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# MATHEMATICAL MODEL FOR ANALYTICAL TRIANGULATION OF SPACE PHOTOGRAPHY

Dr. M.E.O. Ali
Gendron, Lefebvre Inc.
1, Place Laval
Laval
Québec, (Qué.) Canada

and

Dr. A.J. Brandenberger
Department of Photogrammetry
Laval University
Québec, (Qué.)
Canada

## Abstract

A mathematical model and a computer program were developed by the authors to perform analytical aerial triangulation for space photography. In this paper, the mathematical model, the main headlines of the computer program as well as the results of a few tests are given.

## Introduction

Large portion of the world are still very poorly mapped even at scales of 1:250,000 or smaller. The policy of making space photography available almost at no cost to the user, makes mapping from space photography more and more attractive. The authors of this paper were involved in a research program to investigate the maximum possible accuracy for aerial triangulation using SKYLAB photography combined with very high altitude aircraft photography and utilizing SKYLAB orbital parameters. Two versions of the bundle adjustment technique and two associated computer programs were developed in connection with the study. The mathematical models, the computer programs as well as a few test results to show the efficiency of the programs are described in the paper.

The idea of the bundle adjustment is to use the well known collinearity equations to establish two equations for each measured photo point and further to obtain a unique solution for the system of observation equations by least squares methods. The linearized form of the collinearity equations may be given by:

$$A\delta + BV + W = 0 \qquad (1)$$

where

- $\delta$ is the correction vector to the approximate values set for the unknowns,
- $V$ is the residual vector, i.e. the correction vector to the observations,
- $W$ is the misclosure vector,
- $A, B$ are two matrices whose elements are the partial derivatives of equation (1) with respect to the unknowns and to the observations respectively.

## Least Squares Solution

The principle of the least squares method requires the minimizing of the quadratic form $V'PV$, where $P$ is the weight matrix whose elements are the weights associated with each of the observations. The least squares solution of an equation similar to the linear form of the collinearity conditions equations given by (1) can be written as:

$$\delta = N^{-1}U \qquad (2) \quad , \text{ where:}$$

$$N = A'M^{-1}A \qquad (3) \qquad U = -A'M^{-1}W \qquad (4) \quad ,$$

$$M = BP^{-1}B' \qquad (5)$$

By applying the least squares technique to solve the system of linearized observation equations (1), one can see that two matrices have to be inverted, namely the matrix M (equation (5)) to form the normal equations and the matrix N (equation (3)) to solve for the unknowns. A direct method of computing and inverting such large matrices is not practical due to both the excessive amount of computer time and memory required and also because of the rounding off of errors in machine calculations.

Several algorithms and computer programs have been developed using the method of adopting single bundles of rays as a unit to adjust a block of aerial photography; see for example, (4, 5, 6, 7 and 8), Keller (1967), Wong (1971), Schut (1978). To overcome the difficulties of calculating and inverting such large matrices (like M and N), the computer programs associated with these algorithms only solve special cases of the general observation equations (1). Also, one of the main goals of the previous programs is to adjust simultaneously as large a number of photographs as possible in the most economic way. But one should consider that for space photography, the cost portion for adjusting the photogrammetric measurements is negligible compared with the cost for the entire mission. The main goal is to achieve the best possible accuracy for the aerial triangulation by using few photographs only. This reasoning indicates the need for a new algorithm using:

1. measurements of space photography, and
2. ground control points which are few in number and inferior in quality,
3. camera parameters for each exposure station, and a simultaneous least squares adjustment. It was hoped that such a system would produce better estimates for the measured coordinates of the ground control points and would supply accurate coordinates for the pass points.

Two algorithms and their computer programs were developed. Although these algorithms are also special cases of equation (1), they are more suitable for the case of space photography.

## The First Algorithm

In this algorithm the two collinearity condition equations are used to calculate two equations for each measured photo point. In these equations all the camera parameters are used as unknowns, while the ground coordinates are treated in two different ways:

i) the coordinates of the control points are used as observations,
ii) the coordinates of the pass points are used as unknowns.

