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EXPERIMENTAL ARRAY ALGEBRA MODELING ALGORITHMS

ABSTRACT:

The powerful principles of array algebra DTM were implemented in some general purpose filtering and compaction algorithms to evaluate different math models using a typical 1201 X 601 elevation grid of AS-11B-X data. The most accurate math models were then implemented in specialized algorithms revealing the computational power of Array Algebra. For example, 180 901 parameters were fitted to 721 801 observed values in an algorithmic CPU time of 30 seconds with 9 K bytes of minicomputer core space allocated for the data. Also 721 801 nodes of an array algebra finite element model were fitted to 721 801 observed values and 1 443 602 continuity equations in 100 seconds with 22 K bytes data allocation. Over 99% of the computations were performed using less than 20 FORTRAN statements making real-time DTM processing feasible. For DTM and image processing the algorithms automatically reveal gross errors, discontinuities, features, good correlation signatures, etc. This modeling philosophy developed since 1967 has been applied in the unbiased self-calibrating triangulation methods as reported by the author in the 1972 and 1976 ISP Congresses. Further specializations may solve the problem of on-line triangulation.

INTRODUCTION

In the 1972 and 1976 ISP Congresses the author presented reviews on the mathematical foundations of a new generalized vector, matrix, tensor and "fast transform" algebra called array algebra, (Rauhala, 1972b, 1976). These reviews were mainly based on the research work published in the author's Ph.D and ScD theses, (Rauhala, 1972, 1974). Recent reviews, refinements and applications of this powerful math technology can be found in Blaha (1977a, 1977b), Kratky (1976, 1978), Rauhala (1975, 1977, 1978a, 1978b, 1979a, 1979b, 1980a, 1980b), Rauhala and Gerig (1976), Snay (1978), Söderlund (1976), Woltring (1976). This paper demonstrates some experimental results of applying these principles in photogrammetric triangulation and digital terrain modeling after first recapitulating the concepts of array interpolation and filtering in one and higher dimensions.

SELF-CALIBRATION OF PHOTOGRAMMETRIC TRIANGULATION

The basic research leading to the development of array algebra started in 1967 from the idea of a simultaneous adjustment of photogrammetric and geodetic observations, (Rauhala, 1968). The idea of loop inverses and array algebra emerged from the empirical self-calibration parameters of the simultaneous adjustment to be described next in one dimension or in the conventional monolinear case, where multilinear problems are treated in the same fashion as the one-dimensional case.

The systematic total effect $\underset{n,1}{L_o}$ of all conceivable error sources at the variable values $x = (x_o)_1, (x_o)_2, \cdots (x_o)_n$ can be considered as n unknown parameters. The math model of the systematic total effect at any variable value x consists of interpolations from $\underset{n,1}{L_o}$ by

$$F(x) = k_1(x)(L_o)_1 + k_2(x)(L_o)_2 + \cdots + k_n(x)(L_o)_n$$

$$= \underset{1,n}{k(x)} \; \underset{n,1}{L_o} , \tag{1}$$

where $k_1(x), k_2(x), \cdots k_n(x)$ are coefficients of any suitable interpolation method. In the bundle method n has to be restricted to the value 3 in each of the coordinate directions of an image; for example using the Lagrangian interpolation we have, (Rauhala, 1972 p. 7),

$$\left[k_1(x), k_2(x), k_3(x)\right] = \underset{1,3}{a(x)} \; \underset{3,3}{A_o^{-1}}$$

$$= \left[1, x, x^2\right] \begin{bmatrix} 1 & (x_o)_1 & (x_o)_1^2 \\ 1 & (x_o)_2 & (x_o)_2^2 \\ 1 & (x_o)_3 & (x_o)_3^2 \end{bmatrix}^{-1} . \tag{2}$$

The same coefficients apply to the y-direction in a self-calibrating independent model adjustment while interpolation in the x-direction (along the base line) is simply trapezoidal, $n = 2$.

The functional model of equation (1) has some superior numerical properties and simplifications in the stage of fitting $\underset{n,1}{L_o}$ to m real observed values, $\underset{m,1}{L}$, located at $x = x_1, x_2, \cdots x_m$. The observation equations become

$$\underset{mn}{K}\ \underset{n,1}{L_o} = \underset{m,1}{L} - \underset{m,1}{V} \quad , \quad \underset{mn}{K} = \begin{bmatrix} k(x=x_1) \\ k(x=x_2) \\ \vdots \\ k(x=x_m) \end{bmatrix} ,$$

$$\underset{mn}{K} = \underset{mn}{A}\ \underset{nn}{A_o^{-1}} \quad , \quad \underset{mn}{A} = \begin{bmatrix} a(x=x_1) \\ a(x=x_2) \\ \vdots \\ a(x=x_m) \end{bmatrix} . \qquad (3)$$

