A GENERALIZED FACTORED STOCHASTIC MODEL FOR THE OPTIMAL GLOBAL REGISTRATION OF LIDAR RANGE IMAGES

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ABSTRACT:

Surveying complex shapes or very large entities by laser scanners often requires the registration of a sufficient number of partial 3-D range images in order to completely reproduce the model of the real object. If redundancy exists among the partial models composing the measured entity, a global adjustment of the model components improves the final accuracy with respect to a simple pairwise registration.
To this regard, a new solution for the optimal least squares registration of range images, based on the Generalised Procrustes Analysis techniques, has been recently developed by the authors and can be found in the literature. The method, using the classical tie point correspondence, has proven to be very efficient since it does not require any prior information of the geometrical relationship existing among the particular reference frames in which the different partial 3-D models are expressed. Considering its computational advantages, it does not involve linearisation of equation systems nor matrix inversions, the only requirement is the singular value decomposition (SVD) of matrices of order 3 x 3.
In this paper a significant analytical enhancement of the Procrustean method is presented, to manage the stochastic properties of the tie point coordinates in a more complete and exhaustive way.
In the previous formulation the possibility to assign a different isotropic weighting factor to the single tie points, according to their specific accuracy, was considered. With the new proposed method, also the positional components, i.e. each coordinate, can be weighted separately. In this way a complete anisotropic and inhomogeneous factored stochastic model can be introduced in the Procrustes procedure.
The generalisation of the stochastic model is recommended for certain practical applications, particularly for joining aerial laser scanners strips produced with low sampling density. In these cases, matching correspondence points of low resolution range images generates poorly accurate tie point coordinate estimation. Indeed, this event introduces an uncertainty in the 3-D position that must be considered anisotropic, i.e. not affecting the three components of the same amount. In fact, considering one tie-point laying on a surface perpendicular to the laser beam, the effective position of the laser footprint on the correspondence element affects the planimetric position more than the related altimetric component. In these situations, the different quality of the tie points position components must be correctly and advantageously preserved, performing the global registration adopting the anisotropic model here presented.
A suitable application is discussed in the paper to illustrate the registration problem and the expected advantages of the method proposed.

1. INTRODUCTION

In a previous paper an original mathematical model for the optimal global registration of different and partially overlapping 3-D range images produced by laser scanners, by way of the classical tie-point correspondence, has been proposed by the authors (Beinat and Crosilla, 2001).
The method, based on the algorithms of the Procrustes analysis, has proven to be particularly efficient. It provides the direct estimation of the similarity transformation parameters (rotation, translation and dilation) reciprocally linking the distinct reference frames in which the various partial 3-D models are expressed, without the need of any prior information of their reciprocal orientation. On these basis, the global adjustment is then obtained by an iterative and fast converging procedure, fitting on turn every partial model with respect to the averaged sum of the remaining ones.
Other important advantages are evident from the computational point of view. The singular value decomposition (SVD) of matrices of order 3 x 3 that characterises the Procrustes procedure avoids the need for linearisation of equation systems and large matrix inversions, thus determining a fast and easy software implementation.

In the formulation already presented an isotropic distribution of the measurement errors has been considered. Since this aspect could be considered a limit of the method for certain applications, in this paper we formulate an anisotropic approach for the global registration problem so to assign in a more accurate and flexible way different weighting factors to the tie points coordinates.
The idea is to consider a factored structure model for the covariance matrix of the tie point coordinates (Goodall, 1991). According to this model a factored structure model of the weight matrix can also be taken into account.
The first weight matrix, already defined in the previous formulation, contains the isotropic weights assigned to the single tie-points of every different model, according to their specific accuracy of measurement or definition. The second one, introduced hereon, accounts for the different accuracy that globally characterises each one of the three coordinates components of the tie points belonging to a specific partial model.
For example, with this formulation a possible registration problem, often arising when dealing with low resolution aerial range images, can be afforded in a more proper way. In fact, when determining the position of one tie-point laying on a surface perpendicular to the laser beam, as simplified in Fig. 1,

for geometrical reasons the real *vs*. the nearest measured location of the laser footprint on the correspondence element affects the planimetric position more than the related altimetric component. In these situations, the different accuracy of the tie points position coordinate components must be correctly and advantageously taken into account, performing the global registration adopting the anisotropic model here presented.

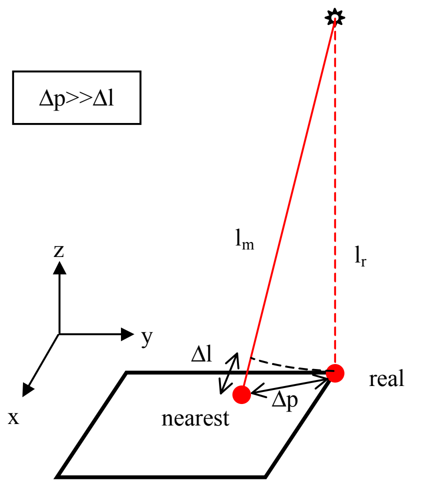


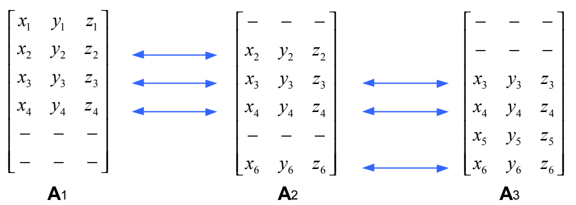
Figure 1. Tie-point position error and its influence on the coordinate components

2. ISOTROPIC ERROR CONDITION

Let we consider m partial models of the same object, produced by a laser scanner at m different base station locations. To link the different point clouds one to each other a total number of p tie-points have been selected.

