ASYMMETRICAL REPRESENTATION OF TOPOLOGICAL RELATIONSHIPS

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ABSTRACT:

Topological relationships are the most important spatial relationships in geographical information science (GIS). There are many models of spatial relationships; the most popular model of them is 9-intersection model (9IM). There are two main problems in 9-intersection model: topological paradox and linear dependency. We solve the problem of topological paradox by using the lower dimensional part (LDP) to replace the boundary of a spatial entity, and the same dimensional part (SDP) to replace the interior of a spatial entity. Linear dependency causes the implementation of 9-intersection model very complex and no calculation formula can be given out. In order to reducing the influence of linear dependency, we only divide one of the two spatial entities into three parts: SDP, LDP and exterior. Next, we use combination of the dimensions of intersections between one entity and the interior (or boundary, exterior) of the other entity to represent the topological relationships between the two spatial entities. This method is simple in realization, and the calculation formulas are given.

1. INTRODUCTION

Topological relationships are the basis of lots of operations executed in GIS, such as including, adjacent, equal, intersecting, connecting and their suitable description and maintenance. There are many models about the definitions and verification of topological relationships, such as 4IM (Pullar, 1988), 9IM (Egenhofer and Herring, 1991), DEM (Clementini et al., 1993), CBM (Clementini et al., 1993), V9I (Chen et al., 2001). Among these models, 9IM provided by Egenhofer is the most popular model. Many researchers have adopted the formal method based on set topology to identify topological relationships (Egenhofer and Herring, 1990; Molenaar, 1998), but the representations of topological relationships have not been studied sufficiently. For example, in 9IM (Egenhofer and Franzosa, 1991), the possible combinations of topological relationships are 512 kinds, but most of them are impossible, so it is very complex to verify the topological relationships. (Zlatanova 1999) applies 9I model to distinguish the topological relationships between 3D spatial entities; in order to eliminate the impossible conditions, 25 constraint rules are given.

In 9IM, each of the two spatial entities is divided into three elements: interior, boundary and exterior, so it is a symmetrical method. This leads to the problems of topological paradox and linear dependency. Linear dependency makes the implementation of 9IM very complex (Chen et al., 2001).

Hence, we provide an asymmetrical representation method of topological relationships. Comparing to 9IM, it has two advantages: (1) there are only two constraint rules to eliminate the impossible conditions, which are very simple and straight; (2) the calculation formulas are given.

This paper is organized as: Section 2 introduces the representation of topological relationships. Section 3 describes the asymmetrical representation method. Examples are given in section 4. The last section concludes the paper.

2. THE REPRESENTATION OF TOPOLOGICAL RELATIONSHIPS

A **topology space** is usually described as a set of any element, in this set the concept of continuity is defined (Clementini et al, 1995). Let X and Y are two topological spaces, then:

Mapping f: $x \to f$ is continuous, if for every open subset V of Y, $f^{1}(V)$ is a open subset of X.

If f is bijection, at the same time, f and f^{1} are continuous, then f is called **topological isomorphism**. Topological isomorphism keeps the adjacent relationships between the mapping points, which includes transform, rotation and zoom. Topological relationships are those relationships that keep invariant under topological isomorphism.

In topological space X, the neighbour of x is a subset of U, and U is an open set that includes x. If set A is a neighbour of its element x, then x is a interior point of A, the set of all the interior points is the interior of A, denoted as **A°** (Kelly 1955). The **boundary** of A is **∂A** := A – A°, the exterior of A is A^{-} = X – A.

We give out the definitions of simple spatial entities in $\mathbb{R}^3$ as follows:

- A simple volume entity is the closure of the connected 3D point set embedded in $\mathbb{R}^3$.
- A simple face entity is the connected 2D point set embedded in $\mathbb{R}^3$, no self-intersection, including one and only one cycle.
- A simple line entity is the connected 1D point set embedded in $\mathbb{R}^3$, no self-intersection, including two and only two end points.

- A simple point entity is the 0D point set in $\mathbb{R}^3$, and composed of only one point.

