Commission III

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ERROR PROPAGATION IN THE ANALYTICAL TRIANGULATION - DEGREE OF THE TRANSFORMATION POLYNOMIALS

Abstract: A brief quotation of image deformations leads to the conclusion about residual systematic errors influencing the relative orientation of the photographs. New formulas are derived for the error propagation from one model to the next one for two cases: 1. the preceding photograph is deformed, 2. the attached photograph is deformed. Formulas both for random and systematic errors propagation in strips and blocks are given. All investigations have aimed to the conclusion about the degree of the polynomials applied for the coordinate transformation.

1. Introduction

Due to deformations of the aerial photographs, caused by various physical factors, the relative orientation of a stereopair cannot be made perfectly. The created model is always more or less deformed. This fact causes problems in the aerial triangulation. A triangulation strip or block has been accomplished if all photographs are relatively oriented. The deformation of one model is transferred to the other models, and the result is that the whole strip or block of models is in some way deformed.

A number of authors have been investigating the error propagation in the strip and/or block triangulation. For exemple Ackermann [1] proved that the resulting accuracy of the coordinates of the new points depends on the shape of the block and on the number as well as the displacement of the ground control points. Ebner [2] investigated, even more in detail, the influence of the random errors on the accuracy, changing the number and the dispacement of the ground control points. Large experiments were carried out in I.T.C in the Netherlands [3], when various systematic deformations were introduced into the matematical model of photographs.

The present author has been investigating the propagation and the accumulation both of the random and systematic errors in the aerial triangulation. His main goal was to justify, or to deny, the application of the polynomials of higher degree for the coordinate transformation. First, he analyzed the deformations of the photographs and the bundles reconstructed of these photographs. He paied attention particulary to those factors causing residual systematic errors, which could not be eliminated even if their influence on the image coordinates was defined (expressed by the equation). Those systematic errors arise, e.g. due to the atmospheric refraction and the lens distortion errors dbz and $d\varphi$ appear in the relative orientation . The crab and the drift of the flight path together with the affine deformation of the film base causes the residual systematic errors dby and $d\varkappa$.

Thereafter, the author derived equations showing the influence of the errors in the relative orientation of the first model on the errors dx, dy, dz of the model coordinates of the points in the attached model. He derived also equations showing the influence of the errors in the relative orientation of the attached model on the errors dx, dy, dz in the same model. If both models, the first one (subscript i) and the attached one (subscript $i+1$), are disturbed by the errors in the relative orientation, the errors dx, dy, dz of the points in attached model are expressed by equations

$$\begin{aligned}
dx = {} & (x+b)\frac{z}{fb}\left(dbx_i - d\varphi_i\right) - y\,d\kappa_i + z\,d\varphi_i - dbx_i \\
& + x\frac{z}{fb}\left\{\frac{b}{f}dbz_{i+1} - f\left(1+\frac{b^2}{f^2}\right)d\varphi_{i+1}\right\} \\
& + \frac{z}{f}\left\{\frac{x}{f}dbz_{i+1} - y\,d\kappa_{i+1} + f\left(1+\frac{x^2}{f^2}\right)d\varphi_{i+1} - \frac{xy}{f}d\omega_{i+1}\right\}
\end{aligned}$$

$$\begin{aligned}
dy = {} & y\frac{z}{fb}\left(dbx_i - d\varphi_i\right) + (x+b)\,d\kappa_i - z\,d\omega_i - dby_i \\
& + y\frac{z}{fb}\left\{\frac{b}{f}dbz_{i+1} - f\left(1+\frac{b^2}{f^2}\right)d\varphi_{i+1}\right\} \qquad (1) \\
& + \frac{z}{f}\left\{-dby_{i+1} + \frac{y}{f}dbz_{i+1} - x\,d\kappa_{i+1} - \frac{xy}{f}d\varphi_{i+1} + f\left(1+\frac{y^2}{f^2}\right)d\omega_{i+1}\right\}
\end{aligned}$$

$$\begin{aligned}
dz = {} & -\frac{zz}{fb}\left(dbx_i - d\varphi_i\right) - (x+b)\,d\varphi_i + y\,d\omega_i - dbz_i \\
& + \frac{zz}{fb}\left\{\frac{b}{f}dbz_{i+1} - f\left(1+\frac{b^2}{f^2}\right)d\varphi_{i+1}\right\} \\
& - \frac{zz}{fb}\left\{\frac{x}{f}dbz_{i+1} - y\,d\kappa_{i+1} + f\left(1+\frac{x^2}{f^2}\right)d\varphi_{i+1} - \frac{xy}{f}d\omega_{i+1}\right\}
\end{aligned}$$

