Commission V

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BUNDLE ADJUSTMENT IN INDUSTRIAL PHOTOGRAMMETRY

Abstract

Two ways of determining initial values for the bundle adjustment in industrial photogrammetry are first explained. One approach is based on solving the 11 parameters of the linear relation between photo and object space coordinates. Those parameters are related to the nine inner and outer orientation elements and two additional parameters. Six control points are needed per photo, or fewer if linear constraints for partly known orientation are added. Another approach is a closed solution for space resection from four control points, or fewer if outer orientation is partly known. It is based on solving the distances between camera station and control points from second degree equations by a search procedure.

The bundle adjustment is performed iteratively: Orientation of photos, intersection of new points, repeated orientation, etc. The method allows for additional parameters, and additional constraints for given spatial data e.g. distances.

Lastly experiences with photogrammetry applied in shipbuilding are given.

Introduction

The bundle method with its general assumptions about camera orientation is becoming more and more popular in analytical photogrammetry. This trend is perhaps more pronounced in industrial than in topographical applications. One reason may be that assumptions such as strip photography with fixed overlaps, near vertical camera axis, constant Z, flat terrain etc., facilitating stepwise adjustment, cannot always be realized in industrial photogrammetry.

Topographic photogrammetry uses also the traditional stepwise aerotriangulation to provide approximate initial values for bundle adjustment. This is due to the fact that no approximate initial values are needed other than those directly derived from the special assumptions such as near vertical photography etc. In addition the stepwise method provides an effective error search.

This paper gives first two approaches for providing approximate initial values for bundle adjustment under more general assumptions. Then an iterative technique to perform the bundle adjustment on a microcomputer is explained. Lastly some practical applications are reported.

The approach of the 11-parameter solution

In [1] the method of direct linear transformation (DLT) between the comparator system x',y' and the object space X,Y,Z is given. The basic equations are:

$$x' = \frac{b_{11}X + b_{12}Y + b_{13}Z + b_{14}}{b_{31}X + b_{32}Y + b_{33}Z + 1}$$

$$y' = \frac{b_{21}X + b_{22}Y + b_{23}Z + b_{24}}{b_{31}X + b_{32}Y + b_{33}Z + 1} \qquad (1)$$

where b's are the 11 transformation parameters. When at least six given control points are imaged in a photo, b's can be solved by least squares (Σvv = min) from the following equations:

$$\begin{aligned} b_{14}+b_{11}X + b_{12}Y + b_{13}Z - x'X\, b_{31} - x'Y\, b_{32} - x'Z\, b_{33} - x' &= v_x \\ b_{24}+b_{21}X + b_{22}Y + b_{23}Z - y'X\, b_{31} - y'Y\, b_{32} - y'Z\, b_{33} - y' &= v_y \end{aligned} \qquad (2)$$

In [2] the functional connection between b's and the orientation parameters is given; in addition two non-linear constraints are derived. In Appendix A these constraints are modified to formulas for direct determination of affinity dM and lack of orthogonality dK, see (A-7). These two additional parameters (with expectations ≈ 0) might be used to detect gross errors. If redudant control is available, gross errors can be located by examining different solutions from different combinations of minimum control. It might, therefore, be an advantage if additional information on inner and outer orientation is available. Introducing this information as linear constraints, would reduce the required number of given control points.

In Appendix A three linear constraints for known camera station (X_0, Y_0, Z_0) are derived, see (A-8). Further, five constraints (with one additional unknown) are derived in Appendix B in the case of known rotations ν,κ (fig.C-3) and inner orientation (including dM,dK). See (B-2),(B-3) and also Table 1. Provided that two axes X,Y of the object system are horizontal, information on ν,κ can be derived directly when the camera is eqiupped with graduation marks to measure rotations. It should be noticed that information on inner orientation alone, does not give linear constraints. Due to this restriction of the 11-parameter solution, another approach might be more favourable.

The approach of closed solution for space resection (known inner orientation)

In [6] a procedure for this solution is given, assuming that the image plane is nearly parallel to the object plane. It is shown in Appendix C how to perform the resection without this assumption, by stepwise determination of

a) distances between camera station and three given control points,
b) camera station, and
c) rotation angles.

A fourth control point must be known to eliminate false solutions. In Appendix D is shown how assumed values of ν,κ can reduce the need for given control points, see also Table 1.

11-parameter solution			Closed solution for resection	
Additional information	Linear constraints	Min. no. control points	Additional information	Min. no. control points
-	-	6	-	4
X_0, Y_0, Z_0	(A-8)	4	X_0, Y_0, Z_0	3
			ν,κ	2
ν,κ,c $x_0=y_0=dM=dK=0$	(B-2),(B-3)	4	ν,κ X_0, Y_0, Z_0	1

Table 1. Minimum control in different cases of additional information

Block triangulation by intersections

In [8] a block triangulation effected by a series of consecutive resections and intersections is proposed. Each iteration cycle solves either the orientation of a single photo (resection) or the coordinates of a single point (intersection). The resections are based on both given points and the points which are determined in the previous intersections. The points are regarded as error free. During the intersections, the orientations from previous resections are regarded as error free. The corresponding series of consecutive solutions converges towards a final solution which is practically equivalent to a simultaneous least squares adjustment. We will show in more detail how this iteration technique can be utilized for bundle adjustment in industrial photogrammetry. On the basis of different approaches of resection and intersection (see [2,9] and also Appendix C), the following procedures are proposed in the two cases of (I) unknown and (II) known inner orientation of a photo.

I) The inner orientation including dM,dK is unknown (the calibration case).