The representation of the topological relationships between two spatial entities can be divided into two steps:
(1) "**Spatial entity→elements**": here the spatial entities is the simple entities like above, the divisions of two spatial entities may be different, for example, spatial entity A may be divided into 2 elements, but spatial entity B may be divided into 3 elements.
(2) "**Combination, operation and evaluation between elements**": we first determine which elements should be combined, and which kind of operation can be used between two elements, and which topological variants are used to evaluate the result of the operation. There are three common topological invariants: intersection invariant, dimension invariant, and connected components invariant.

2.1 The Division of a Spatial Entity

There are many ways to divide a spatial entity, such as according to area, or the topological concepts "interior, boundary and exterior". 9IM uses the latter to divide the spatial entities. The basic topological convention is that the boundary of a spatial entity separates its interior from its exterior. However, the definition of topological convention holds to be true only in a space with a particular dimension. Otherwise, a topological paradox will appear (Li et al., 2000). For example, if a 1-D entity is embedded in 2-D (a higher dimension) space while the same definitions in 1-D are simply adopted, then a paradox appears. The interior of the 1-D entity meets its exterior. This is also the case if a 2-D entity is embedded in a 3-D space while the same definitions in 2-D are simply adopted.

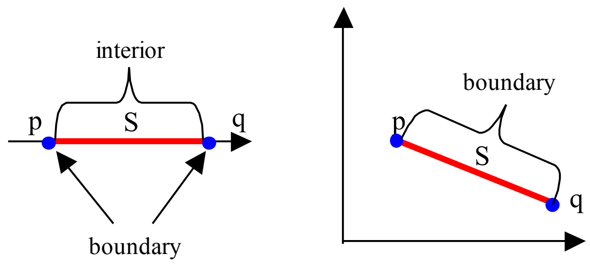


Figure 1. The interior and boundary of a line change according to the dimension of embedded space

To keep the topological consistence, we give another topological interpretation of the concepts "interior, boundary and exterior", that is, we use A° to denote the parts of A that have the points with the dimension equal to dim(A), and ∂A to denote the parts left of A , i.e.:

$$A^\circ = \{x \in A \mid \dim(x) = \dim(A)\}$$
$$\partial A = \{x \in A \mid \dim(x) < \dim(A)\}$$

The dimension of a point can be gotten from the Dimension Theory of Ohliso.

2.2 The Linear Dependence in 9IM

The use of its complement as the exterior of an entity causes the linear dependency between its interior, boundary and exterior. For a spatial entity A and B in space U, we can get:

$$U = A^\circ + \partial A + A^-.$$

$$U = B^\circ + \partial B + B^-$$

We let **op** represent some set operation (such as ∪, ∩ or -). Then,
(1) "A° **op** B°" is influenced by "∂A op B° " and "A^- op B°".
(2) "A° **op** ∂B" is influenced by "∂A op ∂B" and "A^- op ∂B".
(3) "A° **op** B^-" is influenced by "∂A op B^-" and "A^- op B^-".

On the other hand,
(1) "B ° **op** A°" is influenced by "∂ B op A° " and "B^- op A°".
(2) "B ° **op** ∂A" is influenced by "∂ B op ∂A" and "B^- op ∂A".
(3) "B ° **op** A^-" is **influenced** by "∂ B op A^-" and "B^- op A^-".

These conditions cause the judgment of the possible topological relationships is very difficult. However, "A° **op** B°" cannot be determined by "∂A **op** B°" and "A^- **op** B°", that means the exterior contributes to the definition of the topological relationships. Chen et al. uses the Voronoi-regions of an entity to replace its complement as its exterior (Chen et al., 2001). We use the asymmetry method to decrease the influence of linear dependence, that is, we only divide one of the spatial entities and leave the other as a whole. In general, we only divide the entity with higher dimension because we believe the higher the dimension is, the more complex the entity is. For the combination of two elements, we use intersection operation, intersection invariant and dimension invariant.

