DIGITAL ANALYSIS OF ERROR ELLIPSOIDS IN ANALYTICAL PHOTOGRAMMETRIC STEREOTRIANGULATION

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ABSTRACT :

By means of error propagation and error ellipsoids, researche was conducted to perform analyses on the accuracy in three dimensional space of photogrammetric stereotriangulated point positions.
A mathematical model was developped and investigated for error ellipsoids, the main aim being to estatablish an operational procedure for the evaluation of quality of least squares adjustment. Theoretical error study based on fictious photography was also performed, without resorting to tedious sampling techniques (e.g. Monte Carlo method).
The determination of spatial accuracy for selected discrete points requires several steps, involving the evaluation of the characteristic equation of covariance matrix, as well as the computation of their eigenvalues and eigenvectors by an iterative procedure. The semiaxes of rotated ellipsoid were then determined together with their three rotation parameters.
A computer program named EROELIPS was developped and tested with real data to compute the eigenvalues and eigenvectors of each covariance matrix corresponding to selected points, as well as the parameters of standard error ellipsoid in three dimensional space. The program also provides the coordinates of each selected triangulated point in the shifted coordinate system and the parameters of constant probability ellipsoid for different confidence levels.

KEY WORDS: Accuracy, Aerotriangulation, Algorithm, Analytical, Photogrammetry, Theory.

1. INTRODUCTION

The mathematical model developped by the author (Barbalata,1979,1980-a) for multiple station analytical stereotriangulation is based on projective methods (DLT) and the end product of the photogrammetric process is a list of coordinates which define the spatial position of a finite but large number of discrete points. Since the unknown coordinates of the control points were carried in the present solution, the error propagation associated with these also emerges as a byproduct of the solution (Barbalata, 1980-b; Brown D.C.,1955, 1958, 1968, 1976; El Hakim and Faig,1980; El Hakim, 1986; Wong and Elphingstone 1972).

In this context, the method of propagation of variance and covariance has been used to evaluate the accuracy of the quantities determined in a least squares solution (Anderson, 1973; Barbalata, 1978, 1988-a, 1988-b, 1990; Forester, 1975; Gyer and Kenefick, 1970; Kilpela and all, 1981; Kratky and El Hakim, 1983; Li Deren and Shanjie, 1989; Mikhail, 1970; Salsig, 1990).

2. MATHEMATICAL MODEL OF ANALYTICAL TRIANGULATION

If one consider the Bundle Adjustment Method,then the solution is based on the following pair of collinearity equations which express the relation ship between the image coordinates (x,y) of point j on photo i and the corresponding ground coordinates Xj, Yj, Zj of point j:

$$\hat{x}_{ij} = c_i \frac{A_i \Delta X_{ij} + B_i \Delta Y_{ij} + C_i \Delta Z_{ij}}{D_i \Delta X_{ij} + E_i \Delta Y_{ij} + F_i \Delta Z_{ij}}$$
$$\hat{y}_{ij} = c_i \frac{A_i \Delta X_{ij} + B_i \Delta Y_{ij} + C_i \Delta Z_{ij}}{D_i \Delta X_{ij} + E_i \Delta Y_{ij} + F_i \Delta Z_{ij}} \qquad (1\text{-a})$$

where :

$\hat{x}_{ij}$,$\hat{y}_{ij}$ are image coordinates reduced to the approximate principal point and corrected approximately for radial and decentering distortion.

c_i is calibrated principal distance,

A,B,...F are the elements of orthogonal orientation matrix.

$\Delta X_{ij} = X_j - X_i^o$,$\Delta Y_{ij} = Y_j - Y_i^o$, $\Delta Z_{ij} = Z_j - Z_i^o$

in which X_i^o, Y_i^o, Z_i^o are space perspective center coordinates.

