and Li Z., 2003,Algorithm for the generation of Voronoi diagrams on the sphere based on QTM . *Photogrammetric Engineering or Remote Sensing*, 69(1): 79-90.
[6] Cohn A., Bennett B., Goodday J. and Gotts N., 1997, Qualitative Spatial Representation and Reasoning with the Region Connection Calculus. *Geoinformatic*a, 1(3): 1-44.
[7] Egenhofer M., Sharma J., and David M., 1993, A critical comparison of the 4-intersection and 9-intersection models for spatial relations: formal analysis. *Auto-Carto*, 11:1-11.
[8] Frank A. U., 1992,Spatial Concepts, Geometric Data Model and Geometric Data Structures. *Computers & Geosciences*, 18: 409-417.
[9] Goodchild M.F. and Yang Shiren, 1992, A Hierarchical Data Structure for Global Geographic Information Systems, *Computer Vision and Geographic Image Processing,* 54(1): 31-44.
[10] Kong T.Y., 1986, Digital topology with applications to image processing. Doctoral dissertation, *University of Oxford*.
[11] LI Z.L., LI Y.L, CHEN Y.Q., 2000, Basic Topological Models for Spatial Entities in 3-Dimensional Space. *GeoInformatica*, 4(4): 419-433.
[12] Malandain G., Bertrand G., and Ayache N., 1993, Topological segmentation of discrete surfaces. *Int. J. Comp*, 10(2): 183–197.
[13] Rosenfield A., 1975, A characterization of parallel thinning algorithms, *Inform. Control.*, 29: 286–291.
[14] Rosenfield A., 1979, Digital Topology. *Am. Math. Month*, 86: 621-630.
[15] Raskin, G. R., and Fellow, V., 1994, Spatial Analysis on the Sphere: A Review. *NCGIA report*, 94-7, 23pp.
[16] Saha P. K., 1996,Chaudhuri B. B., 3D Digital Topology under Binary Transformation with Applications. *Computer vision and image understanding*, 63(3): 418-429.
[17] White D., Kimmerling J. and Overton W.S., 1992, Cartographic and Geometric Components of a Global Sampling Design For Environment Monitoring, *Cartography & Geographical Information Systems*, 19(1): 5-22.
[18]White, D., Kimberling, A. J. and Song L. 1998. Comparing area and shape distortion on polyhedral-based recursive partitions of the sphere. *International Journal of Geographical Information Systems,* 12(8): 805-827.
[19] Winter S., 1999, Topological Relations in Hierarchical Partitions. *Spatial Information Theory, COSIT'99, Springer*, 141-156.
[20] Zhao X.S., Chen J., and Li Z.L. 2002. A QTM-based algorithm for generation of the voronoi diagram on a sphere. *Advances in Spatial Data Handling*, Published by Springer, Berlin, 269-285.
[21] Zhao X.S., 2002, Spherical Voronoi data model based on QTM. *Ph.D. thesis*, China University of Mining technology (Beijing), Beijing, 105pp.