3. AN ASYMMETRICAL METHOD TO REPRESENT TOPOLOGICAL RELATIONSHIPS

3.1 The Definition of the Asymmetrical Method

Suppose in N dimensional topological space, there are two simple spatial entities A and B, such that 0 < dim(A) ≤ dim(B). (For dim(A)=0, there are only three topological relationships between A and B: in, on and out, so we omit it.) Then the topological relationship R(A, B) between A and B can be represented by a 3-tuple **<En, Bn, In>**, such that:

a) **En**:= $\dim(A \cap B^-) \in \{-1, \dim(A)\} \subset \mathbb{Z}$, that is, the dimension of A intersecting with the exterior of B.

b) **Bn**:= $\dim(A \cap \partial B) \in [-1, \mathrm{MIN}(\dim(A), \dim(B)-1)] \subset \mathbb{Z}$, that is, the dimension of A intersecting with the lower dimensional part of B.

c) **In**:= $\dim(A \cap B^\circ)$, if dim(B)=N, then $In \in \{-1, \dim(A)\} \subset \mathbb{Z}$, else if dim(B)<N, then $In \in [-1, \dim(A)] \subset \mathbb{Z}$, that is, the dimension of A intersecting with the same dimensional part of B.

Note:
(1) For simplicity, we let dim(A)=α, MIN(dim(A), dim(B)-1)=β;

(2) MAX(x_1, …, x_n) denotes the maximal value among x_1, …, x_n; MIN(x_1, …, x_n) denotes the minimal value among x_1, …, x_n.

This method is called "**Asymmetrical Method**" (shortly as **AM**).

3.2 Two Constraint Rules of AM

In the set of all of <En, Bn, In>, not all the combinations are possible. For example, in 2D space, A is a line, B is a rectangle, and then the next two conditions will not appear (Figure 2 and 3).

(1) The intersections between A and B°, A and ∂B, and A and B^{-} are all points.

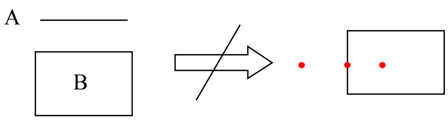


Figure 2. Impossible condition (1)

(2) A intersects with B^{-} and B°, but does not intersect with ∂B.

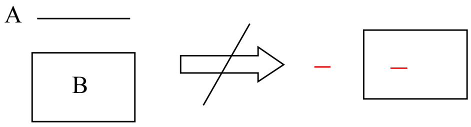


Figure 3. Impossible condition (2)

Here we give out two constraint rules:
(1) ¬(MAX(En, Bn, In)<α), that is, in continuous space, spatial entity O is divided into finite parts o_1, o_2, …, o_n, then dim(O)=MAX(dim(o_1), dim(o_2), …, dim(o_n)). This constraint rule is called "dimensional invariant".

(2) If dim(B)=N, (a)if α=N, then En=α ∧ In=α → Bn=(α-1), and β = α-1; (b)if α<N, then En=α ∧ In=α → Bn∈{α, α-1}, and β=α. This constraint rule shows the separating effect of lower dimensional part, which is called "low dimensional part separating"

Comparing with the constraint rules of 9IM model (Zlatanova, 1999), we can see these two rules are very simple.

Conclusion: In N (N=1, 2, 3) dimensional topological space, there are two spatial entities A and B, dim(A)≤dim(B). The topological relationships between A and B are represented using asymmetrical method <En, Bn, In>. In all of the topological relationships, if one topological relationship violates the two constraint rules, then it must be impossible.

This conclusion can be proved in N (N=1, 2, 3) dimensional topological space through complete induction method.