2. Accumulation of the random and systematic errors in strips

There is possible to make n number of models out of $n+1$ photographs. In all the models, from the first one to the n^{th}, the model coordinates x, y, z will show the errors dx, dy, dz caused by the deformation of the present model. Starting from the second model in the triangulation strip, futher errors dx, dy, dz will appear, caused by the transfer of the deformation of the preceding model.

The random components Δbx, Δby, Δbz, $\Delta\kappa$, $\Delta\varphi$, $\Delta\omega$ as well as the systematic components c_{bx}, c_{by}, c_{bz}, c_{κ}, c_{φ}, c_{ω} of the errors dbx, dby, dbz, $d\varkappa$, $d\varphi$, $d\omega$ of the orientation elements are transferred from one model to the next one in the same way. Therefore the same equation (1) is the starting one for deduction of systematic as well as random deformations of joint models in a triangulation strip. Of course, when more than two models are jointed together, there will be different rules for propagation of the random and systematic errors.

Equations (1) can be simplified if the model, made of a stereopair of photographs, is created in the same scale as the scale of the photographs is. Then can be put $z = f$ into equations (1).

If the errors of the orientation elements comprise only the random elements, then in accordance with (1) can be written

$$
\begin{aligned}
\Delta x = {} & \frac{x+b}{b}\left(dbx_i - d\varphi_i\right) - y\,d\varkappa_i + z\,d\varphi_i - dbx_i \\
& + \frac{x}{b}\left\{\frac{b}{f}\,dbz_{i+1} - z\left(1+\frac{b^2}{z^2}\right)d\varphi_{i+1}\right\} \\
& + \frac{x}{z}\,dbz_{i+1} - y\,d\varkappa_{i+1} + z\left(1+\frac{x^2}{z^2}\right)d\varphi_{i+1} - \frac{xy}{z}\,d\omega_{i+1}
\end{aligned}
$$

$$
\begin{aligned}
\Delta y = {} & -\frac{y}{b}\left(dbx_i - d\varphi_i\right) + (x+b)\,d\varkappa_i - z\,d\omega_i - dby_i \\
& + \frac{y}{b}\left\{\frac{b}{z}\,dbz_{i+1} - z\left(1+\frac{b^2}{z^2}\right)d\varphi_{i+1}\right\} \\
& - dby_{i+1} + \frac{y}{z}\,dbz_{i+1} - x\,d\varkappa_{i+1} - \frac{xy}{z}\,d\varphi_{i+1} + z\left(1+\frac{y^2}{z^2}\right)d\omega_{i+1}
\end{aligned}
\qquad (2)
$$

$$
\begin{aligned}
\Delta z = {} & -\frac{z}{b}\left(dbx_i - d\varphi_i\right) - (x+b)\,d\varphi_i + y\,d\omega_i - dbz_i \\
& + \frac{z}{b}\left\{\frac{b}{z}\,dbz_{i+1} - z\left(1+\frac{b^2}{z^2}\right)d\varphi_{i+1}\right\} \\
& - \frac{z}{b}\left\{\frac{x}{z}\,dbz_{i+1} - y\,d\varkappa_{i+1} + z\left(1+\frac{x^2}{z^2}\right)d\varphi_{i+1} - \frac{xy}{z}\,d\omega_{i+1}\right\}
\end{aligned}
$$

These equations show the influence of the random errors in the orientation elements of the starting model (made of photographs i-1, i), and the influence of the random errors in the elements of the relative orientation of the attached model (made of photographs i, i+1), on the errors in the coordinates x, y, z of the points measured in the attached model. The errors Δx, Δy, Δz in the next attached model (made of photographs i+1, i+2) would show the same equations, only with the difference of the subscript: instead i should be put i+1, and instead of i+1 should be put i+2.