If one consider the Direct Linear Transformation, then the solution is based on the following pair of collinearity equations (Barbalata,1979,1980-a)

$$x_{ij} + \Delta x_{ij} = \frac{a_1 X_j + a_2 Y_j + a_3 Z_j + a_4}{a_9 X_{10} + a_{10} Y_j + a_{11} Z_j + 1} = 0,$$
$$y_{ij} + \Delta y_{ij} = \frac{a_5 X_j + a_6 Y_j + a_7 Z_j + a_8}{a_9 X_j + a_{10} Y_j + a_{11} Z_j + 1} = 0 \qquad (1\text{-b})$$

where:

$x_{ij} + \Delta x_{ij}$; $y_{ij} + \Delta y_{ij}$ are image coordinates reduced to the approximate principal points and corrected approximately for radial and decentering distortion.

X_j, Y_j, Z_j are the ground coordinates of point j,

$(a_1, a_2 , \ldots a_{11})$ are the Projective Parameters of Transformation (PTP).

In both cases of collinearity equations (1-a,1-b) these projective equations may be linearized by first-order Taylor expansion to the general form (Ackermann, 1975,1981; Barbalata, 1979; Fraser, 1980):

$$\tilde{V} + \tilde{B}\tilde{\Delta} = \tilde{E}$$

where:

$$\tilde{V} = \begin{bmatrix} v \\ \mathring{v} \\ \bar{v} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & \bar{B} \\ \mathring{B} & 0 \\ 0 & -1 \end{bmatrix}, \quad \tilde{\Delta} = \begin{bmatrix} \Delta \\ \bar{\Delta} \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} E \\ G \\ \bar{E} \end{bmatrix} \qquad (2)$$

V is the vector of residuals (v from collinearity equations, $\mathring{v}$ from constraints equations and $\bar{v}$ from observation equations arising from a priori knowledge regarding the object ground coordinates),

$\tilde{B}$ are the matrices defined by Jacobians,

$\tilde{\Delta}$ is the correction vector of projective parameters (Δ) and object ground coordinates ($\bar{\Delta}$).

$\tilde{\Sigma}$ = Diag.(Σ, $\mathring{\Sigma}$, $\bar{\Sigma}$) is the covariance matrix associated with the merged observation equations (Σ for collinearity equations, $\mathring{\Sigma}$ for constraint equations and $\bar{\Sigma}$ for suplementary equations concerning the object ground coordinates, where their independence is assumed).

The corresponding normal equations will be:

$$\begin{bmatrix} N + \mathring{N} & \hat{N} \\ \hat{N}^T & \bar{N} + \bar{W} \end{bmatrix}^{1} \begin{bmatrix} \Delta \\ \bar{\Delta} \end{bmatrix} = \begin{bmatrix} C + \mathring{C} \\ \bar{C} - \bar{W}\bar{E} \end{bmatrix} \qquad (3)$$

In actual computation the vector Δ in (3) is first solved from a set of reduced normal equations and then the vectors $\bar{\Delta}_j$ are solved for one at a time, (Helmering, 1977; Grün, 1982), respectively :

$$\Delta = \left[N+\mathring{N}-\hat{N}(\bar{N}+\bar{W})^{-1}\hat{N}^T\right]^{-1}\left[C+\mathring{C}-\hat{N}(\bar{N}+\bar{W})^{-1}(\bar{C}-\bar{W}\bar{E})\right]$$
$$\bar{\Delta}_j = (\bar{N}_j+\bar{W}_j)^{-1}(C_j-\bar{W}_j\bar{E}_j) - (\bar{N}_j+\bar{W}_j)^{-1}\hat{N}^T\Delta \qquad (4)$$

An iterative procedure is used; the iteration is stopped when the corrections in matrices Δ and $\bar{\Delta}$ become negligible small.

A computer program called PROJECT 5 was developped by the author and its formulation is based on the principles of the above paragraph (Barbalata, 1980b).

