LEAST SQUARES SURFACE FITTING TECHNIQUES FOR DIGITAL ELEVATION MODELS

Dr. Mohsen Mostafa Hassan
M.S.D. Cairo , 11712
Egypt
Commission III

ABSTRACT

Six models for processing DEM data using Least Squares are contrived and tested . These models utilize the patchwise surface fitting technique in the moving coordinate system mode . Three methods are used to evaluate the models according to their statistical properties . These methods involve the determination of variance - covariance matrices , measures of scatter and correlation coefficients matrices . A reliability study is also held to determine the rejection matrices and the smallest detectable error for each model .

INTRODUCTION

Surface fitting is a compact way of storing digital elevation data, it facilitates the determination of intermidiate values between reference points by simple direct substitution . When polynomials are used to fit a set of DEM data points , the deviations between the measured and fitted elevations must be kept reasonably small . The simplecity of polynomials permits this goal to be approached using the least squares method . Least squares patchwise polynomial fitting is usually employed for the following reasons :

i) To enable the use of low order polynomials in fitting as many cells as possible in one patch . This technique results in a reduction of the Computer memory needed to store a given DEM data .

ii) To introduce some smoothing or filtering effects to the data by eleminating a part of the random errors as a by-product of the least squares process .

iii) To ènable the use of statistical testing for comparison and judgement of the performance of the models .

In this paper six least squares patchwise surface fitting models are developed and evaluated .

LEAST SQUARES SURFACE FITTING

The employed fitting technique is based on the determination of the most precise unbiased estimates of the unknown parameters of a chosen polynomial as linear functions of the observations . Least squares surface fitting models can be used to represent DEM data, providing that :

i) The reference points are well distributed that they represent the terrain sufficiently accurate .
ii) The observed elevations are statistically uncorrelated .
iii) The horizontal coordinates of the DEM points are known without error.
iv) The DEM data are free from blunders .
v) The appropriate form and order of the fitting polynomial is chosen .

In order to chose an appropriate surface fitting model , it is necessary to consider the following four parameters :

i) The order and form of the mathematical representation of the surface .
ii) The number and distribution of the reference points .
iii) The type and point of application of the boundary conditions (if any).
iv) The coordinate system used for planimetry .

The well known mathematical model of least squares can be summarized in the following equations :

$$\underset{n\times 1}{V} + \underset{n\times 1}{Z} = \underset{n\times m}{T} \; \underset{m\times 1}{C} \tag{1}$$

$$C = (T^t \; P \; T)^{-1} \; T^t \; P \; Z \tag{2}$$

$$\hat{Z} = T \; C \tag{3}$$

$$V = \hat{Z} - Z = T \; C - Z \tag{4}$$

Where :

V is the vector of residuals .
Z is the vector of the input data, which consists of elèvations and boundary conditions .
T is the matrix of coefficients of the unknowns, which is derived by the partial differentiation of the polynomial w.r.t. the unknowns.
C is the vector of the unknown coefficients .
n is the number of the observation equations .
m is the number of the unknown coefficients .
P is the weight matrix .
$\hat{Z}$ is the vector of the computed values of observations .

THE PLANIMETRIC COORDINATE SYSTEM

Digital Elevation Models data are usually sampled under the same conditions of accuracy and precision. Therefore, all observed elevations can be considered to have the same variance , in which case the weight matrix (P) will be substituted by the Identity matrix . This assumption reduces eqn.(2) to :

$$C = (T^t \; T)^{-1} \; T^t \; Z \tag{5}$$

The term $(T^t \; T)^{-1} \; T^t$ in the above equation depends totally on the choice of the coordinate system. (Hassan,1982) .Valuable computational advantages can be achieved if a local coordinate system is assumed for each patch with

unit spacing between reference points in both directions .
The local coordinate system approach gives rise to the patchwise process which resembles the following merits :

i) Matrix (T) will be the same for every patch , therefore the determination of the term $(T^t\ T)^{-1}\ T^t$ will be performed only once .

ii) The planimetric coordinates of reference and densified points will be the same for all patches , which eleminates the effort of generating new sets of coordinates for each patch .

iii) The part of the statistical analysis which includes the determination of the Cofactor matrices is carried out only once .

