MATHEMATICAL MODEL FOR FRONTOGENESIS IN TURBULENT FLOW THROUGH POROUS MEDIA

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ABSTRACT

The effects of horizontal density gradient and nonlinear drag offered by the solid particles on turbulent flow through geological media are analysed analytically in the presence of a vertical gravitational field using Reynolds stress analysis. The basic equations are governed by nonlinear Darcy-Forchheimer model and the fluid, which is at rest initially, is accelerated by the baroclinic density field. A purely horizontal motion develops as the isopyenals rotate towards the horizontal. The vertical density gradient decreases exponentially with time but the horizontal density gradient remains unchanged. In the case of linear theory governed by Darcy equation the horizontal velocity has uniform shear but decreases exponentially with time whereas in the case of nonlinear theory governed by Darcy-Forchheimer equation the horizontal velocity has a variable shear both in time and space. It is shown that the flow is stable because the gradient Richardson number decreases monotonically with time to ½.

1 INTRODUCTION

In many geophysical applications like contaminant transport and mobility of petrol involving gravitational effects, the Reynolds number is large of order of 10^5 because they involve gigantic length scale and complicated geometry. At that high Reynolds number the flow in porous media generated instantaneously by horizontal density gradient is turbulent no matter how small it is. The resulting turbulent motion affects the viscosity of the fluid in porous media and the effects of Darcy resistance and Forchheimer drag may either increase or decrease the density gradient. In the extreme case this density gradient may increase to such an extent that an effective discontinuity or front may develop as in the case of non-viscous flow in the absence of porous media (Simpson and Linden 1989). The turbulent flow in the absence of porous media has been extensively investigated both analytically and numerically using direct integration of Navier - Stokes equations. The work on turbulent flow in porous media is very sparse. Recently Rudraiah et al (1983,84,85,86,88,99) and Takatsu and Masuoka (1998) have studied this turbulent problem using Darcy-Lapwood equation. The use of Darcy-Lapwood equations to study flow through porous media poses the problem of under specified system (Beck 1972) when the basic flow is non-quiescent. This can be overcome by using Darcy-Forchheimer equation (Rudraiah and Shivakumara 1999). Study of this turbulent model using Reynolds stress analysis is the main object of the present paper.

2 FORMULATION OF THE PROBLEM

we consider two-dimensional motion of unbounded fluid saturated porous media in the(x, z) – plane with x-axis horizontal having velocity u and z-axis vertical, i.e., anti-parallel to gravity g having velocity w. The basic equations for this incompressible heterogeneous Boussinesq two-dimensional fluid through porous media are the Darcy-Forchheimer equation.

$$\frac{\partial q_i}{\partial t} + \frac{C_b}{\sqrt{k}} \left|q_i\right| q_i = -\frac{1}{\tilde{n}_0}\frac{\partial p}{\partial x_i} - \frac{\tilde{n}}{\tilde{n}_0} g\, \ddot{a}_{i3} - \frac{\acute{\imath}}{k} q_i \tag{1}$$

$$\frac{\partial q_i}{\partial x_i} = 0 \tag{2}$$

$$\frac{\partial \tilde{n}}{\partial t} + q_i \frac{\partial \tilde{n}}{\partial x_i} = 0 \tag{3}$$

The fluid which is initially at rest so that $q_i = 0$ and released from it with uniform initial gradients of density in the horizontal and vertical, with the initial density given by

$$\tilde{n} = \tilde{n}_0 \left(1 - \acute{a}x - \hat{a}z\right), \quad \acute{a} > 0, \quad \hat{a} > 0 \quad \text{at} \quad t = 0 \tag{4}$$

where $\acute{a} > 0$ implies that fluid is heavy to the left and $\hat{a} > 0$ ensures static stability
we use the Reynolds decomposition namely,

$$q_i = \bar{q}_i + q_i^{'}, \qquad \tilde{n} = \bar{\tilde{n}} + \tilde{n}^{'}, \qquad p = \bar{p} + p^{'} \tag{5}$$

where the bar denotes the mean and prime denotes the fluctuation

$$\text{Now} \quad \left|q_i\right| q_i = \left|\bar{q}_i + q_i^{'}\right| \left(\bar{q}_i + q_i^{'}\right) \tag{6}$$

$$\text{But we know that} \quad \left|\bar{q}_i + q_i^{'}\right| \le \left|\bar{q}_i\right| + \left|q_i^{'}\right| \tag{6a}$$

In this paper, for convenience we force equality (for example it will be valid when $\left|q_i^{'}\right| = \lambda_i\, \bar{q}_i$ where $\ddot{e}_i > 0$ or when $q_i^{'}$ and $\bar{q}_i$ both have the same sign)