In this case the formulas used in the initial resections and intersections can also be used in the iterative adjustment. Thus, the procedure is:

a) In the resection cycle, the 11 parameters b of a single photo are solved from equations (2) with weight (4), utilizing available control including the triangulation points determined in previous intersections. Initial determination of b's may utilize approximate additional data, see Table 1.

b) In the intersection cycle, the coordinates of the points are solved, point by point, from the following equations derived from (1):

$$
\begin{aligned}
(x'b_{31} - b_{11})X + (x'b_{32} - b_{12})Y + (x'b_{33} - b_{13})Z + x' - b_{14} &= v_x \\
(y'b_{31} - b_{21})X + (y'b_{32} - b_{22})Y + (y'b_{33} - b_{23})Z + y' - b_{24} &= v_y
\end{aligned}
\qquad (3)
$$

$$
P^{-1} = \begin{cases} 1 \text{ if initial determination} \\ (b_{31}X)^2 + (b_{32}Y)^2 + (b_{33}Z)^2 + 1 \end{cases} \qquad (4)
$$

P is weight given to (3). b's in (3) and (4), and X,Y,Z in (4) are known values derived in previous resections and intersections.

II) Known inner orientation (the orientation case).

1. Initial determination

a) The initial resections are performed as closed solutions for space resection, to determine initial orientation data ($X_0,Y_0,Z_0,\kappa,\nu,\alpha$) and the corresponding orientation matrix **a** for a single photo. (Appendixes C,D).

b) The proceeding intersections are executed by the method given in [9] to determine initial coordinate values X,Y,Z of a single point.

2. Final block adjustment by repeated resections and intersections.

Each iteration cycle uses the following linearized equations derived from (A-1), to solve either (a) corrections $dX_0,dY_0,dZ_0,d\kappa,d\nu,d\alpha$ to the previous determined orientation of a single photo (resection), or (b) corrections dX,dY,dZ to the previous determined coordinates of a single point (intersection):

$$v_x = -(x - x_0) + f_x + \frac{\delta f_x}{\delta X_0} dX_0 + \ldots\ldots \frac{\delta f_x}{\delta X} dX \ldots\ldots$$

$$v_y = -(y - y_0) + f_y + \frac{\delta f_y}{\delta X_0} dX_0 + \ldots\ldots \frac{\delta f_y}{\delta X} dX \ldots\ldots \qquad (5)$$

Remarks

- The approach of stage (1a) in the orientation case can be used for initial resections in the calibration case too, if approximate values of inner orientation are available. Initial values of b's are then derived from (A-5).
- In the calibration case, the resection cycle can introduce any given parameter of inner orientation through additional constraints and linearization with approximate values of b's from previous resections. (See Appendix A).
- It is preferable to perform all the initial resections and intersections before entering the final adjustment. An example is given in fig. 1.

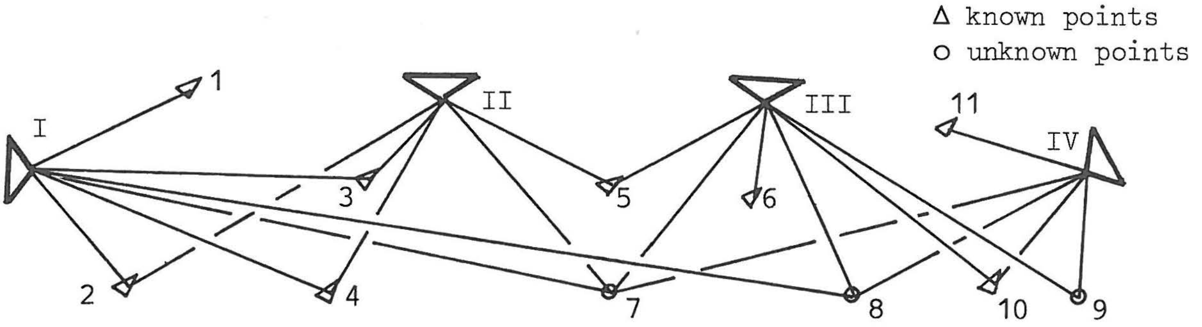

Fig. 1. Initial determination of orientations and points in the sequence: I, II, 7, III, 8, IV, 9. Four control points are needed per photo.

Additional parameters and additional constraints for given spatial data.

The equations (5) to be used in the resection cycle, can be modified to include additional parameters dM_i, dK_i, which are specified for each individual photo i (i = 1,2..n). (See (A-3),(A-4). In the procedure proposed for the calibration case, dM_i, dK_i are related to b's through (A-7)). However, if it is a fundamental assumption, dM_i, dK_i can be forced to obtain the same expected value for all photos by adding following weighted constraints when repeating the resection cycle (for the sake of simplicity, only affinity is regarded):

$$V_{i_k} + d\bar{M}_k = dM_{i_k} , \qquad \text{weight} = P_k$$

where

$$d\bar{M}_k = \begin{cases} 0 & ; k = 1 \\ d\bar{M}_{k-1} + \sum_{i=1}^{n} V_{i_{k-1}}/n & ; k > 1 \end{cases}$$

$$P_k^{-1} = \begin{cases} \text{some introduced value} & ; k = 1 \\ \Sigma V_{i_{k-1}}^2 /(n \cdot m_0^2) & ; k > 1 \end{cases}$$

$d\bar{M}$ = "observation"
V = "observational error"
m_0 = assumed value of standard error of unit weight
k = index for repetition of the resection cycle

Spatial data, G, as distances between points (X,Y,Z) might be introduced in the adjustment as follows:

Let

$$G = f(X,Y,Z) \tag{5}$$

After every repetition of the intersection cycle, the values obtained for X,Y,Z are further corrected by dX,dY,dZ to fulfil (5) and the minimizing condition

$$\Sigma dXdX + \Sigma dYdY + \Sigma dZdZ = \min$$

Practical application

Inspired by the latest efforts to adapt photogrammetry to measurement of marine constructions [3,4,5], the Norwegian Institute of Technology and some Norwegian shipyards decided in 1978 to examine the potentials for applying this measuring technique within the Norwegian shipbuilding industry. In the first phace of the project, this examination has been consentrated on application of analytical photogrammetry in checking the dimensional quality of offshore platforms during construction. The first experiment work was carried out at the Fredrikstad Mekaniske Verksted where an indoor construction of an offshore platform took place. Similar realization of this kind of photogrammetric measurement and obtainable accuracy are reported in [5].