iv) Once the elements of the matrix $(T^t\ T)^{-1}\ T^t$ are determined , a simple algorithm can be written to calculate the coefficients of the polynomial for each patch directly from the elevations of the reference points of this patch . The main statements of this algorithm will be of the form :

$$C = \sum_{i=1}^{n} K_i \quad Z_i \tag{6}$$

Where K_i ($i = 1,2,\ldots\ldots,n$) are known coefficients derived from the elements of the above mentioned matrix . These coefficients posses the following properties :

$$\left.\begin{array}{llll} \sum_{i=1}^{n} K_i & = 1 & \text{for} & C_1 \\ \sum_{i=1}^{n} K_i & = 0 & \text{for} & C_j \quad (\ j = 2,3,\ldots\ldots,m) \\ \sum_{i=1}^{n} |K_i| & = 1 & \text{for} & C_j \quad (\ j = 2,3,\ldots\ldots,m) \end{array}\right\} \tag{7}$$

These properties are not applicable for the cases in which boundary conditions are involved .

MATHEMATICAL MODELS AND REFERENCE POINTS PATTERNS

A general form of a polynomial that can be used for DEM surface fitting may be written in the form :

$$Z = \sum_{ij} C_{ij} \quad X^i \quad Y^j \tag{8}$$

In practice some of the terms of this polynomial are used . Selected terms can be used for specific problems to achieve ease of computations and to avoid singularity . Polynomials used in the present research are :

i) The plane polynomial

$$Z = C_1 + C_2\ X + C_3\ Y \tag{9}$$

ii) The bilinear polynomial

$$Z = C_1 + C_2\ X + C_3\ Y + C_4\ X\ Y \tag{10}$$

iii) The 6-term biquadratic polynomial

$$Z = C_1 + C_2 X + C_3 Y + C_4 X^2 + C_5 Y^2 + C_6 X Y \quad (11)$$

iv) The 9-term biquadratic polynomial

$$Z = C_1 + C_2 X + C_3 Y + C_4 XY + C_5 X^2 + C_6 Y^2 + C_7 X^2 Y + C_8 XY^2 + C_9 X^2 Y^2 \quad (12)$$

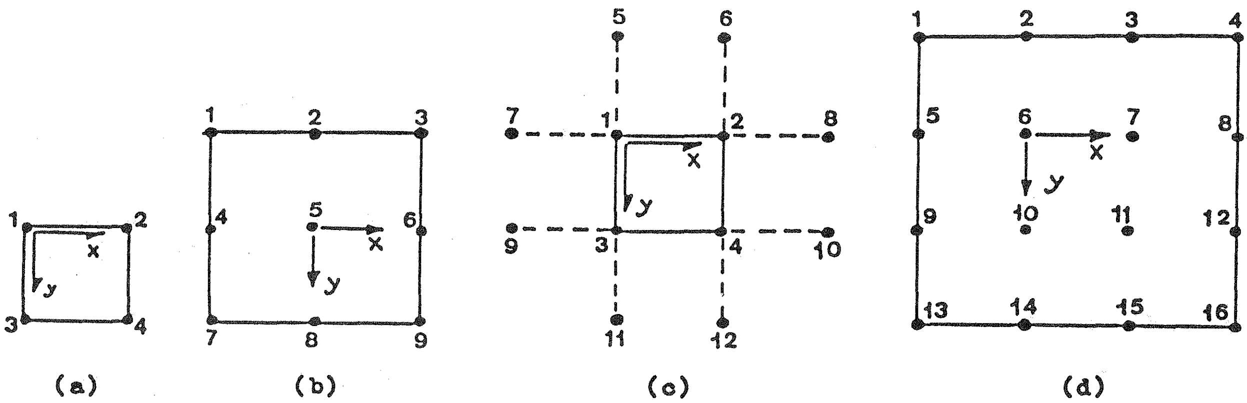

Fig. (1) Reference points patterns and coordinate systems

Model No.(1) :

A plane polynomial of the form of eqn.(9) is fitted to the four elevations of one patch using the coordinate system shown in fig.(1)a

Model No.(2) :

The bilinear polynomial given in eqn.(10) is used to fit the data of the four cells shown in fig.(1)b .

Model No.(3) :

The biquadratic polynomial of eqn.(11) is used to fit the elevations of the nine points shown in fig.(1)b .