The problem analyst has a complete freedom in the choice of the location of L_o with respect to L. In aerial photogrammetry most of the observed values are located in the vicinity of the scaled image coordinates $x = -1, 0, 1$; $y = -1, 0, 1$. Therefore these locations are natural for L_o. Now the design matrix $\underset{nn}{K_o}$ of the subset L_{om} of all observed values of an image, falling on these node locations simply becomes a unit matrix! Thus the interpolation coefficients, equation (2), need to be derived only for the redundant observed values, L_ε, located outside the node points, and the observation equations, equation (3), are specialized to

$$\begin{aligned} \underset{nn}{I}\ \underset{n,1}{L_o} &= \underset{n,1}{L_{om}} - \underset{n,1}{V_o} \\ \underset{r,n}{K_\varepsilon}\ \underset{n,1}{L_o} &= L_\varepsilon - V_\varepsilon \end{aligned} \qquad (3a)$$

$$r = m - n. \qquad (3b)$$

To enable practical experiments of the new self-calibrating triangulation methods, an ideal array test field was built during 1969-70 at the Royal Institute of Technology in Stockholm such that introduction of any modeling bias through K_ε in (3b) was completely avoided. The control data (known points) were restricted to the patterned pass points which all were ideally imaged on the gridded node locations of L_o. Now the discrete mean total effect L_o of all error sources at the nodes becomes estimable without any need to know $F(y,x)$ outside these points, i.e., the modeling bias is completely avoided. Furthermore, a reseau camera was used for the experiments such that film errors, lack of flatness etc. could be discretely eliminated by choosing the location of L_o to coincide with the reseau crosses. Thus 9 parameters, $\underset{9,1}{L_{oy}}$, for systematic y-errors and 9 parameters, $\underset{9,1}{L_{ox}}$, for x-errors are introduced. Three parameters of each group can be deleted because they can be compensated for by the exterior coordinate system. Thus a total of 6 + 6 = 12 effective unbiasedly estimable parameters remain for the bundle method and 4 + 4 = 8 parameters for the independent model adjustment. The unbiasedness of these remaining free parameters is fully unchallenged such that they necessarily require no a priori constraints, (Brown, 1958), in contrast to their ill-conditioned or biased transform domain counterparts $X = A_o^{-1} L_o$ used by most authors on this subject since 1972.

As a corollary to the above paragraph a serious short-coming and paradox is pointed out in the control data transfer of a conventional block adjustment with few (but theoretically sufficient) control points on the periphery of the block; the few image measurements of the control points outside the node locations require an explicit introduction of a math model for interpolating between L_0, i.e., systematic modeling errors have to be introduced. Furthermore, because the location of the image coordinates of the sparse control varies randomly, repeated samples of the systematic total effect at these interpolated points are not possible. In a sharp contrast, a repeated sample of the total effect of the systematic errors at the 9 node points is performed for every image greatly reducing the effect of random errors on L_0. In conclusion, the paradox is that both systematic and random errors are allowed to enter the conventional block adjustment at the crucial moment of transferring the external control to the photogrammetric system. The frequent complaints of "bad control data" can partly be explained by the above reasoning.

In the following section the above modeling technique is extended to the more restricted, but computationally powerful, bi-linear array form. Its application in a photogrammetric or geodetic triangulation requires some generalized array algebra solutions and a rethinking of the theoretical and technical foundations of these and some new surveying technologies beyond the scope of this paper. Some practical applications of the "regular" array solutions in digital terrain modeling and image processing are demonstrated in the last two sections.

ARRAY ALGEBRA IN DIGITAL TERRAIN MODELING

Before the introduction of the multidimensional array filtering the monolinear observation equations (3) are supplemented with some judicious a priori constraints of L_0 by

$$\underset{p,p}{\sqrt{P}}\ \underset{p,n}{C}\ \underset{n,1}{L_0} = \underset{p,1}{0} - \underset{p,1}{V_c} . \tag{3c}$$

For example, in the method of 1-D finite elements interpolation, (Ebner and Reiss, 1978), these constraints consist of the following continuity equations

$$\sqrt{p}\,\left((L_0)_{i-1} - 2(L_0)_i + (L_0)_{i+1} \right) = 0 - \sqrt{p}\,(v_c)_i \quad , \quad i = 2, 3, \cdots n-1 ,$$

$$1/p = 6 . \tag{4}$$

The complete set of observation equations, (3 a, b, c), in parameters L_0 has the least squares solution

$$\boxed{\underset{n,1}{\hat{L}_0} = \underset{n\,m}{H}\ \underset{m,1}{L}} . \tag{5}$$

The linear operator $\underset{n\,m}{H}$ is called filter matrix representing the covariance matrix between $\hat{L}_0$, $\hat{L}$ and having the expression

$$H = (K^T K + C^T P C)^{-1} K^T . \tag{6}$$

In the practical special case of $L_o \in L$, (equations 3a, 3b, 4), it becomes $H = (I + K_6^T K_6 + 1/_6\, C^T C)^{-1} K^T$.