From practical experiences gained so far in the experiment work, and theoretical considerations as given in this paper, the following conclusions are made:

- Flexibility in planning the camera directions, and the positions of camera stations is important to obtain a fast and non-hindering photographing.
- It is difficult to make geodetic measurements of control points without obstructing the construction work, and also to obtain free sights from the camera stations to the targets of those points.
- Therefore, the analytics of the bundle adjustment should, as far as possible, not force restrictions on planning the photography and the control, other than those of pure accuracy reasons.
- During the comparator measurements, on-line computations are almost necessary to ensure a fast and reliable computational reconstruction.

When a single photo has been measured, the following on-line computations are proposed:

a) Averaging of repeated measurements and transformation on fiducial marks.

b) Resection on the basis of all information obtained so far in the process.

c) Intersection of points.

From these computations, the operator can immediately decide on data rejection or remeasurements.

When all the photos of a suitable sub-block are measured, a preliminary block adjustment by intersections may reveal further (smaller) gross errors.

It is possible to perform the on-line computations on a micro computer.

References

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[8] Shmutter, B. and Perlmuter, A. Block triangulation by intersections. Photogrammetria, 35, No 3 - May 1979.

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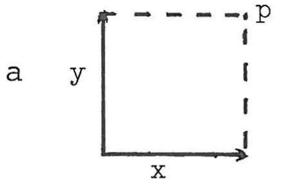
Appendix A. The relationship between the 11 parameters of DLT and the outer and inner orientation elements including two additional parameters.



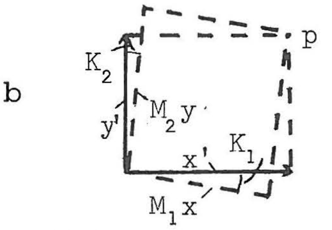


Fig. A-1. a: No image deformation b: Additional image deformation

In fig. A-1a, the orthogonal coordinates x ,y - to be measured if no image deformation is assumed - are related to object space coordinates X,Y,Z as:

$$x - x_0 = (-c)\,\frac{a_{11}(X - X_0) + a_{12}(Y - Y_0) + a_{13}(Z - Z_0)}{a_{31}(X - X_0) + a_{32}(Y - Y_0) + a_{33}(Z - Z_0)}\;(= f_x)$$

$$y - y_0 = (-c)\,\frac{a_{21}(X - X_0) + a_{22}(Y - Y_0) + a_{23}(Z - Z_0)}{a_{31}(X - X_0) + a_{32}(Y - Y_0) + a_{33}(Z - Z_0)}\;(= f_y) \qquad \text{(A-1)}$$

where c,x_0,y_0 are inner orientation elements, X_0,Y_0,Z_0 are camera station coordinates, and a's are the nine elements of an ortogonal matrix dependent on three rotations (see [7] p. 27). The six orthogonality conditions are:

$$\sum_{j=1}^{3} a_{ij} = 1 \;; \qquad i = 1..3$$

$$\sum_{j=1}^{3} a_{ij} \cdot a_{kj} = 0 \;; \qquad (i,k) = (1,2),\ (2,3),\ (1,3) \qquad \text{(A-2)}$$

If, however, the image is deformed in such a way that x ,y are rotated K_1 ,K_2 and multiplied by M_1,M_2 respectively, the measured coordinates are x',y' (fig. A-1b) with following relations to x ,y :

$$x' = M_1\, x\cdot\cos K_1 + M_2\, y\cdot\sin K_2 \qquad y' = -M_1\, x\cdot\sin K_1 + M_2\, y\cdot\cos K_2 \qquad \text{(A-3)}$$

into which the lack of orthogonality dK and affinity dM are introduced by substituting:

$$K_1 = - K_2 = -dK, \qquad M_1 = 1 + dM, \qquad M_2 = 1 - dM \qquad \text{(A-4)}$$

Following equations derived from (1), (A-1) and (A-3), express b's in terms of outer and inner orientation:

$$\lambda = a_{31}X_0 + a_{32}Y_0 + a_{33}Z_0 \qquad \text{(A-5a)}$$

$$b_{1j} = \frac{1}{\lambda}[(c\cdot a_{1j} - x_0 a_{3j})M_1\cos K_1 + (c\cdot a_{2j} - y_0 a_{3j})M_2\sin K_2] \;;\quad j = 1..3 \qquad \text{(A-5b)}$$

$$b_{2j} = \frac{1}{\lambda}[-(c\cdot a_{1j} - x_0 a_{3j})M_1\sin K_1 + (c\cdot a_{2j} - y_0 a_{3j})M_2\cos K_2];\quad j = 1..3 \qquad \text{(A-5c)}$$

$$b_{3j} = -\frac{1}{\lambda}\, a_{3j} \;;\quad j = 1..3 \qquad \text{(A-5d)}$$

$$b_{i4} = - b_{i1}X_0 - b_{i2}Y_0 - b_{i3}Z_0 \;;\quad i = 1,2 \qquad \text{(A-5e)}$$