Model No.(4) :

The biquadratic polynomial of Model No.(3) is used again to fit the elevations and the boundary conditions of the four corners of one square cell (Jancaitis,et.al.,1973). Three observation equations are written for each point as follows :

$$Z_{i,j} = C_1 + C_2 X + C_3 Y + C_4 X^2 + C_5 Y^2 + C_6 X Y \quad (13)$$

$$\frac{\partial Z}{\partial X} = C_2 + 2 C_4 X + C_6 Y = (Z_{i,j+1} - Z_{i,j-1}) / 2 \quad (14)$$

$$\frac{\partial Z}{\partial Y} = C_3 + 2\,C_5\,Y + C_6\,X = (\,Z_{i+1,j} - Z_{i-1,j}\,)\,/\,2 \qquad (15)$$

A total of 12 equations can be written for each patch :

$$\underset{12x1}{V} + \underset{12x1}{Z} = \underset{12x6}{T}\;\underset{6x1}{C} \qquad (16)$$

The vector Z in this case will consist of elevations and slopes . The elements of this vector are sorted point by point with the order of points shown in fig.(1)c .

Model No.(5) :
The 6-term biquadratic polynomial is used once more to fit the elevations of 9 cells. The points pattern and coordinate system used are shown in fig.(1)d

Model No.(6) :
The biquadratic polynomial given in eqn.(12) is fitted to the elevations of nine square cells using the points pattern and coordinte system of fig.(1)d .

STATISTICAL ANALYSIS

The evaluation of the performance of the six models under consideration is carried out using the following statistical concepts :

- Variance covariance matrices .
- Correlation coefficients matrices .
- Measures of scatter .
- Reliability tests .

VARIANCE - COVARIANCE MATRICES

The variances and covariances of a set of random variables are arranged in a matrix called the variance - covariance matrix . This matrix is symmetrical, its main diagonal consists of the variances of the variables and its off-diagonal elements are the covariances of all possible pairs of these variables. Four variance - covariance matrices can be determined for each model .

i) The variance - covariance matrix of observations (U_{zz})
ii) The variance - covariance matrix of unknowns (U_{cc})
iii) The variance - covariance matrix of residuals (U_{vv})
iv) The variance - covariance matrix of the computed values of observations ($U_{\hat{z}\hat{z}}$) .

Applying the general law of propagation of variances and covariances on equations (2) , (3) & (4) ; (Mikhail, 1976) :

$$U_{zz} = \sigma_o^2 \; P^{-1} \qquad (17)$$

$$U_{cc} = \hat{\sigma}_o^2 \;(\,T^t \; P \; T\,)^{-1} = \hat{\sigma}_o^2 \; N^{-1} \qquad (18)$$

$$U_{vv} = \hat{\sigma}_o^2 \;(\,T \;\; N^{-1} \; T^t\,) - \hat{\sigma}_o^2 \; P^{-1} \qquad (19)$$

$$U_{\hat{z}\hat{z}} = \hat{\sigma}_o^2 \; T \,(T^t \; P \; T)^{-1} \; T^t \tag{20}$$

Where σ_o^2 is the apriori variance of unit weight .

$\hat{\sigma}_o^2$ is the apostriori variance of unit weight .

If a variance - covariance matrix is divided by the variance of unit weight the resulting matrix is called a Cofactor matrix (Q) .

$$\text{e.g.} \qquad Q_{cc} = (T^t \; P \; T)^{-1} \tag{21}$$

CORRELATION COEFFICIENTS MATRICES

In order to evaluate the contribution of each term of the polynomial in the fitting process , correlation coefficients are computed from the cofactor matrix of unknown parameters (Q_{cc}) . These coefficients are arranged in a matrix called the correlation coefficients matrix (R) . The elements of this matrix are computed as follows :

$$r_{i,j} = q_{i,j} \; / \; (q_{i,i} \quad q_{j,j})^{1/2} \tag{22}$$

Where r & q are the elements of (R) & (Q) respectively .
The main diagonal of (R) represents the correlation of a variable with itself which is always unity .

MEASURES OF SCATTER

The variance - covariance matrices (U) provide the required information about the precision (deviation from the mean or scatter) of the variables in terms of their variances and covariances . However, in some cases it may be usefull to have a single value to measure the precision . Two such values are :

i) The generalized variance = determinant of (U)

ii) The total variation = trace of (U) .

Generally, large values of these measures indicate high degree of scatter or low precision . Low values represent concentration about the mean.

