Computation of the Redundant Observation Numbers with Full Weight Matrices

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ABSTRACT

The matrix $Q_{\hat{\varepsilon}}P$ has played an important role in analysis of geodetic and/or photogrammetric system reliability. The purpose of this short paper is to show that the diagonal elements of $Q_{\hat{\varepsilon}}P$ can be larger than one, if the weight matrix P has non-zero off-diagonal elements. An alternative to $Q_{\hat{\varepsilon}'}P$ is proposed, and the derived redundancy should be interpreted in terms of the transformed observations. A simple example of photogrammetric relative orientation completes our discussion.

1. INTRODUCTION

The modern theory of observations or errors in geodesy, photogrammetry and remote sensing considers errors under the three categories: random error, systematic error and blunder or gross error (Mikhail & Ackerman 1976). Random observation errors are not predictable and describe the dispersion of the samplings of an observation from its true, generally unknown value. They have been extensively and deeply studied from the statistical point of view. Systematic errors are essentially deterministic. The effect of systematic errors on the quantities of interest is systematic, and cannot be removed by repeat measurement. Proper remodeling is the key to eliminating this type of effect.

Gross errors are not random. They can be practically caused by a variety of environmental effect, operator mistakes and/or failure of measuring instruments. In geodesy, the percentage of gross errors may be about 1% (Krarup & Kubik 1982), though some precautions and checking have been made during in-site measurement. From the statistical point of view, the observations contaminated by gross errors can not be considered as being the samples from a distribution in question. It has been aware of the danger of gross errors, since even an intermediate blunder may cause a complete failure of the least squares adjustment. So far, there are two ways of handling gross errors: rigorous statistical testing and numerical or robust estimation. For more detail, see e.g. Baarda (1968), Huber (1981), Hampel et al. (1986), Li (1988).

It is of at least equal importance to consider and design some measures to assess possible effect of gross errors at the phase of building adjustment models. Probably, the study of the problem resulted in Baarda's well respected theory of reliability (Baarda 1968, 1973). The reliability of an adjustment model describes the resistance ability of the model against gross errors. It consists of two aspects: internal and external reliability. The former is the measure of the minimum detectable gross error by the (mathematical and stochastic) model to be built, while the latter describes the effect of the maximum undetectable gross errors on the model parameters.

The reliability has since become one of the basic quality measures or criteria in design of geodetic and determination networks (e.g. Schaffrin 1985; Wimmer 1982).

When the weight matrix P of an adjustment model is diagonal, Baarda's internal reliability measure depends on p_i and r_{ii}, in addition to the pre-selected significance level and testing power. Here r_{ii} is the ith diagonal element of the matrix $Q_{w}P$, and is called the redundancy by Förstner (1979). It has since been widely employed as an alternative and important measure of the internal reliability (see e.g. Niemeier 1985; Li 1988). Thus the matrix $Q_{w}P$ is now often named as the reliability matrix. Shan (1988) developed a fast algorithm to compute it.

The purpose of this note is to show that the diagonal elements r_{ii} of the matrix $Q_{w}P$ are not necessarily falling between 0 and 1, if the weight matrix P has off-diagonal elements. In this case, r_{ii} can no longer serve as the redundancy, neither is it proper to call the matrix $Q_{w}P$ the reliability matrix. A tentative alternative is proposed for the redundancy. A simple photogrammetric example completes our discussion.