3. ERROR PROPAGATION

Since the unknown coordinates of control points were carried in the present solution, the error propagation associated with these also emerges as a by-product of the solution.
From the partition employed in (3) it follows that the inversion of the general coefficient matrix N of the normal equations can be accomplished by the method of submatrices, i.e.:

$$\tilde{N}^{-1} = \begin{bmatrix} N + \mathring{N} & \hat{N} \\ \hat{N}^T & \bar{N} + \bar{W} \end{bmatrix}^{-1} = \begin{bmatrix} Q & \hat{Q} \\ \hat{Q}^T & \bar{Q} \end{bmatrix} \qquad (5)$$

According to the fundamental definition of the variance-covariance matrix, the covariance matrix of all the triangulated points may be simply expressed as:

$$\bar{\Sigma} = \sigma_0^2 \bar{Q} \qquad (6)$$

where: σ_0^2 is the unit variance estimated from the adjustment and $\bar{Q}$ is the cofactor matrix of all the triangulated coordinates.

From (5) it may be shown easily that the matrix $\bar{Q}$ is given by:

$$\bar{Q} = \bar{N}^{-1} + \bar{N}^{-1}\hat{N}^T Q \hat{N} \bar{N}^{-1} \qquad (7)$$

$$\text{where} \quad Q = \left[N + \mathring{N} - \hat{N}\bar{N}^{-1}\hat{N}^T\right]^{-1} \qquad (8)$$

It is seen from (7) that the cofactor matrix $\bar{Q}$ may be expressed as:

$$\bar{Q} = \bar{N}^{-1} + U \qquad (9)$$

$$\text{in which : } U = \bar{N}^{-1}\hat{N}^T Q \hat{N} \bar{N}^{-1} \qquad (10)$$

The evaluation of the entire cofactor matrix $\bar{Q}$ presents some difficulties. However if one is not interested in the correlations existing between coordinates of different triangulated points, attention may be confined to the evaluation of the cofactor matrix $\bar{Q}_j$ of the coordinates of the individual point j.
For the j-th control point, the cofactor matrix Q would be:

$$\bar{Q}_j = \bar{N}_j^{-1} + \bar{N}_j^{-1} N_j^T Q N_j \bar{N}_j^{-1} \qquad (11)$$

and the covariance matrix :

$$\bar{\Sigma}_j = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{YZ} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 \end{bmatrix}_j = \sigma_0^2 Q_j \qquad (12)$$

4. ERROR ELLIPSOIDS

Error ellipsoids are used to evaluate the accuracy of photogrammetric determination of position in three-dimensional space (Mikhail, 1976; Wong, 1975, 1976).

Although least squares theory of adjustment does not require a specified distribution, most of the statistical testing following the adjustment is concerned with multinormal distribution of random vectors with density function:

$$f(x_1,x_2,\ldots x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{|\Sigma|}} \exp - \frac{(X-\mu_X)^T\Sigma^{-1}(X-\mu_X)}{2} \qquad (13)$$

with mean vector μ_X and covariance matrix Σ.

Referring to equation (13), the function $(X-\mu_X)^T \Sigma^{-1}(X-\mu_X)$, which is a positiv definite quadratic form, represents a family of hyperellipsoids of constant probability.

In the case of three dimensions, the density function becomes:

$$f(x_1x_2x_3) = \frac{1}{(2\pi)^{3/2}\sqrt{|\Sigma|}} \exp - \frac{(x_1x_2x_3-\mu_X)^T\Sigma^{-1}(x_1x_2x_3-\mu_X)}{2} \qquad (14)$$

and the ellipsoid equation for the j-th point, assuming for simplicity $\mu_X=0$, becomes:

$$\text{Hip } (X,Y,Z)_j = \begin{bmatrix} X & Y & Z \end{bmatrix}_j \bar{\Sigma}_j^{-1} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_j = K^2 \qquad (15)$$

The expression $Hip(X,Y,Z)_j=K^2$ from equation (15), for a specific value of K, is an ellipsoid bounded by a parallelipiped with the dimensions $2K\sigma_X$ the direction of X, $2K\sigma_Y$ in the direction of Y and $2K\sigma_Z$ in the direction of Z. Such an ellipsoid is known as an ellipsoid of constant probability and the value of such probability depends on the selected value of K.