RELIABILITY VALUES

Reliability studies are used to evaluate least squares models according to :

i) Their performance w.r.t. the possibility of detecting blunders .

ii) The influence of an undetected gross error on the solution .

According to (Amer , 1981) , it is possible to determine the residuals after the least squares fitting , due to a single known error in the input elevations as follows :

$$\begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{bmatrix} = \begin{bmatrix} & & \\ & Q_{vv} & \\ & & \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{23}$$

Equation (23) determines the residuals (e_i) at the model reference points due to a single gross error (E) at one point only . The application of this equation on model No.(2) with an error (E) at point 3 , gives :

$e_1 = e_9 = -0.14\ E$	$e_2 = e_6 = 0.28\ E$
$e_3 = 0.31\ E$	$e_4 = e_8 = -0.06\ E$
$e_5 = 0.11\ E$	$e_7 = 0.03\ E$

In this model the largest residual occures at the location of the gross error, which gives hope that errors in observations can be detected by the analysis of the residuals . However , in practice this detection can not be atchieved directly due to the following reasons :

i) Blunders may occure in more than one point .

ii) Matrix Q_{vv} is singular, therefore it is not possible to compute the value of the gross error or its point of application from eqn.(23) .

In order to evaluate the fitting models used in this paper from the point of view of blunder detection , a quantity called the boundary value is determined. The boundary value (b_i) at any point (i) is the smallest detectable error at that point .

Another measure for the evaluation of models with respect to the possibility of localizing blunders is the so called Rejection matrices . These matrices are computed by introducing each boundary value as a blunder at the corresponding point and rejecting the residuals greater than a specified value .

RESULTS AND ANALYSIS

Usually the evaluation of the performance of DEM conversion processes depends on the sampling density and terrain type . The least squares technique enables the use of statistical concepts that depends only on the mathematical model . Three Cofactor matrices are determined for each model , namely : Q_{cc} fig.(2) , $Q_{\hat{z}\hat{z}}$ and Q_{vv} . Correlation coefficients between the elements of Q_{cc} are computed and correlation coefficients matrices ,fig.(3) , are constructed . The total variation is computed for the cofactor matrices of each model as shown in table (1) . This table shows that the total variation of (Q_{vv}) is equal to the redundancy of the system , while that of (Q_{zz}) is equal to the number of terms in the polynomial and that of (Q_{zz}) is equal to the number of reference points used . The only usefull values are those of (Q_{cc}) .

Models (1),(3) & (4) have nearly the same degree of scatter while Models (2) & (5) posses the highest precision in determining the unknowns .

The boundary values shown in fig.(4) indicate that errors in elevations at the model inner points can be detected more efficiently than those of the outer points, in other words smaller errors can be detected in the inner points . Model (2) is the best model in its performance with respect to gross error detection .

Model (1) :

$$\begin{bmatrix} .75 & -.5 & -.5 \\ & 1.0 & 0.0 \\ & & 1.0 \end{bmatrix}$$

Model (2) :

$$\begin{bmatrix} .11 & 0 & 0 & 0 \\ & .17 & 0 & 0 \\ & & .17 & 0 \\ & & & .25 \end{bmatrix}$$

Model (3) :

$$\begin{bmatrix} .55 & 0 & 0 & -.34 & -.34 & 0 \\ & .17 & 0 & 0 & 0 & 0 \\ & & .17 & 0 & 0 & 0 \\ & & & .5 & 0 & 0 \\ & & & & .5 & 0 \\ & & & & & .25 \end{bmatrix}$$

Model (4) :

$$\begin{bmatrix} .378 & -.156 & -.156 & .000 & .000 & .111 \\ & .561 & .111 & -.250 & .000 & -.222 \\ & & .561 & .000 & -.250 & -.222 \\ & & & .250 & .000 & .000 \\ & & & & .250 & .000 \\ & & & & & .444 \end{bmatrix}$$

Model (5) :

$$\begin{bmatrix} .215 & .032 & .032 & -.062 & -.062 & .010 \\ & .123 & .010 & .000 & -.062 & -.020 \\ & & .123 & -.062 & .000 & -.020 \\ & & & .062 & .000 & .000 \\ & & & & .062 & .000 \\ & & & & & .040 \end{bmatrix}$$