2. COMPUTING THE REDUNDANCY

2.1 The Independent Observation Model

Let the starting linear (or linearized) photogrammetric (and/or geodetic) adjustment model be

$$
\begin{aligned}
&E(Y) = AX, Y = AX + \varepsilon, \\
&\sigma_{y_i}^2 = \sigma^2/p_i = \sigma_{\varepsilon_i}^2, \\
&\operatorname{cov}(y_i, y_j) = 0, i \neq j = 1,2,\cdots,n
\end{aligned}
\tag{1}
$$

Here $E(\cdot)$ is the operator of mathematical expectation, Y is the observation vector with n elements y_i, A is the design matrix, X

is the unknown vector, ε is the observation error vector, $\sigma_{y_i}{}^2$ is the variance of the ith observation, σ^2 is the (known or unknown) variance component, $\mathrm{cov}(\cdot,\cdot)$ is the operator of statistical correlation. Since the correlations between any two observations are all equal to zero, the observations Y are statistically independent. The linear model (1) can be rewritten in matrix form as follows,

$$\begin{aligned} &E(Y) = AX, Y = AX + \varepsilon \\ &D(Y) = P^{-1}\sigma^2 \end{aligned} \tag{2}$$

where P is the diagonal matrix.

For the linear model (2), the least squares adjusted vector $\hat{Y}$ of the observations Y and the residuals

$$\hat{\varepsilon} = Y - \hat{Y} \tag{3}$$

are estimable. They are invariant with respect to any generalized inverse of the normal matrix $(A^T PA)$. On the other hand, the major interest of this paper is in computation of the redundancy. Without loss of generality, we shall assume in the following that the design matrix A has full column rank. Then (3) becomes

$$\begin{aligned} \hat{\varepsilon} &= Y - AN^{-1}A^T PY \\ &= (I - AN^{-1}A^T P)\varepsilon \\ &= (P^{-1} - AN^{-1}A^T)P\varepsilon \end{aligned} \tag{4}$$

Here $N = A^T PA, Q_{\hat{\varepsilon}} = P^{-1} - AN^{-1}A^T$.

It is obvious from (4) that the residuals $\hat{\varepsilon}$ depends linearly on the observation errors ε. The elements of the matrix $Q_{\hat{\varepsilon}}P$ may reflect to some extent the effect of the errors ε on the residuals $\hat{\varepsilon}$. In particular, the diagonal elements of $Q_{\hat{\varepsilon}}P$ may be a numerical indicator of the effect of ε_i on $\hat{\varepsilon}_i$. Therefore, $Q_{\hat{\varepsilon}}P$ has been usually called the reliability matrix, and the diagonal elements r_{ii} the redundancy, since $0 \le r_{ii} \le 1$ if the weight matrix P is diagonal.

2.2 The Correlated Observation Model

The matrix form of the correlated observation model is

$$\begin{aligned} &E(Y) = AX, Y = AX + \varepsilon \\ &D(Y) = P^{-1}\sigma^2 \end{aligned} \tag{5}$$

where P is the weight matrix with non-zero off-diagonal elements.

Let

$$\begin{aligned} (r_{ij})_{n\times n} &= Q_{\hat{\varepsilon}}P \\ &= \begin{bmatrix} r_1 & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & r_2 & r_{23} & \cdots & r_{2n} \\ r_{31} & r_{32} & r_3 & \cdots & r_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & r_{nn} \end{bmatrix} \end{aligned} \tag{6}$$

where

$$r_{ij} = \sum_{k=1}^{n} q_{ik}p_{kj} = \begin{pmatrix} q_{i1} & q_{i2} & \cdots & q_{in} \end{pmatrix} \begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{pmatrix} \tag{7}$$

with q_{ik} being the elements of the matrix $Q_{\hat{\varepsilon}} (= P^{-1} - AN^{-1}A^T)$.

Since $Q_{\hat{\varepsilon}}P$ is the projection matrix, we have $Q_{\hat{\varepsilon}}P = (Q_{\hat{\varepsilon}}P)(Q_{\hat{\varepsilon}}P)$ or for the ith diagonal element of $Q_{\hat{\varepsilon}}P$,

$$r_i = \sum_{j=1}^{n} r_{ij}r_{ji} = \sum_{j=1, j\neq i}^{n} r_{ij}r_{ji} + r_i^2 \tag{8}$$