Then (6) becomes

$$\left|q_i\right| q_i = \left\{\left|\bar{q}_i\right| + \left|q_i^{'}\right|\right\} \left(\bar{q}_i + q_i^{'}\right) \tag{6b}$$

$$\left|q_i\right| q_i = \left|\bar{q}_i\right| \bar{q}_i + \left|\bar{q}_i\right| q_i^{'} + \left|q_i^{'}\right| \bar{q}_i + \left|q_i^{'}\right| q_i^{'} \tag{6c}$$

Applying Reynolds averaging on this we get

$$\overline{|q_i||q_i|} = \overline{|\bar{q}_i||\bar{q}_i|} + \overline{|\bar{q}_i||q_i^{'}|} + \overline{|q_i^{'}||\bar{q}_i|} + \overline{|q_i^{'}||q_i^{'}|}$$
$$= |\bar{\bar{q}}_i||\bar{\bar{q}}_i| + |\bar{\bar{q}}_i|\overline{|q_i^{'}|} + \overline{|q_i^{'}|}\,\bar{\bar{q}}_i + \overline{|q_i^{'}||q_i^{'}|} \tag{6d}$$

By Definition $\bar{\bar{q}}_i = \bar{q}_i$, and $\overline{q_i^{'}} = \overline{q_i - \bar{q}_i} = \bar{q}_i - \bar{\bar{q}}_i = \bar{q}_i - \bar{q}_i = 0$ (6e)

So that (6d) becomes

$$\overline{|q_i||q_i|} = |\bar{q}_i||\bar{q}_i| + \overline{|q_i^{'}||q_i^{'}|} \tag{6f}$$

For the Closure problem, we use the Gradient Diffusion Model namely

$$\overline{|q_i^{'}||q_i^{'}|} = k_m \nabla \bar{q}_i \tag{6g}$$

which after using volume average becomes

$$\overline{|q_i^{'}||q_i^{'}|} = -\frac{k_m}{\sqrt{k}}\bar{q}_i \tag{6h}$$

Then (6f) , using (6h), becomes

$$\overline{|q_i||q_i|} = |\bar{q}_i||\bar{q}_i| - \frac{k_m}{\sqrt{k}}\bar{q}_i \tag{6i}$$

Observing that purely horizontal motion develops when the isopyenals rotate towards horizontal, and hence with no further approximation but under the assumption w=0 equations (1)-(3) with the aid of (5), (6i) and using Reynolds rule of averages become

$$\tilde{n}_0 \frac{\partial \bar{u}}{\partial t} = -\frac{\partial \bar{p}}{\partial x} - \frac{\grave{\imath} K}{k}\bar{u} - \frac{\tilde{n}_0 C_b \bar{u}^2}{\sqrt{k}} \tag{7a}$$

$$\bar{\rho} g = -\frac{\partial \bar{p}}{\partial z}, \tag{7b}$$

$$\frac{\partial \bar{\rho}}{\partial t} = -\bar{u}\frac{\partial \bar{\rho}}{\partial x}, \qquad \frac{\partial \bar{u}}{\partial x} = 0 \tag{7c,d}$$

where $K = \left(1 - \frac{k_m \sqrt{k}\, C_b}{\nu}\right)$ is the modified viscosity due to turbulence. Eliminating $\bar{p}$ between (7a) and (7b) we get

$$\rho_0 \bar{u}_{zt} - g\bar{\rho}_x = -\frac{\mu K \bar{u}_z}{k} - \frac{2\rho_0 C_b \bar{u}\,\bar{u}_z}{\sqrt{k}} \tag{8}$$

where the suffixes t and x denote partial derivatives. Differentiating (8) w.r.t x and using (7d) we get

$$\bar{\rho}_{xx} = 0 \tag{9}$$

This is a necessary condition for the solution of (8) showing that the frontogenesis is associated with the presence of vertical velocity and transverse circulation. We note that the horizontal density gradient may be uniform or non-uniform. For non-uniform horizontal gradient we have to allow vertical velocity in (7c). In this paper however we deal only with constant horizontal density gradient $-\rho_0\alpha$ and see whether frontogenesis is possible. Also, we find the steady state and the nature of the flow when it is departed from the steady state.

3 SOLUTION OF THE PROBLEM

Replacing $\bar{\rho}_x$ in (8) by $-\rho_0\alpha$, and integrating w. r. t z and using the conditions $\bar{u} = \bar{u}_t = 0$ at z=0 we get

$$\bar{u}_t + \frac{\nu K}{k}\bar{u} + \frac{C_b}{\sqrt{k}}\bar{u}^2 = -\alpha g z \tag{10}$$

In the remaining part of this section we find the solution of (10) when the flow is linear valid in the Darcy flow regime and nonlinear flow valid in the Darcy-Forchheimer regime.