To explain this new algorithm assume that one starts to form the normal equations by calculating the contribution of all the observed control points, followed by the contribution of the unknown pass points. Then, the design matrices A, B and W may be given by:

$$
A = \left[\begin{array}{c:c} A_{I} & 0 \\ \hdashline \dot{A}_{II} & \ddot{A} \end{array}\right]
\begin{array}{l} \text{(equations associated with observed control points)} \\ \text{(equations associated with unknown pass points)} \end{array}
$$

$$
B = \left[\begin{array}{c:c:c} \ddot{B} & \dddot{I}_{I} & 0 \\ \hdashline 0 & 0 & \dddot{I}_{II} \end{array}\right]
\begin{array}{l} \text{(equations associated with observed control points)} \\ \text{(equations associated with unknown pass points)} \end{array}
$$

$$
W = \left[\begin{array}{c} W_{I} \\ \hdashline W_{II} \end{array}\right]
\begin{array}{l} \text{(equations associated with observed control points)} \\ \text{(equations associated with unknown pass points)} \end{array}
$$

where

'.', '..', '...' are affixed to any elements related to the camera parameters, ground coordinates and photo coordinates respectively.

'I', 'II' are affixed to the elements associated with the observed ground control points and the unknown pass points respectively.

Accordingly, the elements of the above questions can be explained as follows:

$\dot{A}_{I}$, $\dot{A}_{II}$ are the matrices which represent the partial derivatives of the Collinearity Condition Equations (CCE) with respect to the unknown camera parameters and associated with the observed control points and the unknown pass points respectively,

$\ddot{A}$, $\ddot{B}$ are the matrices which represent the partial derivatives of the CCE with respect to the unknown ground coordinates of the pass points and of the observed ground coordinates of the control points respectively,

$\dddot{I}_{I}$, $\dddot{I}_{II}$ are two unit matrices which represent the partial derivatives of the CCE with respect to the photo measurements and associated with the observed control points and the unknown pass points respectively,

$W_{I}$, $W_{II}$ are the misclosure vectors associated with the observed control points and the unknown pass points respectively.

Moreover, the above matrices can be written in more detailed forms as follows:

$$
\dot{A}_{I} = \left|\begin{array}{c} \dot{a}_{1} \\ \dot{a}_{2} \\ \vdots \\ \dot{a}_{K} \end{array}\right| , \quad \ddot{B} = \left|\begin{array}{cccc} \ddot{b}_{1} & 0 & \cdots & 0 \\ 0 & \ddot{b}_{2} & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{array}\right|
$$

$$\dot{A}_{II} = \begin{vmatrix} \dot{a}_{k+1} \\ \dot{a}_{k+2} \\ \cdot \\ \cdot \\ \dot{a}_{k+n} \end{vmatrix} , \quad \ddot{A} = \begin{vmatrix} \ddot{a}_{k+1} & 0 & \cdots & 0 \\ 0 & \ddot{a}_{k+2} & & \\ \vdots & & \ddots & \\ 0 & & & \ddot{a}_{k+n} \end{vmatrix}$$

$$W_I = \begin{vmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_3 \end{vmatrix} , \quad W_{II} = \begin{vmatrix} w_{k+1} \\ w_{k+2} \\ \cdot \\ \cdot \\ w_{k+n} \end{vmatrix}$$

where

| | |
|---|---|
| $\underset{(neq_i,6m)}{\dot{a}_i}$ | are the partial derivatives of the CCE with respect to the unknown camera parameters and associated with the ground point i, |
| $\underset{(neq_i,3)}{\ddot{a}_i}$ | are the partial derivatives of the CCE with respect to the unknown ground coordinates $X_G$, $Y_G$, $Z_G$ of the pass point i ($\ddot{a}_i=0$ when processing the observed control point, i.e. when $i \leq k$), |
| $\underset{(neq_i,3)}{\ddot{b}_i}$ | are the partial derivatives of the CCE with respect to the observed ground coordinates $X_G$, $Y_G$, $Z_G$ of the control point i ($\ddot{b}_i=0$ when processing the unknown pass points, i.e. when $i>k$), |
| $\underset{(neq_i,\ 1)}{w_i}$ | is the misclosure vector associated with the ground point i. |
| m | is the number of camera stations |
| k,n | are the numbers of points with observed and unknown ground coordinates respectively, |
| $neq_i$ | is the number of equations associated with the ground point i and this number is equal to twice the number of photographs with the image of the ground point. |

One space photography covers a relatively large area, and it may be practically sufficient to adjust only a few of the photographs at a time. Then, if the unknown ground coordinate vectors $\ddot{\delta}_i$' are eliminated, the reduced system of normal equations can be written as:

$$\left| \dot{N} - \sum_{i=k+1}^{k+n} \bar{n}_i \, \ddot{n}_i^{-1} \, \bar{n}_i' \right| \left| \dot{\delta} \right| = \left| \dot{U} - \sum_{i=k+1}^{k+n} \bar{n}_i \, \ddot{n}_i^{-1} \, \ddot{u}_i \right|$$

or in abbreviated form:

$$\underset{(6m,6m)}{N_R} \; \underset{(6m,1)}{\dot{\delta}} = \underset{(6m,1)}{U_R}$$