Following equations to solve x_0, y_0, c, dK, dM as functions of b's are derived from (A-2) and (A-5) (where M_1, M_2, K_1, K_2 are related to dK, dM by (A-4)):

$$\lambda^{-2} = \sum_1^3 b_{3j}^2 \qquad \text{(A-6a)}$$

$$x_0 M_1 \cos\kappa_1 + y_0 M_2 \sin\kappa_2 \;(= x_0') = \lambda^2 \cdot \Sigma b_{1j} \cdot b_{3j} \qquad \text{(A-6b)}$$

$$-x_0 M_1 \sin\kappa_1 + y_0 M_2 \cos\kappa_2 \;(= y_0') = \lambda^2 \cdot \Sigma b_{2j} \cdot b_{3j} \qquad \text{(A-6c)}$$

$$(c \cdot M_1 \cdot \cos K_1)^2 + (c \cdot M_2 \cdot \sin K_2)^2 \;(= c_1^2) = \lambda^2 \cdot \Sigma b_{1j}^2 - x_0'^2 \qquad \text{(A-6d)}$$

$$(c \cdot M_1 \cdot \sin K_1)^2 + (c \cdot M_2 \cdot \cos K_2)^2 \;(= c_2^2) = \lambda^2 \cdot \Sigma b_{2j}^2 - y_0'^2 \qquad \text{(A-6e)}$$

$$-(c \cdot M_1)^2 \sin K_1 \cdot \cos K_1 + (c \cdot M_2)^2 \sin K_2 \cdot \cos K_2 \;(= \Delta) = \lambda^2 \cdot \Sigma b_{1j} \cdot b_{2j} - x_0' y_0' \qquad \text{(A-6f)}$$

First, x_0', y_0', c_1, c_2 and Δ, as introduced in (A-6), are derived as functions of b's. (We notice that $dK = dM = 0$ makes $x_0 = x_0'$, $y_0 = y_0'$, $c = c_1$ or $c = c_2$). Then, eliminating c from (A-6 d,e,f), and introducing (A-4), we get the following equations to solve dM and dK:

$$\sin 2dK = 2\Delta/(c_1^2 + c_2^2)$$

$$dM/(1 + dM^2) = (c_1^2 - c_2^2)/2(c_1^2 + c_2^2)\cos 2dK$$

With good approximation, the solution is

$$dK = \Delta/(c_1^2 + c_2^2) \qquad dM = (c_1^2 - c_2^2)/2(c_1^2 + c_2^2) \qquad \text{(A-7)}$$

Thus, also M_1, M_2, K_1, K_2 are found, see (A-4), so that we can solve c, x_0', y_0' from (A-6 b,c,d).

From (A-5) the outer orientation is found: a's are solved from (A-5 a..d); X_0, Y_0, Z_0 are solved from following equations derived from (A-5 a,d,e) ($b_{34} = 1$):

$$b_{i1}X_0 + b_{i2}Y_0 + b_{i3}Z_0 + b_{i4} = 0 \quad ; \quad i = 1..3 \qquad \text{(A-8)}$$

Remark: The derived expressions $c = f_c(\mathbf{b})$, $x_0 = f_{x0}(\mathbf{b})$, ... can also be considered as additional non-linear constraints between b's for given c, x_0, ...

Appendix B. Derivation of linear constraints used in the 11-parameter solution due to additional information on the orientation.

It will be assumed that ν, κ (fig. (C-3)) and c are known, further that the image coordinates refer to the principal point and are not deformed (see fig. A-1). Thus,

$$x_0 = y_0 = 0 , \qquad K_1 = K_2 = 0 , \qquad M_1 = M_2 = 1 \qquad \text{(B-1)}$$

Consequently, also a_{13}, a_{23}, a_{33} in (A-1) are known, because (see (C-8) and also [7] p. 27):

$$a_{13} = \sin\nu\sin\kappa \qquad a_{23} = \sin\nu\cos\kappa \qquad a_{33} = \cos\nu$$

Thus, introducing (B-1) into (A-5 b,c,d), when j = 3, we get following three linear constraints with one additional unknown $(1/\lambda)$:

$$b_{13} - c\, a_{13}(1/\lambda) = 0 \qquad b_{23} - c\, a_{23}(1/\lambda) = 0 \qquad b_{33} + a_{33}(1/\lambda) = 0 \qquad \text{(B-2)}$$

Introducing (B-1) into (A-5 b,c,d) and using

$$\Sigma a_{i1}\, a_{i3} = 0 \qquad \Sigma a_{i2}\, a_{i3} = 0$$

we obtain the following non-linear constraints:

$$b_{11}\, b_{13} + b_{21}\, b_{23} + c^2 b_{31}\, b_{33} = 0$$

$$b_{12}\, b_{13} + b_{22}\, b_{23} + c^2 b_{32}\, b_{33} = 0$$

However, subsituting b_{13}, b_{23}, b_{33} by their expressions derived from (B-3), we get the following linear constraints:

$$a_{13}\, b_{11} + a_{23}\, b_{21} - c\, a_{33}\, b_{31} = 0$$

$$a_{13}\, b_{12} + a_{23}\, b_{22} - c\, a_{33}\, b_{32} = 0 \qquad \text{(B-3)}$$

Appendix C. Closed solution for resection in space. Determination of (a) distances between camera station and control points, (b) camera position, and (c) rotations.