Now a unit matrix is added on the main diagonal of an already diagonally dominant banded system of $K_6^T K_6 + pC^T C$ resulting in an extremely well-conditioned solution $\hat{L}_o$ in contrast to the often ill-conditioned conventional transform domain solution $\hat{X} = A_o^{-1} \hat{L}_o$ of

$$\hat{X} = (A^T A + A_o^T C^T P C A_o)^{-1} A^T L . \tag{7}$$

Normally matrix $\underset{nm}{H}$ would not be formed explicitely but for some specialized "successive 1-D modeling" of array calculus, one-time simulations of some special matrices H allow very fast 2-D array solutions of the form

$$\boxed{\underset{n_1 n_2}{\hat{L}_o} = \underset{n_1 m_1}{H_1} \; \underset{m_1 m_2}{L} \; \underset{m_2 n_2}{H_2^T}} . \tag{8}$$

The principle of successive 1-D modeling is the genesis of the multidimensional array calculus. The principle can be demonstrated by assuming some profiles measured along y-axis at $x = (x_o)_1, (x_o)_2, \cdots (x_o)_{n_2}$. If the measurements are not originally located at profiles or if the profiles are crooked, it is a simple matter to translocate them into straight profiles in the y, x-system (Rauhala and Gerig, 1976). If y, x are the circular coordinates r, φ then the profiles are radial or circular. To interpolate $F(y,x)$, first the 1-D interpolations $F(y)_{j_2}$, $j_2 = 1, 2, \cdots n_2$ are performed along each profile. The resulting n_2 values of $F(y)$ represent a profile in the x-direction; a repeated 1-D interpolation along this profile yields the desired $F(y,x)$. In the practical cases of long profiles, the repeated 1-D interpolations have a local character, i.e., only a few of the closest values need to be used in the computations. Completely arbitrary a priori weight and point distribution is allowed in the successive 1-D interpolations.

The "successivity" philosophy also applies to filtering or smoothing interpolation which is demonstrated next to create the smoothed elevation array $\underset{n_1 n_2}{\hat{L}_o}$ from the above profile measurements: Separate 1-D least squares solutions $(\hat{L}_o)_{j_2}$, (5), replace the original profiles and the above interpolation now produces smoothed values $\hat{F}(y,x)$. A natural specialization is to place nodes at the <u>same</u> y-coordinate locations in each profile and to measure $m_2 > n_2$ profiles to achieve redundancy also in the x-direction. Then the 1-D solutions along profiles (columns) yield the intermediate "columnwise adjusted" nodes $\underset{n_1 m_2}{\hat{Y}}$ which can be filtered "rowwise" to yield the final compacted array

$$\underset{n_1 n_2}{\hat{L}_o} = \underset{n_1 m_2}{\hat{Y}} \; \underset{m_2 n_2}{H_2^T} . \tag{8b}$$

This compact matrix expression is possible under the restriction that <u>all</u> rows of $\hat{Y}$ have the <u>same</u> filter matrix H_2 simulated, once and for all, with respect to the n_2 x-coordinate locations of profiles in $\underset{n_1 n_2}{\hat{L}_o}$ and the m_2 locations of $\underset{n_1 m_2}{\hat{Y}}$.

The rigorous case of array algebra modeling is found by arranging the profile measurements L into grid intersections. Now the m_2 1-D column filterings can be denoted

$$\underset{n,m_2}{\hat{Y}} = \underset{n,m_1}{H_1} \; \underset{m_1,m_2}{L} \tag{8a}$$

and in a manner similar to the above paragraph, the final elevation array is found by equations (8b), (8). In the special case of excluding the a priori constraints (3c) in the filter matrices H_1, H_2, the elevation array $\hat{L}_o$ of (8) represents the rigorous least squares solution of the bilinear observation equations

$$\underset{m_1,n_1}{K_1} \; \underset{n_1,n_2}{L_o} \; \underset{n_2,m_2}{K_2^T} = \underset{m_1,m_2}{L} - \underset{m_1,m_2}{V}$$

by minimising the norm $\| L - K_1 \hat{L}_o K_2^T \|$. If each of the successive 1-D filterings require b multiplications then $H_1 L$ requires $n_1 m_2 b$ multiplications and $\hat{L}_o = \hat{Y} H_2^T$ the additional $n_1 n_2 b$ multiplications. This is in a high contrast to the $n_1^3 n_2^3$ multiplications of the brute-force numerical procedure for solving $n_1 n_2$ parameters.