3.1 Darcy Flow Regime

In this case, neglecting the nonlinear term compared to Darcy term in (10) and solving the resulting equation using the condition $\bar{u} = 0$ at $t = 0$, we get

$$\bar{u} = \frac{\alpha g k z}{\nu K}\left[e^{-\frac{\nu K t}{k}} - 1 \right] \tag{11}$$

Physically this implies that in the non-steady case the velocity attenuates with viscosity ν, modified viscosity due to turbulence K and small permeability k and varies with the vertical height z in the steady state as $t \rightarrow \infty$. From (11) we see that as $t \rightarrow \infty$ the flow tends to the steady state and hence the system is stable. This can also be proved from the Richardson number analysis as explained below. Solving (7c) using (11) and (4), we get

$$\bar{\rho} = \rho_0 \left(1 - \alpha x - \beta z\right) + \rho_0 \frac{\alpha^2 g k^2}{\nu^2 K^2} z \left[1 - e^{-\frac{\nu K t}{k}} \right] - \rho_0 \frac{\alpha^2 g k}{\nu K} z t \tag{12}$$

This implies that the horizontal density gradient remains at the original value while the vertical stratification continually increase with strength. The isopyenals rotate towards the horizontal with

$$\tan\theta = \frac{\beta}{\alpha} + \frac{\alpha g k}{\nu K}\left(t - \frac{k}{\nu K} \right) + \frac{\alpha g k^2}{\nu^2 K^2} e^{-\frac{\nu K t}{k}} \tag{13}$$

The velocity given by (11) depicted in Figure 1 shows that the vertical shear $\bar{u}_z$ is independent of z but depends on time t only and decreases to zero with time. A measure of the stability of flow is predicted by the gradient Richardson number R_i given by

$$R_i = -\frac{g \bar{\rho}_z}{\rho_0 \bar{u}_z^2} = \frac{\beta \nu^2 K^2 + \alpha^2 g \nu K k t}{g \alpha^2 k^2 \left(e^{-\frac{\nu K t}{k}} - 1 \right)^2} + \frac{1}{\left(e^{-\frac{\nu K t}{k}} - 1 \right)} \tag{14}$$

R_i decreases with time t (although it tends to infinity initially) and reaches 1/ 2 as $t \rightarrow \infty$. Therefore the turbulent flow is linearly stable as in the laminar case (Rudraiah 1999).

3.2 Darcy Forchheimer Regime

In this case (10) after making dimensionless using the scales h, $(\alpha g)^{-1/2}$, $(\alpha g k)^{1/2}$ for length , time and velocity respectively, takes the form

$$\bar{u}_t = -\sigma \eta - \frac{K \sigma \bar{u}}{R_e} - C_b \bar{u}^2, \qquad \text{where } \sigma = \frac{h}{\sqrt{k}},\ R_e = \frac{h\sqrt{\alpha g k}}{\nu},\ \eta = \frac{z}{h} \tag{15}$$

Equation (15) is of the form of Ricatti equation. To solve it, we first let $\bar{u} = a$, a steady state of (15), and obtain

$$a^2 + \text{ã}a + B\text{ç} = 0, \qquad \text{with} \quad \text{ã} = \frac{\text{ó}K}{C_b\ R_e} \quad \text{and} \quad B = \frac{\text{ó}}{C_b} \tag{16}$$

To find the general solution of (15), we use the transformation

$$\bar{u} = a + \frac{1}{v}, \quad \text{where} \quad \frac{1}{v} \ \text{is the departure from the steady state} \tag{17}$$

Then (15), using (16) and (17), becomes

$$v_t - b\ v = C_b, \quad \text{where} \quad b = \frac{\text{ó}K}{R_e} + 2aC_b \tag{18}$$

The solution of (18), using the condition $v = -\frac{1}{a}$ at $t = 0$, is

$$v = \frac{C_b}{b}\left(e^{bt} - 1\right) - \frac{e^{bt}}{a} \tag{19}$$

Finally, the solution of (15), using (16), (17) and (19) is

$$\bar{u} = a\left(1 - \frac{b\,e^{-b\,t}}{d}\right), \quad \text{where} \quad d = a\ C_b\left(1 + e^{-b\,t}\right) + \frac{K\text{ó}}{R_e} \tag{20}$$