In the shifted coordinate system X',Y',Z', considering the ellipsoid with K=1 as a representativ one, the standard ellipsoid is obtained:

$$\frac{X'^2}{\sigma^2_{X'}} + \frac{Y'^2}{\sigma^2_{Y'}} + \frac{Z'^2}{\sigma^2_{Z'}} = 1 \tag{16}$$

the axes of the ellipsoid coincide with the axes of a rotated coordinate system.

The system X',Y',Z' is obtained from the original system X,Y,Z by three rotations ω, φ, $\varkappa$ about the X',Y',Z' axes respectively. With respect to this new system, it becomes clear that $\sigma_{X'}= a$, $\sigma_{Y'} = b$, $\sigma_{Z'}= c$ and the new random variables X',Y',Z' are uncorrelated, such that $\sigma_{X'Y'} =\sigma_{X'Z'}=\sigma_{Y'Z'}=0$. Consequently, through the transformation from X, Y,Z to X',Y',Z' it was possible to replace a set of three dimensions of correlated random variables by another set of the same dimensions of uncorrelated random variables.

The transformation matrix can be constructed by determining the normalized eigenvectors of the covariance matrix and using them as columns. This is equivalent to diagonalizing the covariance matrix Σ_j, since the resulting covariance matrix of the uncorrelated set of random variables X',Y',Z' is always in diagonal form. The elements of this diagonal matrix are the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the original covariance matrix Σ_j and the values of the semiaxes a, b, c, of the ellipsoid can be readily shown to be the square roots of the eigenvalues λ_1, λ_2, λ_3.

In actual computations, the semiaxes a, b, c of error ellipsoid are determined by diagonalizing the covariance matrix Σ_j , respectively:

$$\begin{bmatrix} \sigma^2_{X'} & 0 & 0 \\ 0 & \sigma^2_{Y'} & 0 \\ 0 & 0 & \sigma^2_{Z'} \end{bmatrix} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = R_j^T \bar{\bar{\Sigma}}_j R_j \tag{17}$$

where R is an orthogonal matrix whose columns are the normalized eigenvectors of the matrix Σ_j; λ_1, λ_2, λ_3 are the eigenvalues of the matrix Σ_j and the X',Y',Z' is a rotated coordinate system such that the random variables in the directions of its axes are uncorrelated.

The computational flow itself, used to evaluate the accuracy of photogrammetric determination of position in threedimensional space, is actually quite short and consists in the following major steps :

Let $\bar{\bar{\Sigma}}_j$ be a covariance matrix of the coordinates of the triangulated j point obtained from the least squares adjustment (12). Furthermore, let the eigenvalue equation be defined by the characteristic matrix equation:

$$(\bar{\bar{\Sigma}}_j - \lambda I)X = 0 \tag{18}$$

which represents a set of homogeneous linear equations.

The characteristics equation of the covariance matrix $\bar{\bar{\Sigma}}_j$ is evaluated as:

$$(-\lambda)^3_j + Tr(\bar{\bar{\Sigma}})\lambda^2_j - \left(\begin{vmatrix} \sigma^2_X & \sigma_{XY} \\ \sigma_{YX} & \sigma^2_Y \end{vmatrix}_j + \begin{vmatrix} \sigma^2_Y & \sigma_{YZ} \\ \sigma_{ZY} & \sigma^2_Z \end{vmatrix}_j + \begin{vmatrix} \sigma^2_X & \sigma_{XZ} \\ \sigma_{ZX} & \sigma^2_Z \end{vmatrix}_j \right)\lambda + |\bar{\bar{\Sigma}}|_j = 0 \tag{19}$$

where :
$Tr(\bar{\bar{\Sigma}})=\sigma^2_X+\sigma^2_Y+\sigma^2_Z$ - trace of the covariance matrix
$|\bar{\bar{\Sigma}}|$ = determinant of the $\bar{\bar{\Sigma}}_j$ covariance matrix