In some important practical cases, the filter matrices H_1, H_2 become circular, i.e., with exception of few first and last rows, all rows contain the <u>same</u> filter coefficients $h_o, h_1, h_2, \cdots h_b$. For example, by coinciding the nodes with every second evenly distributed observed value, the multiplication $\underset{n,1}{\hat{L}_o} = \underset{n,2n}{H} \; \underset{2n,1}{L}$ can be expressed by the convolution

$$(\hat{l}_o)_i = h_o l_i + h_1 (l_{i+1} + l_{i-1}) + h_2 (l_{i+2} + l_{i-2}) + \cdots + h_b (l_{i+b} + l_{i-b}),$$

$$i = b, b+2, b+4, \ldots m-b-2, m-b. \tag{10}$$

The following set of coefficients

$$\underset{1,2b+1}{h} = \left[h_b, h_{b-1} \cdots h_2, h_1, h_o, h_1, h_2 \cdots h_{b-1}, h_b \right] \tag{11}$$

corresponds to the "impulse response" in the terminology of signal processing and its Fourier transform is known as the "transfer function". The following experimental results demonstrate the computational power of array filtering. The results are so convincing that no explicit comparisons to the conventional methods are sought.

DTM EXPERIMENTS OF 4:1 COMPACTION

The experiments of performing the filtering

$$\underset{601,301}{\hat{L}_o} = \underset{601,1201}{H_1} \; \underset{1201,601}{L} \; \underset{601,301}{H_2^T} \tag{12}$$

of a typical elevation data of AS-11B-X are detailed in Rauhala (1979b, 1980b). In these experiments several interpolation models were tested to find the most accurate and efficient impulse response h, eq. (11). It turned out that the best model was the simplest form of a 5 X 5 trapezoidal filter yielding the RMS of 0.30 for the 8-bit integer data of L. Thus the 1-D filterings (10) were truncated after the term h_2.

The CPU time of the minicomputer SEL 32/55 was clocked at 7 min. for a 15 X 15 (b=7) general purpose filtering of equation (12). The time includes reading the data from the disc, performing the filterings, outputting $_{601,301}\hat{L}^{o}$ together with node residuals $_{601,301}\hat{V}^{o}$ on the disc, and bringing the residuals back to the core memory for a "picture-like" output on the line printer. A specialized algorithm was designed for the optimal 5 X 5 filter reducing the CPU time to 1 min. 35.5 sec. Without the line printing of residuals the time was 58.5 sec. The purely algorithmic CPU time, excluding the read/write operations from/to the disc, was 30 seconds for the rigorous least squares solution of 180 901 parameters to 721 801 observed values. The core space requirement, excluding the program, was 9 K bytes for the specialized 5 X 5 filter algorithm. The total length of the algorithm is 120 FORTRAN statements, excluding the residual print.

DTM EXPERIMENTS OF 1:1 FINITE ELEMENT FILTERING

The fictitious continuity equations of (4) were included in the filter matrices to allow filtering when the nodes are coincident with the observed values. To study the effect of the a priori constraints, the continuity equations (4) were extended from the 2nd difference condition to the similar conditions of 3rd and 4th differences. The following RMS's were found at the 1201 X 601 nodes using p X p finite element filter widths:

p	2nd diff.	3rd diff.	4th diff.
3	.27	-	-
5	.21	.18	.15
7	.21	.18	.17
∞	.21	.18	.17

(13)

The "picture-like" residuals (one hexadecimal number representing the small integer residual allowing the print of a strip with 132 consecutive columns of an array) confirmed the excellent modeling accuracy of case p = 5 with the 4th difference condition. The boundaries of the discontinuous zones of the data became evident in the residual prints with an extraordinary clarity.

It turned out that the 2 X 601 X 301 fictitious continuity equations contained in $_{601,301}\hat{L}^{o}$ could replace the 3 X 601 X 301 real redundant observed values resulting in 75% savings in the data collection. The above conclusion was reached by extracting $_{601,301}L$ from the original 1201 X 601 grid by deleting every second row and every second column. The resulting RMS was 0.32 or very close to the RMS of 0.30 of the 4:1 compaction and filtering of the original data.

The algorithmic CPU time of a specialized finite element program was 100 seconds for fitting 1201 X 601 = 721 801 nodes to 721 801 observed values and 2 X 1201 X 601 = 1 443 602 continuity equations. The data allocation was 22 K bytes. Over 99% of the computations are performed using 17 FORTRAN statements and only 3 statements in one subroutine. These experimental results confirmed the expectation that array algebra properly implemented offers indeed a very powerful tool for several related numerical problems such as real-time digital terrain modeling, image processing, change detection, feature-, gross error-, and discontinuity extraction, correlation, photogrammetric and geodetic net adjustments, etc.

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