PERFORMING SPACE RESECTION USING TOTAL LEAST SQUARES

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ABSTRACT:

In space resection computing，the error equations are based on the collinearity equations. Because the coordinates of the ground point and the image point all exist errors. So we use the Total Least Squares (TLS) method. The method of Total Least Squares is one of several linear parameter estimation techniques that have been devised to compensate for data errors. Furthermore, TLS is the method of fitting that is appropriate when there are errors in both the observation vector *b* and the variable matrix *A*. So it can establish a more practical and suitable model so-called Error-in -Variables model. On the base of this model, the errors in the observation vector *b* (*e*) and the variable matrix *A* (E_A) can be corrected at the same time.

1. INTRODUCTION

The space resection method for photogrammetric applications is based on the collinearity equations. It is the determination of the exterior orientations. Two collinearity equations are written for each image point. With three control points, the resulting six equations allow for the unique solution of the six unknown parameters. However, if additional points are available, the Least Squares (LS) adjustment method can be performed to estimate better results and to allow for point measurement editing. However, in the classical Least Squares model, also known as Gauss-Markov model, the measurement *A* of the variables are assumed to be error-free or fixed, so it does not need correction, and the vector of normally distributed errors *e* are confined to the observation vector *b*.

However, in many cases such as sampling errors, human errors, modeling errors and instrument errors may imply inaccuracies of the data matrix *A* as well. So the assumption is frequently unrealistic where errors exist in both the observation vector *b*, and the matrix of variables *A*. That is why the Total Least Squares (TLS) method is introduced here. The method of Total Least Squares is one of several linear parameter estimation techniques that have been devised to compensate for data errors. With the Error-in -Variables model, errors in both the observation vector b and the variable matrix A can be corrected [2].

The examples in this paper prove that with TLS method the more accurate and reliable parameter values can be obtained. Furthermore, when the number of control points is reduced, the TLS solutions are more robust.

2. BASIC PRINCIPLE OF SPACE RESECTION AND TLS

2.1 Mathematical Model of Space Resection

Space Resection is the determination of an image's position and orientation parameters with respect to an object space coordinate system in which a certain amount of ground control points are reasonably distributed. Because both the object coordinates (X, Y, Z) and the image coordinates (x, y) of the control points are known, with the collinearity equations, the 6 exterior orientation elements (X_S, Y_S, Z_S, φ, ω, κ) can be calculated. If the interior orientations (x_0, y_0, f) are also known, the collinearity equations are employed as follow:

$$x - x_0 = -f\frac{a_1(X-X_s)+b_1(Y-Y_s)+c_1(Z-Z_s)}{a_3(X-X_s)+b_3(Y-Y_s)+c_3(Z-Z_s)} \qquad (1)$$
$$y - y_0 = -f\frac{a_2(X-X_s)+b_2(Y-Y_s)+c_2(Z-Z_s)}{a_3(X-X_s)+b_3(Y-Y_s)+c_3(Z-Z_s)}$$

Where a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 are the elements of the rotation matrix which is formed by the rotation angles φ, ω, κ. With the Taylor series approximations, the collinearity equations can be linearized as:

$$x-(x)=\frac{\partial x}{\partial X_s}dX_s+\frac{\partial x}{\partial Y_s}dY_s+\frac{\partial x}{\partial Z_s}dZ_s+\frac{\partial x}{\partial \varphi}d\varphi+\frac{\partial x}{\partial \omega}d\omega+\frac{\partial x}{\partial \kappa}d\kappa$$
$$y-(y)=\frac{\partial y}{\partial X_s}dX_s+\frac{\partial y}{\partial Y_s}dY_s+\frac{\partial y}{\partial Z_s}dZ_s+\frac{\partial y}{\partial \varphi}d\varphi+\frac{\partial y}{\partial \omega}d\omega+\frac{\partial y}{\partial \kappa}d\kappa \qquad (2)$$