Since the matrix $\bar{\bar{\Sigma}}_j$ (12) is symmetric, then the eigenvalues are real and the eigenvectors are all mutually orthogonal, that is:

$$X_i^T X_j = X_j^T X_i = 0 \tag{20}$$

Let the three eigenvectors of the covariance matrix $\bar{\bar{\Sigma}}$ be denoted as $X\lambda_1$, $X\lambda_2$, $X\lambda_3$, respectively:

$$X_{\lambda_1} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_{\lambda_1}, \quad X_{\lambda_2} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_{\lambda_2}, \quad X_{\lambda_3} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_{\lambda_3} \tag{21}$$

For an eigenvalue λ_i, (i=1,2,3), we solve the set of homogeneous linear equations:

$$\begin{bmatrix} \sigma^2_X-\lambda_i & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{YX} & \sigma^2_Y-\lambda_i & \sigma_{YZ} \\ \sigma_{ZX} & \sigma_{ZY} & \sigma^2_Z-\lambda_i \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_{\lambda_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

The eigenvalues and eigenvectors for the matrix $\bar{\bar{\Sigma}}_j$ are evaluated by an iterative procedure which con sists in the following steps:

A. From Eq.(22), for i=1 it results:

$$\begin{aligned} &a)\ \lambda_1 X_1 = \sigma^2_X X_1 + \sigma_{XY} X_2 + \sigma_{XZ} X_3 \\ &b)\ \lambda_1 X_2 = \sigma_{YX} X_1 + \sigma^2_b X_2 + \sigma_{YZ} X_3 \\ &c)\ \lambda_1 X_3 = \sigma_{ZX} X_1 + \sigma_{ZY} X_2 + \sigma^2_Z X_3 \end{aligned} \tag{23}$$

At the first iteration, for $X_1 = 1$, $X_2 = 1$, $X_3 = 1$ from Eq.(23c) we obtain:

$$\lambda_1 = \sigma_{ZX} + \sigma_{ZY} + \sigma^2_Z \tag{24}$$

At the second iteration the computed value of can then be substituted into Eqs.(23a,b) to solve for X_1 , X_2 and λ_1 as below:

$$\begin{aligned} X_1 &= \frac{1}{\lambda_1}(\sigma^2_X + \sigma_{XY} + \sigma_{XZ}) \\ X_2 &= \frac{1}{\lambda_1}(\sigma_{YX} + \sigma^2_Y + \sigma_{YZ}) \\ \lambda_1 &= \sigma_{ZX} X_1 + \sigma_{ZY} X_2 + \sigma^2_Z \end{aligned} \tag{25}$$

The iterations are stopped after the solutions have been checked to be convergent. At the end of the k-th iteration, the solutions are:

$$\begin{aligned} X_1^{1,k} &= \frac{1}{\lambda_1}(\sigma^2_X X_1^{(1,k-1)} + \sigma_{ZY} X_2^{(1,k-1)} + \sigma^2_{ZX}) \\ X_2^{1,k} &= \frac{1}{\lambda_1}(\sigma_{YX} X_1^{(1,k-1)} + \sigma^2_Y X_2^{(1,k-1)} + \sigma_{YZ}) \\ X_3^{1,k} &= 1;\ \lambda_1 = \sigma_{ZX} X_1^{(1,k-1)} + \sigma_{ZY} X_2^{(1,k)} + \sigma^2_Z \end{aligned} \tag{26}$$