The <u>standard errors</u> σx, σy, σz in the n^{th} model of the triangulation strip would be expressed as the quadratic sum of all errors Δx, Δy, Δz in each of the models starting with the second one in a strip.

$$\sigma x^2 = \sum_{j=2}^{n}(\Delta x^2) \qquad \sigma y^2 = \sum_{j=2}^{n}(\Delta y^2) \qquad \sigma z^2 = \sum_{j=2}^{n}(\Delta z^2) \quad (3)$$

where j is the number of the order of the model in the strip. The quadrats of the partial errors Δx, Δy, Δz in equations (3) will lead to the second powers and products of dbx, dby, ..., $d\omega$. Because the random errors of the orientation elements are absolutely very small, the powers and the products of the errors are negligible. Then the errors σx, σy, σz of the points in the n^{th} model will be

$$\sigma x = \Delta x\sqrt{n-1} \qquad \sigma y = \Delta y\sqrt{n-1} \qquad \sigma z = \Delta z\sqrt{n-1} \quad (4)$$

It can be concluded that no systematic deformation of the joint model in a strip is caused by the random errors of the orientations elements.

If the errors of the orientation elements comprise only the systematic components, which are in the starting as well as in the attached model equal, the systematic errors c_x, c_y, c_z of the points in the n^{th} model are

$$
\begin{aligned}
c_x = & \sum_{j=2}^{n}\Big(\Big\{ (x+b)\frac{1}{b}(dbx_i - d\varphi_i) - y\,d\kappa_i + z\,d\varphi_i - dbx_i \Big\} \\
& \qquad + \frac{x}{b}\Big\{ \frac{b}{z}dbz_i - z\Big(1+\frac{b^2}{z^2}\Big)d\varphi_i \Big\}\Big) \\
& + \sum_{j=1}^{n}\Big\{ \frac{x}{z}dbz_i - y\,d\kappa_i + z\Big(1+\frac{x^2}{z^2}\Big)d\varphi_i - \frac{xy}{z}d\omega_i \Big\} \\
c_y = & \sum_{j=2}^{n}\Big(\Big\{ \frac{y}{b}(dbx_i - d\varphi_i) + (x+b)d\kappa_i - z\,d\omega_i - dby_i \Big\} \\
& \qquad + \frac{y}{b}\Big\{ \frac{b}{z}dbz_i - z\Big(1+\frac{b^2}{z^2}\Big)d\varphi_i \Big\}\Big) \\
& + \sum_{j=1}^{n}\Big\{ -dby_i + \frac{y}{z}dbz_i - x\,d\kappa_i - \frac{xy}{z}d\varphi_i + z\Big(1+\frac{y^2}{z^2}\Big)d\omega_i \Big\} \\
c_z = & \sum_{j=2}^{n}\Big(\Big\{ \frac{z}{b}(dbx_i - d\varphi_i) - (x+b)d\varphi_i + y\,d\omega_i - dbz_i \Big\} \\
& \qquad + \frac{z}{b}\Big\{ \frac{b}{z}dbz_i - z\Big(1+\frac{b^2}{z^2}\Big)d\varphi_i \Big\}\Big) \\
& + \sum_{j=1}^{n} - \frac{z}{b}\Big\{ \frac{x}{z}dbz_i - y\,d\kappa_i + z\Big(1+\frac{x^2}{z^2}\Big)d\varphi_i - \frac{xy}{z}d\omega_i \Big\}
\end{aligned}
\tag{5}
$$

It is possible to introduce some specifications into equations (5). In these equations there was put a different coordinate system for each model, with the origin in the projection center of the right photograph. However, in analytical triangulation there is usual to apply only one coordinate system for the whole strip with the origin in the projection center of the first photograph in the strip. The z axis corresponds with the camera axis of the first photograph, the x axis directs to the projection center of the second photograph, being perpendicular to z. The y axis is perpendicular to the xy plane and forms with the x,y axes the left-handed system. It is possible to anticipate that all bases b in a strip of photographs with 60% overlap are approximately of the same length. Then the errors c_x, c_y, c_z in the n^{th} model of the strip are