3.3 The Calculation Formulas to Calculate the Number of Possible Topological Relationships

According to the definition of AM and two constraint rules, we give the possible conditions of R(A, B) under dim(B)=N and dim(B)<N separately:

(1) If dim(B)=N, then En∈{-1, α}, Bn∈[-1, β], In∈{-1, α}. Hence, all of the possibilities are:

2×(β+2)×2 All the combinations
- 1×(MIN(α-1, β)+2)×1 (Remove the conditions violating "dimension invariant")
- IFF(α=N, β+1, β) (Remove the conditions violating "lower dimensional part separating")
= **4β + 6 - MIN(α-1, β) - IFF(α=N, β+1, β)** (1)

(2) If dim(B)<N, then En∈{-1, α}, Bn∈[-1, β], In=-1. Hence, all of the possibilities are:

1×(β+2)×2+1×(β+2)×(α+2) (All of the combinations)
- (MIN(α-1, β)+2) (Remove the conditions violating "dimensional invariant")
=**(β+2)×(α+4)- (MIN(α-1, β)+2)** (2)

3.4 The Asymmetrical Method of Dividing the Lower Dimensional Entity

In the discussion above, we are all based on the division of the higher dimensional entity. Although in applications it is few to divide the lower dimensional entity and compare it with the elements of the higher dimensional entity, it does exist. For example, " a room is at the end of the road" (Figure 4). It will turn back to the previous condition if we abstract the room to a point.

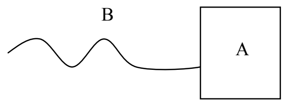


Figure 4. An example of dividing the lower dimensional entity

Suppose in N dimensional topological space U, there are two spatial entities A and B, N ≥ dim(A) > dim(B), then the topological relationships between the two entities have such characters as following:

1. Because a higher dimensional entity cannot be included by a lower dimensional entity, then A ⊈ B, and A∩B≠∅, that is, En is always dim(A). Then, the 3-tuples can be simplified to a 2-tuples - <Bn, In>.
2. Because dim(A)>dim(B), the points in B° are all dim(A) touchable, then the second constraint rule does not work. That is, there is only "dimensional invariant", no "low dimensional part separating'.
3. Because dim(B)<N, R(A, B) is not changed according to the embedding space.
4. Because dim(A)>dim(B), the scope of the dimension of the intersection between A's LDP and SDP is determined by the dimension of B, that is, the number of possible topological relationships is determined by B.

According to the analysis above, here the calculation formula to calculate the possible topological relationships is given as following:

Because Bn ∈ [-1, dim(B)-1], In ∈ [-1, dim(B)], the number of possible topological relationships is:

$$(\dim(B)-1-(-1)+1) \times (\dim(B)-(-1)+1)$$
$$=(\dim(B)+1) \times (\dim(B)+2) \qquad (3)$$

Here is an example:

1. If B is a line, then the number of possible topological relationships is $(1 + 1) \times (1 + 2) = 6$.
2. If B is a face, then the number of possible topological relationships is $(2 + 1) \times (2 + 2) = 12$.

We can see from the analysis above, it is relatively simple to represent the topological relationships if the lower dimensional entity is divided.

Next we illustrate the topological relationships involving at least one simple volume in 3D space, for the constraint of pages, we omitted other conditions.

4. EXAMPLES IN 3D TOPOLOGICAL SPACE

4.1 A Simple Line – a Simple Volume

En$\in${-1, 1}, Bn$\in${-1, 0, 1}, In$\in${-1, 1}, α=1, β=1, so the possible relationships are (Table 1):

4×1+6-MIN(1-1, 1) – IFF(1=3, 1+1, 1)=4+6-0-1=9.

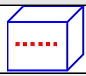
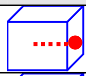
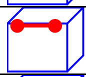
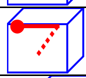
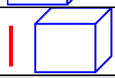
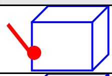
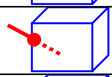
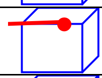
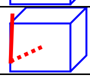
SN	En	Bn	In	relationship
	-1	-1	-1	Violate rule (1)
(1)	-1	-1	1	
	-1	0	-1	Violate rule (1)
(2)	-1	0	1	
(3)	-1	1	-1	
(4)	-1	1	1	
(5)	1	-1	-1	
	1	-1	1	Violate rule (2)
(6)	1	0	-1	
(7)	1	0	1	
(8)	1	1	-1	
(9)	1	1	1	

Table 1. Topological Relationships between a simple line and a simple volume

4.2 A Simple Face - a Simple Volume

En$\in${-1, 2}, Bn$\in${-1, 0, 1, 2}, In$\in${-1, 2}, α=2, β=2, so the possible relationships are (Table 2):

4×2+6-MIN(2-1, 1) – IFF(2=3, 2+1, 2)=8+6-1-2=11.