Model (6) :

$$\begin{bmatrix} .302 & .082 & .082 & .022 & -.137 & -.137 & -.038 & -.038 & .062 \\ & .248 & .022 & .067 & -.137 & -.038 & -.038 & -.113 & .062 \\ & & .248 & .067 & -.038 & -.137 & -.113 & -.038 & .062 \\ & & & .202 & -.038 & -.038 & -.113 & -.113 & .062 \\ & & & & .137 & .062 & .037 & .062 & -.062 \\ & & & & & .137 & .062 & .037 & -.062 \\ & & & & & & .113 & .062 & -.062 \\ & & & & & & & .113 & -.062 \\ & & & & & & & & .062 \end{bmatrix}$$

Fig.(2) Cofactor matrices for unknown parameters Q_{cc}

Model (1) :

$$\begin{bmatrix} 1 & -.577 & -.577 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

Model (3) :

$$\begin{bmatrix} 1 & 0 & 0 & -.63 & -.63 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}$$

Model (2) :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

Model (4) :

$$\begin{bmatrix} 1 & -.339 & -.339 & 0 & 0 & .271 \\ & 1 & .240 & -.81 & 0 & -.540 \\ & & 1 & 0 & -.81 & -.540 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}$$

Model (5) :

$$\begin{bmatrix} 1 & .2 & .2 & -.54 & -.54 & .11 \\ & 1 & .08 & -.71 & 0 & -.29 \\ & & 1 & 0 & -.71 & -.29 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}$$

Model (6) :

$$\begin{bmatrix} 1 & .3 & .3 & .09 & -.67 & -.67 & -.20 & -.20 & .45 \\ & 1 & .09 & .30 & -.75 & -.20 & -.22 & -.67 & .50 \\ & & 1 & .30 & -.20 & -.75 & -.67 & -.22 & .50 \\ & & & 1 & -.22 & -.22 & -.75 & -.75 & .56 \\ & & & & 1 & .45 & .30 & .50 & -.67 \\ & & & & & 1 & .50 & .30 & -.67 \\ & & & & & & 1 & .56 & -.75 \\ & & & & & & & 1 & -.75 \\ & & & & & & & & 1 \end{bmatrix}$$

Fig.(3) Correlation coefficients matrices for (Q_{cc})

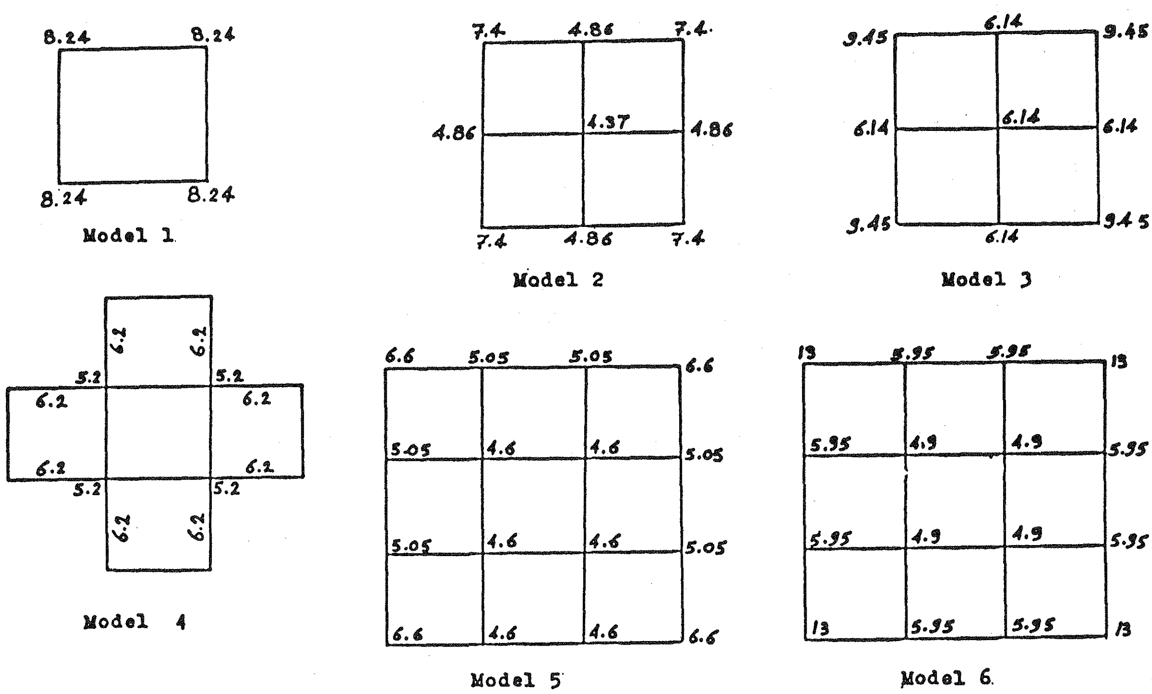

Fig.(4) Smallest detectable gross error (boundary value)