B. For the second eigenvector, $X\lambda_1$, from the condition of ortogonality of eigenvectors $X\lambda_1$ and X we can readily write:

$$x_1^{(1,k)}x_1^{(2)} + x_2^{(1,k)}x_2^{(2)} + x_3^{(1,k)}x_3^{(2)} = 0 \qquad (27)$$

from which it results:

$$x_3^{(2)} = -(x_1^{(1,k)}x_1^{(2)} + x_2^{(1,k)}x_2^{(2)}) \qquad (28)$$

For $x_2^{(2)} = 1$, the computed value of $x_3^{(2)}$ from Eq. (28) can then be substituted into Eq. (23b). Hence

$$x_1^{(2)} = \frac{1}{\lambda_2}\left[(\sigma_X^2 - \sigma_{XZ}x_1^{(1,k)})x_1^{(2)} - \sigma_{XZ}x_2^{(1,k)} + \sigma_{XY}\right.$$
$$\lambda_2 = (\sigma_{YX} - x_1^{(1,k)})x_1^{(2)} + \sigma_Y^2 - x_2^{(1,k)} \qquad (29)$$

The system is solved iteratively and for $x_1^{(2,0)} = 1$ it results:

$$\lambda_2^{(0)} = \sigma_{YX} - x_1^{(1,k)} + \sigma_Y^2 - x_2^{(1,k)} \qquad (30)$$

At the second iteration, substituting Eq. (30) into Eqs.(29) it yields:

$$x_1^{(2,1)} = \frac{1}{\lambda_2^{(0)}}(\sigma_X^2 - \sigma_{XZ}x_1^{(1,k)} - \sigma_{XZ}x_2^{(1,k)} + \sigma_{XY}$$
$$x_3^{(2,1)} = -(x_1^{(1,k)}x_1^{(2,1)} + x_2^{(1,k)}) \qquad (31)$$
$$\lambda_2^{(1)} = (\sigma_{YX} - x_1^{(1,k)})x_1^{(2,1)} + \sigma_Y^2 - x_2^{(1,k)}$$

At the end of the k-th iteration, the solutions are:

$$x_1^{(2,k)} = \frac{1}{\lambda_2^{(k-1)}}(\sigma_X^2 - \sigma_{XZ}x_1^{(1,k)})x_1^{(2,k-1)} - \sigma_{XZ}x_2^{(1,k)} + \sigma_{XY}$$
$$x_2^{(2,k)} = 1\;;\; x_3^{(2,k)} = -(x_1^{(1,k)}x_1^{(2,k)} + x_2^{(1,k)}) \qquad (32)$$
$$\lambda_2^{(k)} = (\sigma_{YX} - x_1^{(1,k)})x_1^{(2,k)} + \sigma_Y^2 - x_2^{(1,k)}$$

C. For the third eigenvector $X\lambda_3$, we can readily write from the conditions of orthogonality of eigenvectors $X\lambda_1$ and $X\lambda_3$, respectively $X\lambda_2$ and $X\lambda_3$:

$$x_1^{(1,k)}x_1^{(3)} + x_2^{(1,k)}x_2^{(3)} + x_3^{(1,k)}x_3^{(3)} = 0$$
$$x_1^{(2,k)}x_1^{(3)} + x_2^{(2,k)}x_2^{(3)} + x_3^{(2,k)}x_3^{(3)} = 0 \qquad (33)$$

For $x_1^{(3)} = 1$, it results directly:

$$x_2^{(3)} = \frac{x_1^{(1,k)}x_3^{(2,k)} - x_1^{(2,k)}x_3^{(1,k)}}{x_2^{(1,k)}x_3^{(2,k)} - x_2^{(2,k)}x_3^{(1,k)}}\;,\quad x_3^{(3)} = \frac{x_1^{(1,k)}x_2^{(2,k)} - x_1^{(2,k)}x_2^{(1,k)}}{x_2^{(1,k)}x_3^{(2,k)} - x_2^{(2,k)}x_3^{(1,k)}} \qquad (34)$$

It can be easily derived from Eq.(23c), for i=3, that the eigenvalue λ_3 may be expressed as follows

$$\lambda_3 = \frac{\sigma_{ZX} + \sigma_{ZY}x_2^{(3)} + \sigma_Z^2 x_3^{(3)}}{x_3^{(3)}} \qquad (35)$$