Since the size of the $N_R$ matrix is relatively small, the unknown corrections to the camera parameters $\dot{\delta}$ can be easily calculated as follows:

$$\underset{(6m,1)}{\breve{\delta}} = N_R^{-1} \; U_R$$

where

$$\dot{N} = \sum_{i=1}^{k+n} \breve{a}_i' \, m_i^{-1} \, \acute{a}_i$$

$$\ddot{n}_i = \ddot{a}_i' \, m_i^{-1} \, \ddot{a}_i \quad , \quad i=k+1,\ k+2,\ \ldots,\ k+n$$

$$\bar{n}_i = \dot{a}_i' \, m_i^{-1} \, \ddot{a}_i \quad \text{and } i=k+1,\ k+2,\ \ldots,\ k+n$$

$$\dot{U} = \sum_{i=1}^{k+n} - \dot{a}_i' \, m_i^{-1} \, w_i$$

$$\ddot{u}_i = -\ddot{a}_i' \, m_i^{-1} \, w_i \quad \text{and } i=k+1,\ k+2,\ \ldots,\ k+n$$

and

$\underset{(neq_i,neq_i)}{m_i}$ is the nonzero submatrix of the M matrix associated with the ground point i,

$m_i = \ddot{b}_i \ddot{p}_i^{-1} \ddot{b}_i + \dot{p}_i^{-1}$ for i=1, 2, ..., k (associated with control points)

or

$m_i = \dddot{p}_i^{-1}$ for i=k+1, k+2, ..., k+n (associated with pass points)

$\underset{(3,3)}{\ddot{p}_i}$ is the weight matrix for a ground control point i

$\underset{(neq_i,neq_i)}{\dddot{p}_i}$ is the weight coefficient matrix for the measured photo coordinates associated with the ground point i.

Then, the unknown corrections ($\ddot{\delta}_i$) to the ground coordinates for any point i, can be calculated by back substitution from equations similar to the following equations:

$$\dot{\delta}_i = \ddot{n}_i^{-1} \, \ddot{u}_i - \ddot{n}_i^{-1} \, \bar{n}_i' \, \dot{\delta}$$

## The Second Algorithm

In this algorithm, the two collinearity condition equations are also used to form two equations for each measured photo point. The application of these equations is as follows:

1. all the camera parameters are used as observations,
2. the ground coordinates are used in two different ways when compared with the first algorithm:
   a) the coordinates of the control points are used as observations,
   b) the coordinates of the pass points are used as unknowns.

To form the normal equations for the least squares adjustment, one has to calculate the M matrix given by equation (5). Here, in the second algorithm, the B matrix contains partial derivatives of the collinearity condition equations corresponding to both the coordinates of the ground control points and the camera parameters. Hence, it is impossible to calculate the contributions to the normal equations for either points or photos independently of the others. Accordingly, the whole A, B, W and $M^{-1}$ matrices have to be calculated before it is possible to calculate any contributions to the normal equations. However, to reduce the computing time and memory space, all the ground control points are used as one group followed by all the pass points as another group. In this way, the M matrix is partitioned into four smaller submatrices which can be calculated one after another.

Assuming that one starts to calculate the observation equations associated with all the control points and thereafter calculates the observation equations of all the pass points, then the design matrices A, B and W can be written as follows:

$$A = \begin{bmatrix} 0 \\ \hline \ddot{A} \end{bmatrix} \begin{array}{l} \text{(equations associated with the control points)} \\ \text{(equations associated with the pass points)} \end{array}$$

$$B = \begin{bmatrix} \dot{B}_I & \ddot{B} & I_I & 0 \\ \hline \dot{B}_{II} & 0 & 0 & I_{II} \end{bmatrix} \begin{array}{l} \text{(equations associated with the control points)} \\ \text{(equations associated with the pass points)} \end{array}$$

$$W = \begin{bmatrix} \underset{(NEQ_I,1)}{W_I} \\ \hline \underset{(NEQ_{II},1)}{W_{II}} \end{bmatrix} \begin{array}{l} \text{(equations associated with the control points)} \\ \text{(equations associated with the pass points)} \end{array}$$

where:

$\ddot{A}$, $\ddot{B}$, $W_I$, $W_{II}$ have the same definitions and detailed expressions as for the first algorithm,

$j$ is the number of stations with observed camera parameters.