$$\begin{aligned} c_x = {} & \left\{-(n-1)dbx + nzd\varphi\right\} + x\left\{(n-1)\frac{dbx}{b} - (n-2)\frac{dbz}{z}\right. \\ & \left. -(n-1)\frac{z}{b}\left(1 + \frac{1}{z} + \frac{b^2}{z^2}\right)d\varphi\right\} - ynd\varkappa - xy\frac{d\omega}{z} + x^2\frac{d\varphi}{z} \end{aligned}$$

$$\begin{aligned} c_y = {} & \left\{- ndby - (n-2)zd\omega\right\} + x(n-2)\, d\varkappa \\ & + y\left\{(n-1)\frac{dbx}{b} + n\frac{dbz}{z} - (n-1)\frac{z}{b}\left(1 + \frac{1}{z} + \frac{b^2}{z^2}\right)d\varphi\right\} \\ & - xy\frac{d\varphi}{z} + y^2\frac{d\omega}{z} \end{aligned} \tag{6}$$

$$\begin{aligned} c_z = {} & (n-1)\frac{z}{b}dbx - \frac{z^2}{b}\left\{n + (n-1)\left(\frac{1}{z} + \frac{b^2}{z^2}\right)d\varphi\right\} \\ & - x\left\{\frac{dbz}{b} + (n-1)d\varphi\right\} + y\left\{\frac{z}{b}d\varkappa + (n-1)d\omega\right\} + xy\frac{d\omega}{b} - x^2\frac{d\varphi}{b} \end{aligned}$$

It is evident that the errors c_x, c_y, c_z will comprise constant as well as variable elements with regard to the modle coordinates x,y. The variable element will have linear parts and quadratic parts. It means that the errors c_x, c_y, c_z will grow not only linearly with the growing x,y. Equations (6) comprise also terms with the second powers or products x^2, y^2, xy of the model coordinates. It is possible to state that due to the errors of the relative orientation "quadratic" deformations of the strip of models will apear.

3. Model deformation in blocks of photographs

As mentioned in the introduction, the present author investigated the deformations of the photographs, particulary paying attention to those factors, whose systematic influence could not be eliminated. He concluded that no one of the known physical factors caused the systematic error c_{ω} of that kind. Then it is allowed to let out the terms with $d\omega$ in equations(6).

The coefficients of x and y in equations (6) can be replaced by general symbols. Then the following polynomials can be written, which show how the errors $dx = c_x$, $dy = c_y$, $dz = c_z$ will grow in dependence on the model coordinates x,y in triangulation strip.

$$\begin{aligned} dx &= a_1 + a_2x + a_3y + a_4x^2 \\ dy &= b_1 + b_2x + b_3y + b_4xy \\ dz &= c_1 + c_2x + c_3y + c_4x^2 \end{aligned} \tag{6a}$$

The aerial photographs with 60% end and side overlaps are very suitable for the block triangulation. There is possible to orient the photographs with adjustment. It was proved [4], [5] that even in the smallest subblock, that is in the quadruplet (two and two photographs of two adjacent strips), the relative orientation as well as the scaling could be adjusted. The adjustment reduces the influence of the random errors in the orientation elements. It means that for the standard errors σx, σy, σz can be applied equations (4) even in the block triangulation. Among the systematic errors of the orientation elements also c_{ω} takes the place in blocks. Now it is not allowed to let out the terms with $d\omega$ in equations (6). On the cotrary, detailed analysis would lead to conclusion that it is necessary to introduce further terms of the second degree.

$$
\begin{aligned}
dx &= a_1 + a_2x + a_3y + a_4xy + a_5x^2 + a_6y^2 \\
dy &= b_1 + b_2x + b_3y + b_4xy + b_5x^2 + b_6y^2 \\
dz &= c_1 + c_2x + c_3y + c_4xy + c_5x^2 + c_6y^2
\end{aligned}
\tag{7}
$$

As long as the systematic errors will be kept small, there appear no terms of higher degree than second in equations (7).

The conclusion of this short contribution is as follows: It is justified to use polynomials of the second degree for the coordinate transformation in the aerial triangulation. It is not useful to apply polynomials of higher degree than second; it would only enhance the number of the transformation coefficients and consequently would enhance the necessary number of the ground control points.

References:

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