Denote

$$c_i = \overline{q}_i^T(\overline{p}_j \otimes \overline{p}_i)\overline{q}_j \tag{9}$$

where

$$\overline{p}_j = \begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{pmatrix}, \overline{q}_j = \begin{pmatrix} q_{j1} \\ q_{j2} \\ \vdots \\ q_{jn} \end{pmatrix}$$

hence (8) becomes

$$r_i = c_i + r_i^2 \tag{10}$$

It can be easily proven that if the absolute value of r_i is less than or at most equal to the unity, we must have

$$c_i \ge 0 \tag{11}$$

which is true, only if the weight matrix P is diagonal. Since (7) reduces in this case to

$$r_{ij} = q_{ij}p_j \tag{12}$$

and

$$\begin{aligned} c_i &= \sum_{j=1, j\neq i}^{n} r_{ij}r_{ji} = \sum_{j=1, j\neq i}^{n} q_{ij}p_j q_{ji}p_i \\ &= \sum_{j=1, j\neq i}^{n} p_i p_j q_{ij}^2 \ge 0 \end{aligned} \tag{13}$$

Unfortunately, (11) is generally not true, if the observations are correlated, since $c_i = \overline{q}_i^T(\overline{p}_j \otimes \overline{p}_i)\overline{q}_j$ is not a positive definite binomial. Thus the matrices $Q_{\hat{\varepsilon}}P$ can not serve as a proper reliability matrix, nor are r_i called the redundancy, since r_i can fall outside 0 and 1.

2.3 An Alternative to $Q_{\hat{\varepsilon}}P$

Assume that the weight matrix P is positive definite. Then we have

$$P = U\Lambda U^T = U\Lambda^{1/2}\Lambda^{1/2}U^T \tag{14}$$

Where U is the orthonormal matrix, Λ is the eigenvalue matrix with positive diagonal elements. Use of U and Y is made to form the following new observations

$$Y' = \Lambda^{1/2}U^TY \tag{15}$$

which are uncorrelated with the unit variance-covariance matrix.

Hence the cofactor matrix of the new residuals $(Y' - \hat{Y}')$ is

$$Q_{\hat{\varepsilon}'} = I - \Lambda^{1/2}U^TAN^{-1}A^TU\Lambda^{1/2} \tag{16}$$

which is clearly a kind of rearrangements of the matrix $Q_{\hat{\varepsilon}}P$. It is clear that $Q_{\hat{\varepsilon}'}$ has all the properties of $Q_{\hat{\varepsilon}}P$, i.e.,

(a) $Q_{\hat{\varepsilon}'}$ is an idempotent matrix,

$$Q_{\hat{\varepsilon}'}Q_{\hat{\varepsilon}'} = Q_{\hat{\varepsilon}'}$$

(b) The total redundant observation number r is equal to the trace of $Q_{\hat{\varepsilon}'}$,

$$r = tr(Q_{\hat{\varepsilon}'}) = tr\{I - \Lambda^{1/2}U^TAN^{-1}A^TU\Lambda^{1/2}\} = n - t = tr(Q_{\hat{\varepsilon}}P)$$

More important is that $Q_{\hat{\varepsilon}'}$ is semi-positive definite, and

$$0 \le Q_{\hat{\varepsilon}'} \le I \tag{17}$$

holds true for any positive definite weight matrix P. Therefore, one may call $Q_{\hat{\varepsilon}'}$ the reliability matrix of the adjustment model (5), and the diagonal elements $r_{ii} = q_i$ the redundancy. One should be careful, however, in interpreting the result. r_{ii} now give a picture of the redundancy of the new, uncorrelated observations y_i' instead of y_i.