$\dot{B}_I$, $\dot{B}_{II}$ are two matrices which represent the partial derivatives of the Collinearity Condition Equations (CCE) with respect to the observed camera parameters and associated with the observed ground control points and ground pass points respectively.

The only unknowns in the case of the second algorithm are the ground coordinates of the pass points $\ddot{\delta}$ which can be calculated directly from:

$$\underset{(3n,1)}{\ddot{\delta}} = N^{-1}U$$

where:

$$N = \ddot{A}'\,\hat{M}_{22}\ddot{A} \quad , \quad U = -(\ddot{A}\;\hat{M}'_{12}\;W_1 + \ddot{A}'\;\hat{M}_{22}W_{II})$$

$$\hat{M}_{22} = (M_{22} - M'_{12}\;M_{11}^{-1}\;M_{12})^{-1}$$

$$\hat{M}_{12} = -M_{11}^{-1}\;M_{12}\;\hat{M}_{22} \quad ,$$

$$M_{11} = \dot{B}_I\;\dot{P}^{-1}\;\dot{B}'_I + \ddot{B}\;\ddot{P}^{-1}\;\ddot{B}' + \dddot{P}_I^{-1} \quad ,$$

$$M_{12} = \dot{B}_I\;\dot{P}^{-1}\;\dot{B}'_{II}$$

$$M_{22} = \dot{B}_{II}\;\mathring{p}^{-1}\;\dot{B}'_{II} + \dddot{p}_{II}^{-1}$$

and $\mathring{p}$ is the weight matrix for the measured camera parameters.

Since the collinearity condition equations are linear with regard to the unknowns, only one iteration is necessary to reach the final solution.

## Aerial Triangulation Test Results

Several aerial triangulation tests were performed using the two computer programs to adjust some of the SKYLAB space photography (scale 1:2,900,000), & a combination of SKYLAB and very high altitude aerial photography (scale 1:120,000) with and without utilization of SKYLAB orbital parameters. Although some test results were given in previous publications (1) and (2), the full details of all the tests and their analysis are given in the author's thesis (3). Here, only a few tests will be described in order to show the efficiency and the capability of the developed programs.

One model of SKYLAB photography (S-190 A camera) covering the areas of Windsor in Canada and parts of the State of Michigan, in the U.S.A., was adjusted using the developed bundle adjustment programs. Seventy six points were identified in the model and their ground coordinates were measured from either 1:25,000 or 1:24,000 scale maps of Canada and the U.S.A. respectively. It was assumed that space photography would not be used for aerial triangulation in areas for which large scale maps exist. Consequently and to simulate real conditions, map coordinates were rounded off to the nearest 100, 200, ... to 1,000 m. The differences (or residuals) between the map coordinates and their rounded values, as well as the Root Mean Square Errors (RMSE) for each case were calculated. Table 1 shows the RMSE's as well as the Absolute Values of the Maximum Residuals (AVMR) for each case.

The map coordinates and their rounded values for 28 well distributed points were used as control in a series of eleven aerial triangulation tests. The differences between the adjusted ground coordinates of the control and pass points and the map coordinates and the RMSE's of these differences were calculated for all tests. Tests no. 1 to 11 in Table 2 show the RMSE's and the AVMR's for the eleven tests when the coordinates of the 28 control points were either the map coordinates or their values rounded to the nearest 100, 200, ... or 1,000 m respectively. Test number 12 in Table 2 shows the RMSE's and AVMR's for one test using the second bundle adjustment algorithm and when the coordinates of the control points were rounded to the nearest 500 m.

## Conclusions

Appropriate applications for space photography are for areas where there is no ground control as such, except point coordinates obtained from small scale maps. Accordingly, in such aerial triangulations the coordinates of the ground control must be treated as observed values and must be adjusted when solving the photogrammetric system. This idea led to the developing of two new bundle adjustment algorithms, which could be efficiently used for the aerial triangulation of space photography and when the available ground control is of inferior quality. In the first algorithm the coordinates of the ground control as well as the photo measurements are used as observations in the collinearity condition equations. Then, point by point, the contributions to the elements of the normal equations are calculated. The system of the normal equations can then be solved yielding the corrections to both the unknown camera parameters and the unknown ground coordinates of the pass points. Due to the nonlinearity of the collinearity condition equations with respect to the unknowns, more than one iteration is necessary to reach the final solution. Finally, point by point, the residuals of the photo-measurements, and the ground measurements (if the points are control points) can be calculated.