The eigenvectors of the covariance matrix $\bar{\Sigma}j$ are determined with given coefficients of proportionality. Consequently, all the solutions of the system (22) are as follows:

λ_i	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$
λ_1	$x_1^{(1,k)}\ c_1$	$x_2^{(1,k)}\ c_1$	c_1
λ_2	$x_1^{(2,k)}\ c_2$	c_2	$x_3^{(2,k)}$
λ_3	c_3	$x_2^{(3,k)}\ c_3$	$x_3^{(3,k)}$

(36)

D. Finally, for checking up the correctness of the computation, one may use the equality:

$$\lambda_1 + \lambda_2 + \lambda_3 = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2 = \mathrm{Tr}(\bar{\Sigma}_j) \qquad (37)$$

For each symmetric matrix $\bar{\Sigma}j$, there exists a rotation orthogonal matrix R_j such that $R_j^T\bar{\Sigma}_j R_j$ is a diagonal matrix where the elements are the corresponding eigenvalues of $\bar{\Sigma}j$. The columns of R are the normalized eigenvectors $X\lambda_1$, $X\lambda_2$, $X\lambda_3$ of $\bar{\Sigma}j$. Thus:

$$R_j = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}_j = \begin{bmatrix} X\lambda_1 & X\lambda_2 & X\lambda_3 \end{bmatrix}_j^N \qquad (38)$$

where:

$$X\lambda_1^N = \begin{bmatrix} x_1^N \\ x_2^N \\ x_3^N \end{bmatrix}_{\lambda_1} = \begin{bmatrix} x_1/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_2/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_3/\sqrt{x_1^2 + x_2^2 + x_3^2} \end{bmatrix}_j^{(1,k)},$$
$$X\lambda_2^N = \begin{bmatrix} x_1^N \\ x_2^N \\ x_3^N \end{bmatrix}_{\lambda_2} = \begin{bmatrix} x_1/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_2/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_3/\sqrt{x_1^2 + x_2^2 + x_3^2} \end{bmatrix}_j^{(2,k)}, \qquad (39)$$
$$X\lambda_3^N = \begin{bmatrix} x_1^N \\ x_2^N \\ x_3^N \end{bmatrix}_{\lambda_3} = \begin{bmatrix} x_1/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_2/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_3/\sqrt{x_1^2 + x_2^2 + x_3^2} \end{bmatrix}_j^{(3)}$$

and

$$R_j^T\bar{\Sigma}_j R_j = \begin{bmatrix} x_{1.1} & x_{2.1} & x_{3.1} \\ x_{1.2} & x_{22} & x_{3.2} \\ x_{1.3} & x_{2.3} & x_{3.3} \end{bmatrix}_j^N \begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{YX} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 \end{bmatrix}_j *$$
$$* \begin{bmatrix} x_{1.1} & x_{1.2} & x_{1.3} \\ x_{2.1} & x_{2.2} & x_{2.3} \\ x_{3.1} & x_{3.2} & x_{3.3} \end{bmatrix}_j^N = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_j \qquad (40)$$

Furthermore, let the orientation of ellipsoid be defined by the rotations ω, φ and κ about the X', Y' and Z' axes, respectively.