3. A SIMPLE EXAMPLE OF PHOTOGRAMMETRIC RELATIVE ORIENTATION

Given an image pair, photogrammetric relative orientation is to solve for the five elements of relative orientation between the left and right photos. The starting point is the coplane equation of three points. The example of this note is to deal with the relative orientation of the consecutive photo connection. The observation equation of a point is given by

$$q = \Delta b_y + {y'}/{f}\,\Delta b_z + {x'y'}/{f}\,\Delta\varphi' + (f + {y'^2}/{f})\Delta\omega' + x'\Delta\kappa' + \varepsilon_q \tag{18}$$

Let n points be chosen to solve for the five unknowns. Then the matrix form of the observation equations is

$$Y = AX + \varepsilon \tag{19}$$

Generally, six points are chosen, and the observations are uncorrelated. In order to illustrate the computation of the redundancy with full weight matrices, however, we employ sixteen points and assume on purpose that the weight matrix for this relative orientation problem has non-zero off-diagonal elements. It is generated by using the correlation function,

$$f(x) = \exp\{-2.5 \times 0.05x^2\} \tag{20}$$

No.	diagonal elements of $Q_{\hat{\varepsilon}}P$	diagonal elements of $Q_{\hat{\varepsilon}'}$
1	0.301785	0.001882
2	0.474687	0.211125
3	**1.148344**	0.034564
4	0.573400	0.277935
5	0.667905	0.123565
6	0.495162	0.722902
7	0.396784	0.852520
8	**1.099681**	0.996166
9	0.781673	0.806883
10	0.804491	0.995447
11	0.980285	0.978314
12	0.427515	0.999783
13	0.756525	0.998937
14	0.273823	0.999977
15	**1.156935**	0.999999
16	0.661005	0.999999
sum	11.000000	11.000000

Table 1 . diagonal elements of matrix $Q_{\hat{\varepsilon}}P$ and $Q_{\hat{\varepsilon}'}$

Listed in column two of Table 1 are the sixteen diagonal elements of the matrix $Q_{\hat{\varepsilon}}P$. Three diagonal elements $(Q_{\hat{\varepsilon}}P)_{ii}$ are larger than unity, with the value of 1.14, 1.09, and 1.15, respectively. Thus, in this case, the diagonal elements of the matrix $Q_{\hat{\varepsilon}}P$ cannot be interpreted in terms of redundancy. $Q_{\hat{\varepsilon}}P$ can no longer serve as the reliability matrix.

Column three of Table 1 gives the sixteen redundancies of the transformed observations. They are clearly smaller than one. It is interesting to note that eight r_{ii} are close to one, which means that only half of the new derived independent observations are employed to solve for the five relative orientation parameters, and another half is almost completely redundant. This is probably due to the high correlation of the original observations.

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REFERENCE

References from Journals:

Baarda W. (1968): A testing procedure for use in geodetic networks, Neth. Geod. Comm., Vol.2, No.5

Baarda W. (1973): S-transformations and criterion matrices, Neth. Geod. Comm., Publ. Geod., New Series, Vol.5, No.1

Förstner W. (1979): Das Programm TRINA zur Ausgleichung und Gütebeurteilung geodätiscer Lagenetze, ZfV, 104, 61-72

Krarup T. & Kubik K. (1983): The Danish method: experience and philosophy, DGK, A98, pp. 131-134

Milbert D. (1985): Network adjustment of correlated coordinate difference observations, In: Proc. first Int. Symp. on Precise

Positioning with GPS, Maryland, USA, pp. 657-665
Niemeier W. (1985): Netzqualität und Optimierung, In: H. Pelzer (ed.), Geodä-tische Netze in Landes- und Ingenieurvermessung II, Konrad Wittwer, Stuttgart, pp. 153-224
Shan J. (1989): A fast recursive method for the repeated computation of the reliability matrix $Q_{vv}P$, Photogrammetia, 43, 337-346.

References from Books:

Deren, Li.(1988): Error Processing and reliability theory, Beijing.
Huber P.J. (1981): Robust statistics, John Wiley & Sons, New York.
Mikhail E.M. & Ackerman F. (1976): Observations and least squares, T.Y. Crowell Company, Inc., New York
Weibin, Huang.(1992): Advanced adjustment theory and its applications, Beijing.
[12] Zhizhou, Wang.(1986): The Principle of Photogrammetry, Beijing.