This method is described in many textbooks, among them Edward M. Mikhail, James S. Bethel and J. Chris McGlone (2001, p. 79)[3] and Li De-ren, Zhou Yue-qin and Jin Wei-xian (2001, p. 34) [4]. Assuming the observation vector *b*, the coefficient matrix *A* and the parameter vector ξ with:

$$b = \begin{bmatrix} x - (x) \\ y - (y) \end{bmatrix}$$
$$A = \begin{bmatrix} \frac{\partial x}{\partial X_s} & \frac{\partial x}{\partial Y_s} & \frac{\partial x}{\partial Z_s} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \omega} & \frac{\partial x}{\partial \kappa} \\ \frac{\partial y}{\partial X_s} & \frac{\partial y}{\partial Y_s} & \frac{\partial y}{\partial Z_s} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \omega} & \frac{\partial y}{\partial \kappa} \end{bmatrix} \quad (3)$$
$$\xi = \begin{bmatrix} dX_s & dY_s & dZ_s & d\varphi & d\omega & d\kappa \end{bmatrix}^T$$

If the number of control points is n, then *b* denotes the 2n×1 vector, A the 2n×6 coefficient data matrix, ξ the 6×1 fixed vector of parameters. When the number of the error equations is more than 6, a standard Least Squares adjustment method can be employed to estimate the parameter vector ξ within the Gauss-Markov model. But there is a basic assumption that only observations are affected by random errors *e*, however, the coefficient matrix *A* is considered as fixed or error-free. So the observation equations can be expressed as:

$$\begin{aligned} b - e &= A \cdot \xi \\ rank(A) &= m < n \end{aligned} \quad (4)$$

However, the assumption that all the random errors are confined to the observation vector *b* is frequently not true. Various types of error exist almost in any measured quantity. Errors due to modelling errors, human errors, and faulty measuring instruments and so on all contribute to the fact that the coefficient matrix *A* includes unknown errors. In this case, the Total Least-Squares approach is the proper method with which a more suitable model can be established to treat problems where all the data are affected by random errors [2].

2.2 Mathematical Model and Solution of Total Least Squares

Different from the classical Gauss-Markov model where only the observation vector *b* is subjected to random errors, the Total Least Squares problem provides that both the observation vector *b* and the data matrix *A* are subjected to random errors and thus need to be adjusted. With this assumption, the Error-in-Variables (EIV) model can be established as follow:

$$\begin{aligned} &(A - E_A)\xi = b - e, \quad n > m = rank(A) \\ &\begin{bmatrix} e \\ vecE_A \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \sigma_0^2 \begin{bmatrix} I_n & 0 \\ 0 & I_m \otimes I_n \end{bmatrix} \right) \end{aligned} \quad (5)$$

Where E_A are error matrix existed in the coefficient matrix *A*. In total least squares method, the $E_A \equiv 0$. $\Sigma_0 = \sigma_0^2 \cdot I_{m+1}$ is an $(m+1)\times(m+1)$ matrix with an unknown variance component σ_0^2 , I_n is a $n \times n$ identity matrix, and $\otimes$ denotes the "Kronecker - Zehfuss product" of matrices, expressed as $M \otimes N := \left[m_{ij} \cdot N \right], M = \left[m_{ij} \right]$, the "vec" operator stacks one column of a matrix under the other, moving from left to right. So the aim of equally Weighted Total Least Squares (WTLS) principle is to minimize the objective function:

$$e^T e + (vecE_A)^T (vecE_A) = \min \quad (6)$$

The singular value decomposition (SVD) is always used to solve the TLS problem [6]. So with the SVD method, the augmented matrix [A, b] can be decomposed as:

$$\left[A, b \right] = U \Sigma V^T \quad (7)$$

Where $U = \left[u_1 \quad u_2 \quad \ldots \quad u_n \right] \in R^{n\times n}$ is a matrix with n columns that are eigenvectors of $[A, b][A, b]^T$ and $V = \left[v_1 \quad v_2 \quad \ldots \quad v_{m+1} \right] \in R^{(m+1)\times(m+1)}$ is a matrix with (m+1) columns that are eigenvectors of $[A, b]^T[A, b]$. $\Sigma = diag\left[\sigma_1 \quad \sigma_2 \quad \ldots \quad \sigma_{m+1} \right]$ is an n×(m+1) matrix with the diagonal elements equal to the singular values and off diagonal elements equal to zero.