The three rotation parameters can be obtained by identifying the matrices:

$$R = \begin{bmatrix} c\varphi\, c\kappa & c\omega\, s\kappa + s\varphi\, s\omega\, c\kappa & s\omega\, s\kappa - s\varphi\, c\omega\, c\kappa \\ -c\varphi\, s\kappa & c\omega\, c\kappa - s\varphi\, s\omega\, s\kappa & s\omega\, c\kappa + s\varphi\, c\omega\, s\kappa \\ s\varphi & -c\varphi\, s\omega & c\varphi\, c\omega \end{bmatrix} \qquad (41)$$

from which:

$$\varphi = \arcsin r_{3.1}\;;\; \omega = \mathrm{arctg}\left(-\frac{r_{3.2}}{r_{3.3}}\right);\; \kappa = \mathrm{arctg}\left(-\frac{r_{2.1}}{r_{1.1}}\right)$$
$$c = \cos,\quad s = \sin \qquad (42)$$

In accord with (17), the semiaxes a,b,c of the rotated ellipsoid are the square roots of the eigenvalues λ_1, λ_2, λ_3, respectively:

$$a = \sigma_{X'} = \sqrt{\lambda_1}\;;\quad b = \sigma_{Y'} = \sqrt{\lambda_2}\;;\quad c = \sigma_{Z'} = \sqrt{\lambda_3} \qquad (43)$$

For the probability that the point be situated inside or on the ellipsoid defined by (15), where $a = K\sigma_{X'}$, $b = K\sigma_{Y'}$, $c = K\sigma_{Z'}$, the expression is:

$$P\left[\left(\frac{X'^2}{\sigma^2_{X'}}+\frac{Y'^2}{\sigma^2_{Y'}}+\frac{Z'^2}{\sigma^2_{Z'}}\right)<K^2\right]=P\left[\chi^2_{f=3}<K^2\right]=1-\alpha \quad (44)$$

For the standard ellipsoid (K = 1):

$$P\left[\chi^2_{f=3}<1\right]=1-\alpha=0.199 \quad (45)$$

which is obtained from distribution with three degrees of freedom; so the probability that the point be situated inside the standard ellipsoid is $\approx$ 20%.

In order to establish confidence regions, we select the α level and compute the multiplier K. For $\alpha = 0.05$:

$$P\left[\chi^2_{f=3}<\chi^2_{0.05,\,f=3}\right]=P\left[\chi^2_{f=3}<7.815\right]=0.95 \quad (46)$$

from which it results $K = \sqrt{7.815} = 2.7955$. Consequently, the probability that the point be situated within or on an ellipsoid with the axes $a= 2.7955\sqrt{\lambda_1}$, $b=2.7955\sqrt{\lambda_2}$, $c=2.7955\sqrt{\lambda_3}$ is 95%.

Other typical values are:

a) for P=99%: $a=3.368\sqrt{\lambda_1}$, $b=3.368\sqrt{\lambda_2}$, $c=3.368\sqrt{\lambda_3}$

b) for P=99.9%: $a=4.037\sqrt{\lambda_1}$, $b=4.037\sqrt{\lambda_2}$, $c=4.037\sqrt{\lambda_3}$

For each one-dimensional marginal normal distribution, the probability that each variable X, Y, Z lies in the region within plus and minus one standard deviation ($\pm\sigma_X$ or $\pm\sigma_Y$ or $\pm\sigma_Z$), from the normal distribution function, is 68.27%. By contrast, the probability for joint event, which is falling within the standard ellipsoid, is considerable less, being only $\approx$ 20%.

A computer program called EROELIPS was developped by the author and its formulation is based on the above principles. The output of the program EROELIPS provides the eigenvalues, the eigenvectors of each matrix Σ_j and the parameters of standard error ellipsoid in three-dimensional space (a, b, c, ω, φ, κ).

The program also provides the coordinates of the j-th triangulated point in the shifted coordinate system X', Y', Z' and the parameters of ellipsoids of constant probability for different confidence levels.

5. CONCLUSIONS

By means of error propagation and error ellipsoids presented above, it is possible to evaluate the accuracy of photogrammetric determination of positions in three-dimensional space and to conduct theoretical error studies based upon fictitious photography without resorting to tedious sampling techniques (e.g. Monte Carlo method).

From such investigations, the potential accuracy of the multiple station analytical stereotriangulation developped from Bundle Adjustment Method or from Direct Linear Transformation in various situations can be ascertained and also the influence of various distributions of control can be determined.

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