MODEL	TOTAL VARIATION(TRACE U) / $\hat{\sigma}_o^2$			
	Q_{xx}	Q_{cc}	$Q_{\hat{z}\hat{z}}$	Q_{vv}
1	4	2.75	3	1
2.	9	0.69	4	5
3	9	2.139	6	3
4	12	2.444	6	6
5.	16	0.625	6	10
6	16	1.562	9	7

Table (1) Total variations as computed from the Cofactor matices

The correlation coefficients matrices fig.(3) are used to evaluate the chosen mathematical models. Larger values of the elements of this matrix indicate strong correlation between the corresponding terms of the polynomial. Models are analysed as follows :

i) Model No.(1) :

As expected in this case, no correlation occures between the second and third terms of the polynomial. However, relatively strong correlation is observed between the constant term and both the X and Y terms. This correlation indicates that it is possible to eleminate the first term, in which case the degrees of freedom will increase to two and the eqn. will be of the form : $Z = C_1 X + C_2 Y$

The cofactor matrix of unknowns in this case will be :

$$Q_{cc} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

The correlation matrix can then be determined as follows :

$$R = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

This analysis shows that no considerable gain is achieved by the adaptation of this model.

ii) Model No.(2) :

This is a perfectly designed model with no correlation between any of its terms, however, this does not mean that this model should give superior results in practice. In this case the analysis of the variance - covariance matrices will be more usefull.

iii) Model No.(3) :

The value of -.63 may be considered high, in which case the terms including X^2 and Y^2 may be eleminated from the mathematical model which will give Model (2).

iv) Model No.(4) :

The high correlation between the second and fourth terms and also between the third and fifth terms suggests the elemination of the fourth and fifth terms, which reduces the polynomial to the bilinear form with the following cofactor and correlation matrices :

$$Q_{cc} = \frac{1}{9}\begin{bmatrix} 3.4 & -1.4 & -1.4 & 1 \\ & 2.8 & 1 & -2 \\ & & 2.8 & -2 \\ & & & 4 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -.454 & -.454 & -.271 \\ & 1 & .357 & -.598 \\ & & 1 & -.598 \\ & & & 1 \end{bmatrix}$$

v) Models (5) & (6) :

These two models can be treated simillarly, however, as the redundancies are high, the elemination of any term is not practical.

CONCLUDING REMARKS

The analysis of the variance - covariance matrices in not practical in patchwise DEM processing . The apostriori variance of unit weight must be determined for each patch , the mean value of this variance will not give a true representation of the precision of the system . However , it is possible to use the Cofactor and correlation matrices to establish a statistical base for the judgement of the performance of the different models of the patchwise fitting . Correlation coefficients matrices for the cofactors of the models are useful in the analysis and design of the least squares system ; namely , the mathematical model and geometry of the model . The interpretation of the correlation coefficients matrices show that the choice of the number and degree of the polynomial terms is strongly related to the geometry of the system, i.e. the number and arrangement of the reference points .

From the point of view of gross error detection , patches that contain more than one cell are better than those which include only four elevations and boundary conditions for one cell . The more cells included in the patch the higher the probability of gross error detection . The smallest possible detectable gross error occures at the central portion of the patch . Reliability increases with the increase in redundancy and also with the increase of the values of the elements of the main diagonal of Q_{vv} .

REFERENCES

(1) Amer,F.A.,1981, " Theoretical reliability of elementary Photogrammetric procedures " , ITC journal -3 , p: 278 - 307 .

(2) Hassan,M.M.,1982, " An Investigation into the performance of some conversion processes in Digital Elevation Models ", Ph.D. thesis , Cairo University .

(3) Jancaitis,J.R. and Junkins,J.L.,1973, " Modeling Irregular surfaces ", Photogrammetric Engineering,4, p: 413 - 420 .

(4) Mikhail,E.M.,1976, " Observations and Least Squares ", IEP - Dun Donnelly, Harper & Row , New York .