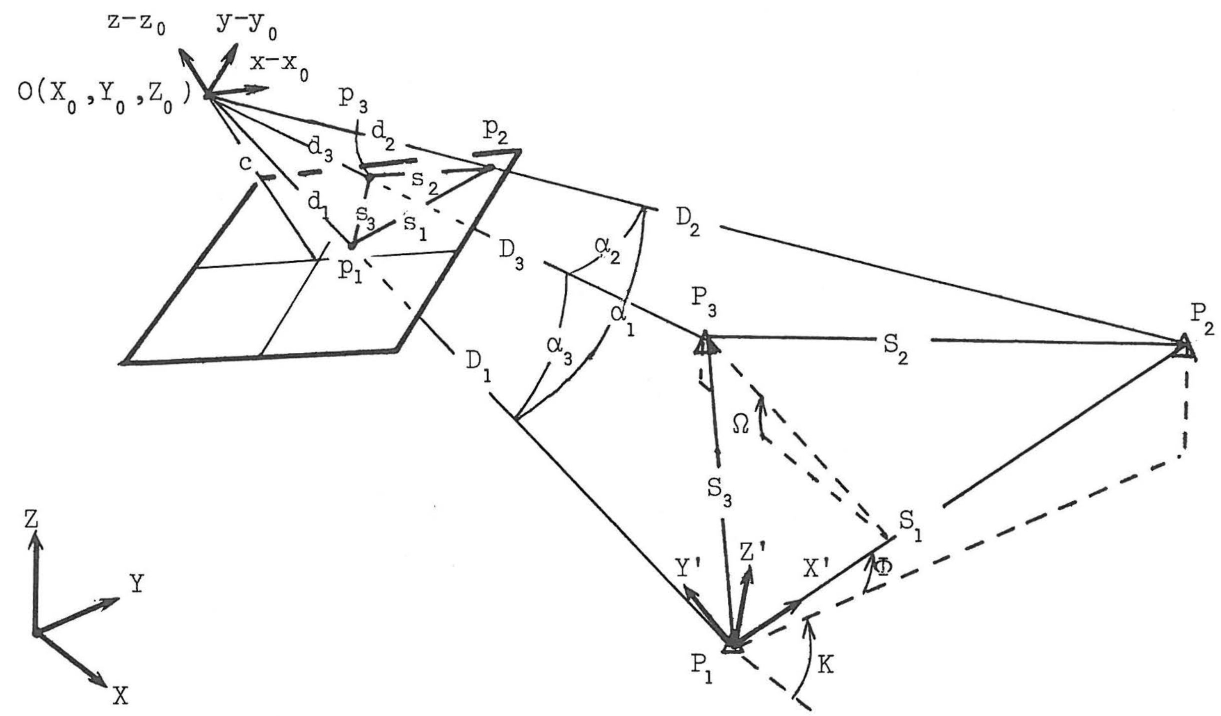


Fig. C-1 Geometry of resection in space

a) From fig. C-1, the following 2. degree equations to solve D's are derived:

$$S_1^2 = D_1^2 + D_2^2 - 2D_1\, D_2\, \cos\alpha_1 \qquad \text{(C-1a)}$$

$$S_2^2 = D_2^2 + D_3^2 - 2D_2\, D_3\, \cos\alpha_2 \qquad \text{(C-1b)}$$

$$S_3^2 = D_1^2 + D_3^2 - 2D_1\, D_3\, \cos\alpha_3 \qquad \text{(C-1c)}$$

where α's are derived from measured image points (x,y) and inner orientation x_0,y_0,c, whilst S's are derived from control points (X,Y,Z). Thus, α_1 and S_1 are:

$$\cos\alpha_1 = (d_1^2 + d_1^2 - s_1^2)/2d_1d_2 \tag{C-1d}$$

with $d_i^2 = (x_i - x_0)^2 + (y_i - y_0)^2 + c^2 \quad ; \quad i = 1,2$

$s_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$$S_1^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 \tag{C-1e}$$

Simular formulas are derived for α_2,α_3 and S_2,S_3.

One approximate distance is known

D's can be solved iteratively from (C-1) with initial approximate value of one distance; say D_1. Solving D_2 and D_3 from (C-1a) and (C-1c), we get:

$$D_2 = D_1\cos\alpha_1 + i\sqrt{-D_1^2\sin^2\alpha_1 + S_1^2} \quad ; \quad i = \pm 1$$

$$D_2 = D_1\cos\alpha_3 + j\sqrt{-D_1^2\sin^2\alpha_3 + S_3^2} \quad ; \quad j = \pm 1 \tag{C-2}$$

Introducing (C-2) into (C-1b), we get one equation of the type:

$$S_2^2 = f(D_1,i,j) \tag{C-3}$$

or linarized

$$S_1^2 - f(D_1,i,j) - \frac{\partial f}{\partial D_1}\Delta D_1 = 0$$

where
$$\frac{\partial f}{\partial D_1} = 2[S_2^2 + (D_3\cos\alpha_2 - D_2)S_1^2\cdot i(-D_1^2\sin^2\alpha_1 + S_1^2)^{-\frac{1}{2}} + (D_2\cos\alpha_2 - D_3)S_3^2\cdot j(-D_1^2\sin^2\alpha_3 + S_3^2)^{-\frac{1}{2}}]/D_1$$

It should be noticed that i,j must be determined by looking for which of the four cases (i,j)=(1,1),(-1,1),(1,-1),(-1,-1) fulfils the inequality condition

$$S_2^2 - f(D_1,i,j) < \Delta S^2 \tag{C-4}$$

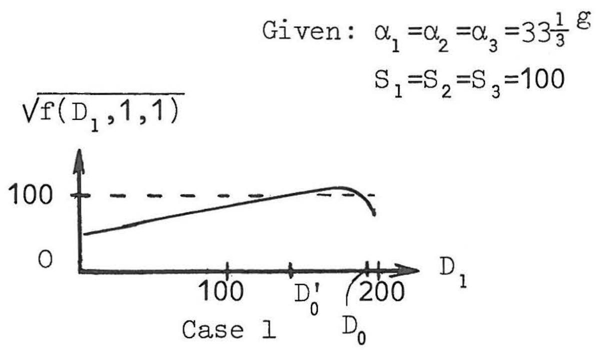
where ΔS is a chosen test value dependent on the accuracy of D_1. In case of ambiguity (see fig. C-2), a fourth control point is needed. If (C-4) is not fulfilled in either of the four cases, some mistake has been made.