This is numerically computed for different values of K and results are depicted in Figure 2. Here also we see that as $t \to \infty$, $\bar{u} \to a$ the steady state and hence it is stable. From (7c) using (20) and after integrating the resulting equation, and using (4) ,we get

$$\bar{\text{ñ}} = \left(1 - \text{á}x - \text{â}z\right) + \frac{\text{á}a\,t}{\text{ó}} + \frac{\text{á}}{\text{ó}C_b}\log\left[1 + \frac{aC_b}{b}\left(e^{-bt} - 1\right)\right] \tag{21}$$

We note that even in this nonlinear case the horizontal density gradient remain at its original value while the vertical stratification varies continually with z and decays exponentially with space and time. Because of the non-linear nature of $\bar{\text{ñ}}$ with z, it is cumbersome to find the angle at which the isopyneals rotate towards the horizontal. Further R_i defined in (14) takes the form

$$R_i = \frac{b\left[-\frac{b\hat{a}}{\acute{a}} - \frac{2}{b} + t + \left(\frac{1+(1-2aC_b t)e^{-bt}}{aC_b e^{-bt} + b - aC_b}\right)\right]}{f_2^2} \tag{22}$$

where $f_2 = 1 - \frac{\left(b + 2aC_b - 2aC_b bt\right)e^{-bt}}{C + aC_b e^{-bt}} - \frac{abC_b e^{-bt}\left[1+(1-2aC_b t)e^{-bt}\right]}{\left(C + aC_b e^{-bt}\right)^2}$, $C = b - 2aC_b$

We note that in contrast to the linear case given by (14), the R_i in the nonlinear theory depends exponentially both on z and t and approaches a constant value as $t \to \infty$. In this case the flow may be stable or unstable depending on the value of R_e.

CONCLUSIONS

Both linear and nonlinear turbulent flow are discussed analytically using Reynolds stress approximation. The effect of turbulence is found to decrease the viscosity effect by an amount given by K. We found that horizontal motion prevails both in the linear and nonlinear cases as isopyenals rotate towards the horizontal because the horizontal density gradient remains unaltered while the vertical density gradient decreases exponentially with time. Analysis reveal that the vertical shear is uniform in space in the case of linear theory while it is non-uniform in space in the case of nonlinear theory. We conclude that the flow in the Darcy regime is monotonically stable because it tends to a constant value as $t \to \infty$, while in the case of nonlinear theory the flow is stable or unstable depending on the value of R_e. Finally we conclude that in the case of constant horizontal density gradient frontogenesis cannot occur and to predict frontogenesis we have to allow vertical flow in the equation (7c). The work in this direction is in progress.

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NOMENCLATURE

$x_i = (x,z)$ when i=1 and i=3	rectangular coordinates
$q_i = (u,w)$ when i=1 and i=3	velocity in x and z directions
t	Time
p	Pressure
ρ	Density
g	Gravity
k	Permeability
ν	Kinematic viscosity
C_b	Drag coefficient
δ_{i3}	Kronecker delta
$\bar{\ }$	Mean
$'$	Fluctuation
k_m	Eddy diffusivity
K	Modified viscosity due to turbulence
σ	Porous parameter
Re	Reynolds number
R_i	Richardson number

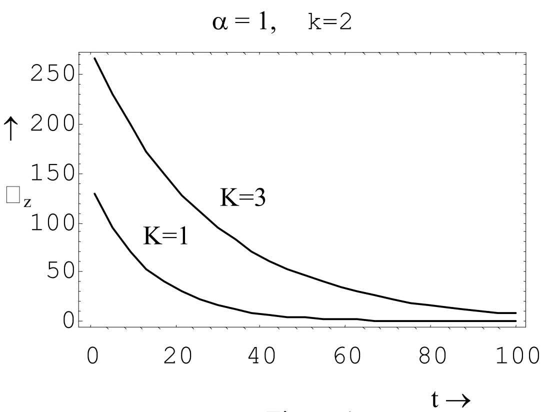

Figure 1

Velocity Vs Time for Linear case

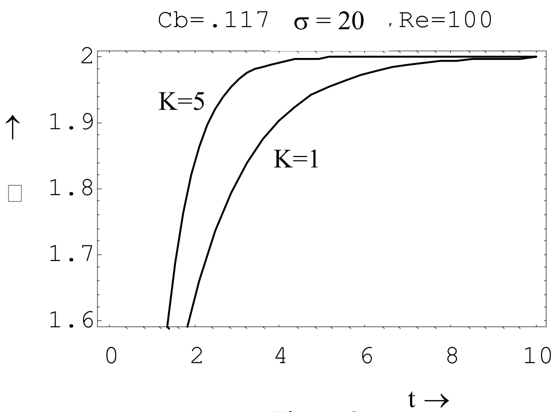

Figure 2

Velocity Vs Time for Nonlinear case