To find the parameter vector $\hat{\xi}$ to minimize the objective function, Equation (5) can be rewritten as follow:

$$\left[\hat{A}, \hat{b} \right] \cdot \begin{bmatrix} \hat{\xi} \\ -1 \end{bmatrix} = 0 \quad (8)$$

Where $\left[\hat{A}, \hat{b} \right]$ and $\hat{\xi}$ are the adjusted values. By definition, vector $\left[\hat{\xi}^T; -1 \right]^T$ is in the nullspace of $\left[\hat{A}, \hat{b} \right]$. So with the SVD and equations (8), the last column of orthogonal matrix V $\left[v_{1,m+1} \quad v_{2,m+1} \quad \cdots \quad v_{m+1,m+1} \right]^T$ spans the nullspace of $\left[\hat{A}, \hat{b} \right]$. Then the solution of TLS is in the following form (proved by Sabine Van Huffel, 1991) [9]:

$$\hat{\xi} = \frac{-1}{v_{m+1,m+1}} \cdot \left[v_{1,m+1}, \mathrm{L} \ , v_{m,m+1} \right] \quad (9)$$

The corresponding TLS residual matrix $\left[\hat{E}_A, \hat{e} \right]$ is as follow:

$$\left[\hat{E}_A, \hat{e} \right] = \left[A, b \right] - \left[\hat{A}, \hat{b} \right] = \sigma_{m+1} \cdot u_{m+1} \cdot v_{m+1}^T \quad (10)$$

Where σ_{m+1} is the singular value, u_{m+1} is the left singular vector and v_{m+1} is the right singular vector.

In addition, an approximate variance–covariance matrix for the TLS is given by [5]:

$$\hat{v} = \hat{e}^T\hat{e} + vec(\hat{E}_A)^T vec(\hat{E}_A)$$
$$\sigma_0^2(TLS) = \frac{\hat{v}}{n-m} \qquad (11)$$

$$D(\hat{\xi}) \approx \hat{\sigma}_0^2 (N - \hat{v}\mathrm{I_m})^{-1} N (N - \hat{v}\mathrm{I_m})^{-1}$$
$$= \hat{\sigma}_0^2[(N - \hat{v}\mathrm{I_m})^{-1} + \hat{v}(N - \hat{v}\mathrm{I_m})^{-2}] \qquad (12)$$
$$= (n-m)^{-1}[\hat{v}(N - \hat{v}\mathrm{I_m})^{-1} + \hat{v}^2(N - \hat{v}\mathrm{I_m})^{-2}]$$

Where $N = A^T A$。

3. CALCULATION PROCEDURE AND EXAMPLES

3.1 Calculation procedure

Performing space resection using total least squares can be realized by the following steps:

(1) Given approximation, where k is the scale denominator calculated by random two points:

$$k = \sqrt{(X(i)-X(j))^2 + (Y(i)-Y(j))^2} / \sqrt{(x(i)-x(j))^2 + (y(i)-y(j))^2}$$
$$X_S{}^0 = \sum X / n \qquad (13)$$
$$Y_S{}^0 = \sum Y / n$$
$$Z_S{}^0 = k \times f + \sum Z / n$$
$$\varphi^0 = \omega^0 = \kappa^0 = 0$$

(2) According to equation (2), with n points, 2n error equations can be established;

(3) With equation (7), (8) and (9), the TLS solution can be calculated, and then the 6 exterior orientations can be obtained;

(4) Evaluate the precision of the adjusted result with equation (10), (11), (12).

3.2 Examples

In this section, the TLS approach will be studied in some space resection examples. At the same time, a comparison between the result of TLS and LS will be presented.

Example 1, the numbers are taken from Li De-ren et al. (1992, p 114) [8]. The interior orientations are known $x_0=y_0=0$, f=153.24mm. The image coordinates and object coordinates of the 4 ground control points are as follows:

Tab. 1 Image Coordinate and Object Space Coordinate of Control Points

No.	Image Coordinate		Object Coordinate		
	x(mm)	y(mm)	X(m)	Y(m)	Z(m)
1	-86.15	-68.99	36589.41	25273.32	2195.17
2	-53.4	82.21	37631.08	31324.51	728.69
3	-14.78	-76.63	39100.97	24934.98	2386.5
4	10.46	64.43	40426.54	30319.81	757.31

The exterior orientations are estimated through TLS and LS approaches as follow:

Tab. 2 exterior orientation elements of TLS and LS

	TLS	LS
X_S(m)	39797.537	39795.452
Y_S(m)	27479.976	27476.462
Z_S(m)	7560.294	7572.686
φ(rad)	-0.004146	-0.003987
ω(rad)	0.001445	0.002114
κ(rad)	-0.066663	-0.067578