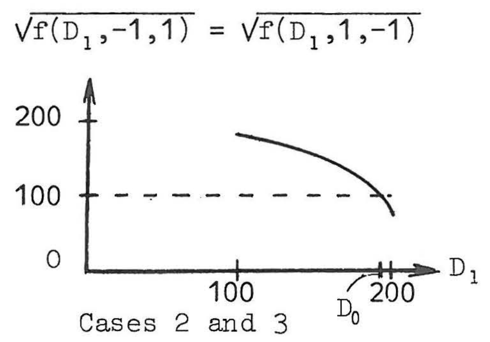
No approximate distances D are known

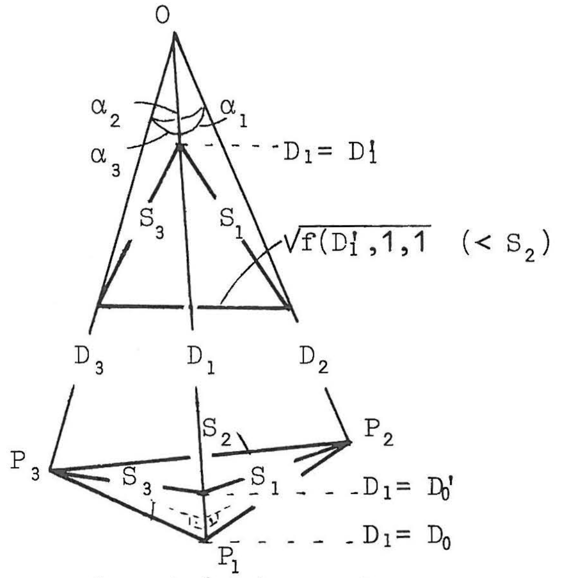
In this case the program itself must search for solutions of D_1,i,j. Because the solutions must make the right hand sides of (C-2) positive and real, the following conditions must be fulfilled:

$$\begin{aligned}
&D_1 < S_1/\sin\alpha_1, \text{ and } i = \pm 1 \text{ if } \alpha_1 < 100^g \\
&D_1 < S_1 \quad, \text{ and } i = 1 \text{ if } \alpha_1 \geq 100^g \\
&D_1 < S_3/\sin\alpha_3, \text{ and } j = \pm 1 \text{ if } \alpha_3 < 100^g \\
&D_1 < S_3 \quad, \text{ and } j = 1 \text{ if } \alpha_3 \geq 100^g \\
&D_1 > S_1 \text{ if } i = -1 \;;\; D_1 > S_3 \text{ if } j = -1 \;;\; \text{else } D_1 > 0
\end{aligned} \tag{C-5}$$

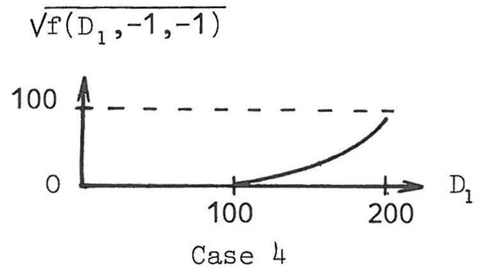
Thus, the search for solutions fulfilling (C-4) might be effected by trying a chosen set of different values for D_1,i,j within the actual range (C-5). To obtain an effective and fast solution, a more sophisticated procedure described in textbooks on numerical analysis could be applied. In fig. C-2, different solutions for D's, obtained in a theoretical example, are shown.







Geometri of case 1



Case	i	j	D_1	D_2	D_3
1	1	1	D_0	D_0	D_0
			D_0'	D_0	D_0
2	-1	1	D_0	D_0'	D_0
3	1	1	D_0	D_0	D_0'
4	-1	-1	No solution		

Solutions in different cases

Fig. C-2. The search for solutions in a theoretical example of resection

b) The coordinates X_0, Y_0, Z_0 of the camera station O are then determined. The corresponding coordinates X_0', Y_0', Z_0' in the auxiliary system X',Y',Z' (fig. C-1) are first solved from the equations:

$$(X_i' - X_0')^2 + (Y_i' - Y_0')^2 + (Z_i' - Z_0')^2 = D_i^2 \; ; \quad i = 1..3$$

where the coordinates X_i', Y_i', Z_i' of the three points P_i are:

P_1 : (0,0,0) $\qquad$ P_2 : $(S_1,0,0)$ $\qquad$ P_3 : $(S_3\cos\beta, S_3\sin\beta, 0)$

with $\cos\beta = (S_2^2 - S_1^2 - S_3^2)/2S_1S_3$. Two solutions for Z_0' are obtained.