In the second algorithm, the camera parameters, the ground coordinates of the control points and the photo measurements are used as observations in the collinearity condition equations. Although the second algorithm can rigorously treat the case of space photography for which all orbital parameters are used, its solution requires more computer time and memory. In this algorithm, the contributions to the normal equations can not be calculated one point after another. Instead, first the contributions of all the control points must be calculated as one unit and then the contributions of all the pass points as another unit. The only unknowns in this algorithm are the ground coordinates of the pass points and hence the collinearity condition equations are linear with respect to these unknowns. Therefore, only one iteration is necessary to reach the final solution. Finally, the residuals for all the measurements and their variance-covariance matrices can be calculated if needed.

During the investigation, the camera parameters supplied to the authors by NASA were not well defined. Moreover, due to some inherent difficulties during the mission the parameters were also not reliable. Accordingly, the camera parameters and their weights resulting from the solution of the first algorithm were used as observations in the second algorithm. Hence, the significant improvement of the accuracy of the results obtained from the second algorithm compared with those of the first algorithm should not be used to draw any final conclusion.

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Table 1 RMSE and AVMR for the rounded control points

| Coord. Rounded to to | Ontario area | | | | | | |
|---|---|---|---|---|---|---|---|
| | Number of points | RMSE | | | AVMR | | |
| | | X (m) | Y (m) | Z (m) | X (m) | Y (m) | Z (m) |
| 100 m | 28 | 27 | 33 | 28 | 48 | 49 | 46 |
| 200 | 28 | 62 | 61 | 64 | 99 | 100 | 97 |
| 300 | 28 | 93 | 85 | 86 | 148 | 140 | 144 |
| 400 | 28 | 122 | 119 | 141 | 190 | 197 | 199 |
| 500 | 28 | 152 | 127 | 182 | 248 | 222 | 246 |
| 600 | 28 | 162 | 174 | 201 | 296 | 300 | 297 |
| 700 | 28 | 210 | 193 | 216 | 346 | 310 | 303 |
| 800 | 28 | 233 | 248 | 240 | 373 | 387 | 372 |
| 900 | 28 | 267 | 254 | 266 | 432 | 449 | 444 |
| 1000 | 28 | 282 | 267 | 278 | 446 | 488 | 461 |
| 500 | 3 | 130 | 157 | 161 | 157 | 222 | 242 |
| 500 | 7 | 165 | 119 | 152 | 247 | 222 | 242 |
| 500 | 14 | 150 | 132 | 154 | 247 | 222 | 242 |
| 500 | 21 | 157 | 132 | 173 | 247 | 222 | 246 |

Table 2 RMSE and AVMR (Ontario model using the first and second bundle adjustment algorithm)

| Serial no. of test | RMSE of control points | | | AVMR of control points | | | RMSE of pass points | | | AVMR of pass points | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | X (m) | Y (m) | Z (m) | X (m) | Y (m) | Z (m) | X (m) | Y (m) | Z (m) | X (m) | Y (m) | Z (m) |
| First bundle adjustment algorithm | | | | | | | | | | | | |
| 1 | 01 | 01 | 00 | 02 | 01 | 00 | 35 | 40 | 148 | 104 | 127 | 394 |
| 2 | 22 | 30 | 28 | 46 | 80 | 45 | 40 | 39 | 162 | 92 | 127 | 394 |
| 3 | 37 | 45 | 62 | 91 | 96 | 99 | 37 | 46 | 128 | 116 | 122 | 374 |
| 4 | 52 | 44 | 84 | 112 | 104 | 151 | 52 | 34 | 148 | 96 | 136 | 397 |
| 5 | 53 | 66 | 138 | 105 | 104 | 220 | 46 | 58 | 194 | 141 | 183 | 532 |
| 6 | 76 | 84 | 175 | 138 | 166 | 243 | 69 | 44 | 130 | 149 | 168 | 325 |
| 7 | 34 | 76 | 196 | 113 | 165 | 292 | 39 | 53 | 191 | 98 | 126 | 385 |
| 8 | 68 | 65 | 209 | 116 | 161 | 313 | 75 | 62 | 205 | 168 | 155 | 386 |
| 9 | 114 | 107 | 228 | 247 | 221 | 337 | 100 | 88 | 210 | 203 | 248 | 406 |
| 10 | 86 | 111 | 241 | 155 | 273 | 396 | 88 | 88 | 193 | 181 | 239 | 385 |
| 11 | 40 | 63 | 280 | 91 | 133 | 474 | 36 | 58 | 286 | 122 | 108 | 588 |
| Second bundle adjustment algorithm | | | | | | | | | | | | |
| 23 | 34 | 53 | 170 | 76 | 159 | 242 | 36 | 41 | 157 | 100 | 129 | 357 |