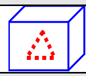
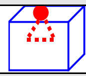
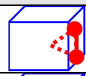
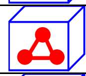
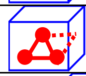
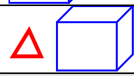
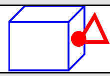
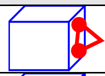
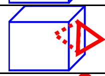
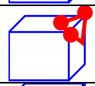
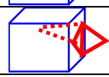
SN	En	Bn	In	relationship
	-1	-1	-1	Violate rule (1)
(1)	-1	-1	2	
	-1	0	-1	Violate rule (1)
(2)	-1	0	2	
	-1	1	-1	Violate rule (1)
(3)	-1	1	2	
(4)	-1	2	-1	
(5)	-1	2	2	
(6)	2	-1	-1	
	2	-1	2	Violate rule (2)
(7)	2	0	-1	
	2	0	2	Violate rule (2)
(8)	2	1	-1	
(9)	2	1	2	
(10)	2	2	-1	
(11)	2	2	2	

Table 2. Topological Relationships between a simple face and a simple volume

4.3 A Simple Volume – a Simple Volume

En$\in${-1, 3}, Bn$\in${-1, 0, 1, 2}, In$\in${-1, 3}, α=3, β=2, so the possible relationships are (Table 3):

4×2 + 6 – MIN(3-1, 2) – IFF(3=3, 2+1, 2) = 8+6-2-3 = 9

5. CONCLUSION AND FUTURE RESEARCH

In order to give a better representation of the topological relationships between the spatial entities, we provide an asymmetrical method, which is not sensitive to the increase of dimensions and easy to be implemented, and the constraint rules are simpler than other methods. The following table gives the number of distinguished topological relationships using different methods in 2D space. In the future research, we will apply this method to improve the capability of V9I model.

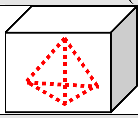
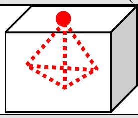
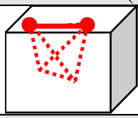
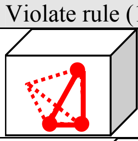
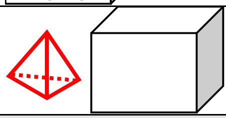
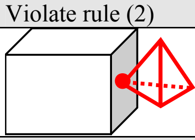
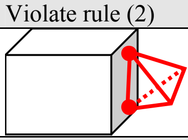
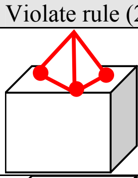
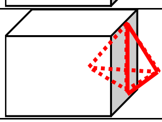
SN	En	Bn	In	relationship
	-1	-1	-1	Violate rule (1)
(1)	-1	-1	3	
	-1	0	-1	Violate rule (1)
(2)	-1	0	3	
	-1	1	-1	Violate rule (1)
(3)	-1	1	3	
	-1	2	-1	Violate rule (1)
(4)	-1	2	3	
(5)	3	-1	-1	
	3	-1	3	Violate rule (2)
(6)	3	0	-1	
	3	0	3	Violate rule (2)
(7)	3	1	-1	
	3	1	3	Violate rule (2)
(8)	3	2	-1	
(9)	3	2	3	

Table 3. Topological Relationships between two simple volumes

Method	A/A	L/A	P/A	L/L	P/L	P/P	Total
4IM	6	11	3	12	3	2	37
9IM	6	19	3	23	3	2	56
DEM	9	17	3	18	3	2	52
DE+9I	9	31	3	33	3	2	81
V9I	13	13	5	8	4	3	46
AM	7	9	3	8	3	2	32

Table 4. Comparing of distinguished topological relationships using different methods (modified from (Clementini and Di Felice, 1995; Chen et al. 2001))

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