X_0', Y_0', Z_0' are then transformed to system X,Y,Z by (C-6) derived from the following three single rotations:

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_K = R_K \begin{Bmatrix} X_2 - X_1 \\ Y_2 - Y_1 \\ Z_2 - Z_1 \end{Bmatrix}, \qquad \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_{K,\Phi} = R_\Phi \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_K , \qquad \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = R_\Omega \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_{K,\Phi}$$

with

$$R_K = \begin{Bmatrix} \cos K & \sin K & 0 \\ -\sin K & \cos K & 0 \\ 0 & 0 & 1 \end{Bmatrix}, \qquad R_\Phi = \begin{Bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{Bmatrix}, \qquad R_\Omega = \begin{Bmatrix} 0 & 0 & 1 \\ \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \end{Bmatrix}$$

$$\mathrm{tg}K = \frac{Y_2 - Y_1}{X_2 - X_1}$$

$$\mathrm{tg}\Phi = \frac{Z_{2_K}}{X_{2_K}} = \frac{Z_2 - Z_1}{(X_2 - X_1)\cos K + (Y_2 - Y_1)\sin K}$$

$$\mathrm{tg}\Omega = \frac{Z_{3_{K\Phi}}}{Y_{3_{K\Phi}}} = \frac{-(X_3 - X_1)\sin\Phi\cos K - (Y_3 - Y_1)\sin\Phi\sin K + (Z_3 - Z_1)\cos\Phi}{-(X_3 - X_1)\sin K + (Y_3 - Y_1)\cos K}$$

Thus,

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = (R_\Omega\, R_\Phi\, R_K)^{-1} \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} + \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} \tag{C-6}$$

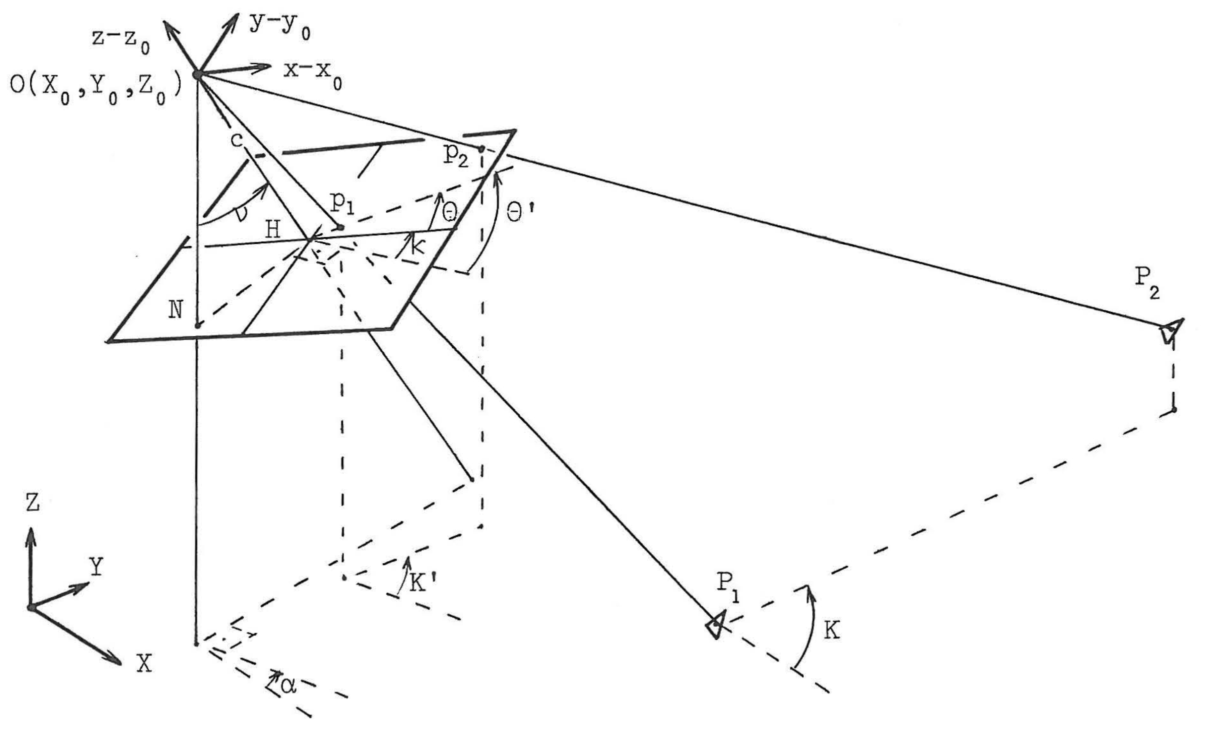


Fig. C-3. Rotations α,ν,κ

c) Now we will show how to derive the rotations α, ν, κ and the corresponding rotation matrix **a**. The image points p_i have the following coordinates in the object space system (see fig.'s C-1 and C-3):

$$X_{p_i} = X_0 + d_i \cos\delta_{x_i}, \quad Y_{p_i} = Y_0 + d_i \cos\delta_{y_i}, \quad Z_{p_i} = Z_0 + d_i \cos\delta_{z_i}; \quad i = 1,3$$

where: $d_i = \sqrt{(x_{p_i} - x_0)^2 + (y_{p_i} - y_0)^2 + c^2}$

$$\cos\delta_{x_i} = (X_0 - X_{P_i})/D_i, \quad \cos\delta_{y_i} = (Y_0 - Y_{P_i})/D_i, \quad \cos\delta_{z_i} = (Z_0 - Z_{P_i})/D_i$$

By the procedure (b) in this appendix, the coordinates X_H, Y_H, Z_H of the principle point H are derived in the object space system on the basis of points $O(X_0, Y_0, Z_0)$ and $p_i(X_{p_i}, Y_{p_i}, Z_{p_i})$, $i = 1..3$, and distances

$$HO = c, \qquad Hp_i = \sqrt{(x_{p_i} - x_0)^2 + y_{p_i} - y_0)^2}; \qquad (i = 1,2)$$

using p_3 to eliminate false solution.