Using equation (10) the residuals for the TLS solution were calculated as:

$$\left[\hat{E}_A, \hat{e}\right] = \begin{bmatrix} 0.0603 & -0.0214 & -0.0046 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.4174 & -0.1482 & -0.0315 & 0.0000 & 0.0000 & 0.0000 & -0.0003 \\ 0.1108 & -0.0393 & -0.0084 & 0.0000 & 0.0000 & 0.0000 & -0.0001 \\ -0.6372 & 0.2263 & 0.0482 & -0.0001 & -0.0001 & -0.0001 & 0.0005 \\ 0.1447 & -0.0514 & -0.0109 & 0.1447 & 0.1447 & 0.1447 & -0.0001 \\ 0.2188 & -0.0777 & -0.0165 & 0.0000 & 0.0000 & 0.0000 & -0.0002 \\ -0.3146 & 0.1117 & 0.0238 & 0.0000 & 0.0000 & 0.0000 & 0.0002 \\ 0.0111 & -0.0040 & -0.0008 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \times 10^{-5} \qquad (14)$$

The estimated variance components are:

Tab. 3 Precision of TLS and LS

	TLS	LS
σ_0(mm)	0.0000065754	0.0072594240
σ_{XS}(mm)	0.0010380868	1.1073850459
σ_{YS}(mm)	0.0011483554	1.2495151993
σ_{ZS}(mm)	0.0004438512	0.4881299565
σ_φ(mrad)	0.0000001651	0.0001786252
σ_ω(mrad)	0.0000001452	0.0001614610
σ_κ(mrad)	0.0000000653	0.0000720382

These results show that with the TLS method, the values of variance components are smaller than the LS results, so the precision of the TLS solution is higher. This indicates that the EIV model is slightly more suitable as it minimizes the overall required changes.

Example 2, the aim of this example is to analyze the effect of control points' reduction on the stability of the space resection solution. The interior orientations are: $x_0=y_0=0$， f=126mm. And the coordinates of the control points are stated in Tab.4.

The first step is to randomly select 7 points as control points, assuming the point No. are 1, 4, 7, 10, 13, 16 and 18. The next step is to reduce the number of control points, such as 5 control points, the point No. are 1, 4, 7, 10 and 13. Finally just leaving 4 points as control points and the point No. are 10, 11, 13 and 14. In these three cases, the results of exterior orientation elements and the precision are stated in Tab.5 and Tab.6.

Beyond the control points, the residual points perform as check points. Compared with LS, with the TLS approach, when the number of control points is 7, 5 and 4, the precision of check points are improved as 0.4508m、0.3270m、0.0042m.

This result indicates that when the number of control points is reduced, with the TLS approach, the result is more stable and with higher accuracy.

Tab. 4 Image Coordinate and Object Space Coordinate of Control Points

No.	Image Coordinate		Object Coordinate		
	x(mm)	y(mm)	X(m)	Y(m)	Z(m)
1	-210.07	-0.07	-3200.266	4306.636	209.298
2	-210.05	-0.05	-3176.741	4306.678	223.278
3	-210.05	-210.05	-3137.499	-712.574	237.258
4	-83.98	210.02	-137.449	9294.694	251.328
5	-84.03	-0.03	-113.611	4315.113	265.278
6	-84.07	-210.07	-89.780	-613.852	279.368
7	0.00	210.00	1854.807	9216.931	293.348
8	-0.07	-0.07	1868.406	4320.678	307.388
9	0.08	-209.92	1881.826	-525.335	321.388
10	83.93	209.93	3780.604	9139.405	335.348
11	84.00	0.00	3783.914	4325.910	349.348
12	84.04	-209.96	3787.153	-437.440	363.408
13	210.08	210.08	6580.542	9058.539	377.408
14	209.93	-0.07	6568.578	4333.508	391.278
15	209.97	-210.03	6556.585	-341.612	405.238
16	-84.09	114.91	-17.573	6887.991	419.238
17	-83.92	-114.92	6.866	1756.648	433.348
18	84.00	115.00	3712.571	6859.282	447.388
19	83.91	-115.09	3716.357	1805.806	461.238