The three single transformations when rotating α, ν, κ are (see fig C-3):

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix}_P = R_\alpha \begin{Bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{Bmatrix}_P, \quad \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}_P = R_\nu \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix}_P, \quad \begin{Bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{Bmatrix}_P = R_\kappa \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}_P \qquad \text{(C-7a)}$$

with

$$R_\alpha = \begin{Bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{Bmatrix}, \quad R_\nu = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & \cos\nu & \sin\nu \\ 0 & -\sin\nu & \cos\nu \end{Bmatrix}, \quad R_\kappa = \begin{Bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

Further we get from fig C-3:

$$\frac{x_p - x_0}{z_p - z_0} = \frac{x_P - x_0}{z_P - z_0}, \quad \frac{y_p - y_0}{z_p - z_0} = \frac{x_P - x_0}{z_P - z_0} \quad \text{with } z_p - z_0 = -c \qquad \text{(C-7b)}$$

Introducing X_H, Y_H, Z_H into (C-7a), we can solve α, ν (see also fig. C-3):

$$tg\alpha = \frac{X_H - X_0}{Y_H - Y_0}, \qquad tg\nu = \frac{Y'_H}{Z'_H} = \frac{-(X_H - X_0)\sin\alpha + (Y_H - Y_0)\cos\alpha}{Z_H - Z_0}$$

Introducing $X_{P_1}, Y_{P_1}, Z_{P_1}$ and $x_{p_1}, y_{p_1}, z_{p_1}$ into (C-7), we can solve κ (see also fig. C-3):

$$tg\kappa = tg(\Theta' - \Theta) = \frac{tg\Theta' - tg\Theta}{1 + tg\Theta' tg\Theta} \qquad \text{with} \qquad tg\Theta = \frac{y_{p_1} - y_0}{x_{p_1} - x_0}$$

$$tg\Theta' = \frac{y'_{p_1}}{x'_{p_1}} = \frac{-(X_{P_1} - X_0)\sin\alpha\cos\nu + (Y_{P_1} - Y_0)\cos\alpha\cos\nu + (Z_{P_1} - Z_0)\sin\nu}{(X_{P_1} - X_0)\cos\alpha + (Y_{P_1} - Y_0)\sin\alpha}$$

Then a (=R) is derived (for more detailed derivation, see [7] p. 27):

$$\begin{Bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{Bmatrix}_P = R \begin{Bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{Bmatrix}_P , \quad R = R_\kappa R_\nu R_\alpha = \begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix} = \begin{Bmatrix} \text{Trig.} \\ \text{funct. of} \\ \alpha,\nu,\kappa \end{Bmatrix} \qquad \text{(C-8)}$$

Introducing (C-8) into (C-7 b) would give (A-1).

Appendix D. Space resection with additional information on outer orientation

Given are the inner orientation (x_0, y_0, c), two control points $P(X_P, Y_P, Z_P)$, corresponding image points $p(x_p, y_p)$ and rotations ν, κ. Unknowns are the camera station $O(X_0, Y_0, Z_0)$ and rotation α. From fig. (C-3) are derived

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}_p = R_\kappa^{-1} \begin{Bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{Bmatrix}_p , \quad \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix}_p = R_\nu^{-1} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}_p , \quad \begin{Bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{Bmatrix}_p = R_\alpha^{-1} \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix}_p , \; z_p - z_0 = -c \qquad \text{(D-1a)}$$

$$\frac{X_P - X_0}{Z_P - Z_0} = \frac{X_p - X_0}{Z_p - Z_0} , \qquad \frac{Y_P - Y_0}{Z_P - Z_0} = \frac{Y_p - Y_0}{Z_p - Z_0} \qquad \text{(D-1b)}$$

where $R_\kappa, R_\nu, R_\alpha$ are the matrices in (C-7). α is derived from (D-1), or fig. C-3 :

$$tg\alpha = tg(K - K') = \frac{tgK - tgK'}{1 + tgKtgK'}$$

$$\text{with } tgK = \frac{Y_{P_2} - Y_{P_1}}{X_{P_2} - X_{P_1}} , \qquad tgK' = \frac{Y'_{p_2} - Y'_{p_1}}{X'_{p_2} - X'_{p_1}} , \; (X'_p, Y'_p \text{ derived by (D-1a)}).$$

Unknown X_0, Y_0, Z_0 are derived from:

$$\frac{X_{P_i} - X_0}{Z_{P_i} - Z_0} = \frac{X_{p_i} - X_0}{Z_{p_i} - Z_0} , \qquad \frac{Y_{P_i} - Y_0}{Z_{P_i} - Z_0} = \frac{Y_{p_i} - Y_0}{Z_{p_i} - Z_0} ; \qquad i = 1,2$$

where $(X_{p_i} - X_0)$, $(Y_{p_i} - Y_0)$, $(Z_{p_i} - Z_0)$ are derived by (D-1a).

In the case of that X_0, Y_0 are given, one control point P_1 is needed. α is:

$$tg\alpha = tg(K - K') = \frac{tgK - tgK'}{1 + tgKtgK'} \quad \text{with } tgK = \frac{Y_{P_1} - Y_0}{X_{P_1} - X_0} , \quad tgK' = \frac{Y'_{p_1}}{X'_{p_1}}$$

where X'_{p_1}, Y'_{p_1} are derived by (D-1a). Z_0 is derived by (D-1b).