Tab. 5 exterior orientation elements of TLS and LS of TLS and LS

TLS	Number of Points	7	5	4
	X_S(m)	1881.1980	1881.3105	1880.3176
	Y_S(m)	4320.9724	4321.1066	4320.1829
	Z_S(m)	3229.0223	3228.7824	3228.5189
	φ(rad)	-0.0039895	-0.0041366	-0.0040834
	ω(rad)	0.0003138	0.0003345	0.0004450
	κ(rad)	0.0026749	0.0027759	0.0027000
LS	X_S(m)	1880.8954	1880.1144	1880.7500
	Y_S(m)	4322.8582	4319.9895	4322.8226
	Z_S(m)	3233.4910	3228.7190	3233.4377
	φ(rad)	-0.0045172	-0.0039876	-0.0045325
	ω(rad)	-0.0002376	0.0004366	-0.0002279
	κ(rad)	0.0025081	0.0026277	0.0025276

Tab. 6 Precision of TLS and LS

TLS	Number	7	5	4
	σ_0(mm)	0.0147339	0.0166678	0.0157712
	σ_{XS}(mm)	0.3877	0.6170	0.5393
	σ_{YS}(mm)	0. 3063	0. 5285	0. 6296
	σ_{ZS}(mm)	0. 2373	0. 3311	0. 5361
	σ_φ(mrad)	0. 0000401	0.0000500	0.0000863
	σ_ω(mrad)	0. 0000609	0.0000915	0.0000854
	σ_κ(mrad)	0. 0000496	0.0000718	0.0001004
LS	σ_0(mm)	0.0535488	0.0674734	0.0645894
	σ_{XS}(mm)	1.3678	2.4632	2.2442
	σ_{YS}(mm)	1.0758	2.0994	2.6165
	σ_{ZS}(mm)	0.8332	1.3044	2.2349
	σ_φ(mrad)	0.0001459	0.0002030	0.0003563
	σ_ω(mrad)	0.0002204	0.0003684	0.0003518
	σ_κ(mrad)	0.0001805	0.0002913	0.0004140

4. CONCLUSION

The Total Least Squares adjustment has been employed to estimate the exterior orientation elements of the space resection problem. Through the examples, we can obtain following conclusions:

(1) Using TLS algorithm, we can indeed created a more symmetrical model. Compared with Gauss-Markov model, the EIV model minimizes the overall required changes, so it is slightly more suitable.

(2) Compared with LS, the residuals and variance components of the TLS solution are smaller. So with the TLS approach we can obtain the result with higher accuracy.

(3) In order to solve the TLS problem, the singular value decomposition (SVD) is used here. With this method, the matrix inversion is avoided. That is why when the number of control points is less the result is still stable.

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REFERENCE

[1] Wang Zhi-Zhuo, 1984. The Principle of Photogrammetry. Wu Han: Surveying Press, pp. 14-122.

[2] Yaron A. Felus, Burkhard Schaffrin, 2005. Performing Similarity Transformations Using the Error-In-Variables Model. ASPRS 2005 Annual Conference Baltimore, Maryland, March, pp.7-11.

[3] Edward M. Mikhail, James S. Bethel, J. , 2001. Chris McGlone. Introduction to Modern Photogrammetry. Wiley, Har/Cdr edition, pp.66-103.

[4] Li De-ren, Zhou Yue-qin, Jin Wei-xian, 2001. The Outline of Photogrammetry and Remote Sensing. Bei Jing: Surveying Press, pp. 15-68.

[5] Burkhard Schaffrin, 2006. A Note on Constrained Total Least-Squares Estimation. Linear Algebra and its Applications, 417 (2006): pp. 245-258.

[6] Golub, H.G., Van Loan, F.C. , 1980. An analysis of the total least squares problem. SIAM Journal on Numerical Analysis, 17(6): pp. 883-893.

[7] Alexander Kukush, Sabine Van Huffel, 2004. Consistency of Elementwise-Weighted Total Least Squares Estimator in a Multivariate Errors-Invariables Model AX =B. Springer-Verlag, Metrika (2004) 59: pp. 75-97.

[8] Li De-ren, Zheng Zhao-bao, 1992. Analytical Photogrammetry. Bei Jing: Surveying Press, pp. 114-128.

[9] Sabine Van Huffel, 1991. The Total Least Squares Problem. Computational Aspects and Analysis. Society for Industrial and Applied Mathematics: Philadephia, pp. 300.