We could therefore define m tie-point coordinate matrices $\mathbf{A}_1 \ldots \mathbf{A}_m$ of size p×3, each one containing the coordinates of the same set of p corresponding 3-D tie-points, defined in their own m different R^3 reference frames.

Since not all the tie-points appear in every point cloud representing one partial model, the $\mathbf{A}_i$ (i = 1 … m) matrix rows corresponding to missing points are left empty (or padded with zeros) and a null weight is assigned to them by way of a Boolean $\mathbf{M}_i$ matrix (Figure 2) (Commandeur, 1991).



$\mathrm{diag}(\mathbf{M}_1) = [1\ 1\ 1\ 1\ 0\ 0]$; $\mathrm{diag}(\mathbf{M}_2) = [0\ 1\ 1\ 1\ 0\ 1]$; $\mathrm{diag}(\mathbf{M}_3) = [0\ 0\ 1\ 1\ 1\ 1]$

Figure 2. Incomplete $\mathbf{A}_i$ p×3 tie-points datasets and resulting $\mathbf{M}_i$ p×p Boolean diagonal matrices.

It has already been demonstrated that the global registration of m point clouds by way of their tie-points requires the registration, by way of a proper similarity transformation of every data matrix $\mathbf{A}_i$ respect to the remaining ones (Beinat and Crosilla, 2001).

Said $\{\mathbf{t}, \mathbf{T}, c\}_{ij}$ (i = 1…m; $i < j$), the unknown transformation parameters *i.e.* the 3×1 translation vector, the 3×3 rotation matrix and the global scale factor respectively with $\mathbf{j}$ as a p×1 predefined auxiliary unitary vector ($\mathbf{j} = [1_1 \ldots 1_p]$), the solution of the global registration problem is obtained by solving the following least squares condition:

$$\mathrm{S} = \mathrm{tr}\sum_{i<j}^{m}\left[\left(c_i\mathbf{A}_i\mathbf{T}_i + \mathbf{j}\mathbf{t}_i\right)-\left(c_j\mathbf{A}_j\mathbf{T}_j + \mathbf{j}\mathbf{t}_j\right)\right]^{\mathrm{T}}\mathbf{M}_i\mathbf{M}_j \left(\sum_{k=1}^{m}\mathbf{M}_k\right)^{-1}\left[\left(c_i\mathbf{A}_i\mathbf{T}_i + \mathbf{j}\mathbf{t}_i\right)-\left(c_j\mathbf{A}_j\mathbf{T}_j + \mathbf{j}\mathbf{t}_j\right)\right] = \min \qquad (1)$$

Equation (1) summarises the Generalised Procrustes problem (GP) that can be numerically solved in iterative way by one of the methods proposed in the literature (Ten Berge, 1977)

The problem can be afforded in an alternative and more efficient way, by estimating an unknown "optimal" matrix $\mathbf{Z}$, also named "consensus", related to every $\mathbf{A}_i$ (i = 1…m) by a proper unknown similarity transformation $\{\mathbf{t}, \mathbf{T}, c\}_i$, unless a random error matrix $\mathbf{E}_i$:

$$\mathbf{Z} + \mathbf{E}_i = \mathbf{A}_i^{\mathrm{p}} = c_i\mathbf{A}_i\mathbf{T}_i + \mathbf{j}\mathbf{t}_i^{\mathrm{T}} \qquad (2)$$

If $\mathbf{E}_i$ has the general definition of:

$$\mathrm{vec}\left(\mathbf{E}_i\right) \simeq N\left\{0, \mathbf{\Sigma} = \sigma^2\mathbf{I}\right\} \qquad (3)$$

where $\mathbf{\Sigma}$ is the covariance matrix and $\mathbf{I}$ is the identity matrix, that corresponds to the case in which all the tie-points have the same precision, the GP problem of Equation (1) can be equivalently reformulated (Kristof & Wingersky, 1971) as:

$$\mathrm{S} = \sum_{i}\mathrm{tr}\left(\mathbf{A}_i^{\mathrm{p}} - \mathbf{Z}\right)^{\mathrm{T}}\mathbf{M}_i\left(\mathbf{A}_i^{\mathrm{p}} - \mathbf{Z}\right) = \min \qquad (4)$$

In this case it can be shown that the geometrical centroid:

$$\hat{\mathbf{Z}} = \left(\sum_{i=1}^{m}\mathbf{M}_i\right)^{-1}\sum_{i=1}^{m}\mathbf{M}_i\mathbf{A}_i^{\mathrm{p}} \qquad (5)$$

corresponds to the least squares estimate of the unknown "consensus" matrix $\mathbf{Z}$.

$\hat{\mathbf{Z}}$ is the p×3 matrix containing the coordinates of the p tie-points expressed into the same mean and common reference frame, and the global registration of every $i = 1 \ldots \mathrm{m}$ point cloud can be finally obtained applying the similarity transformation parameters $\{\mathbf{t}, \mathbf{T}, c\}_i$ linking the original $\mathbf{A}_i$ to $\hat{\mathbf{Z}}$.

3. ANISOTROPIC ERROR CONDITION

The condition of Equation (3) can be further extended. In a previous paper Beinat and Crosilla (2001) have already discussed the following partial anisotropic error condition, expressed by a factored form of the covariance matrix $\mathbf{\Sigma}_i$:

$$\mathrm{vec}(\mathbf{E}_i) \simeq N\left\{0, \mathbf{\Sigma}_i = \sigma^2 (\mathbf{Q}_{Pi} \otimes \mathbf{Q}_{Ki})\right\} \tag{6}$$

with $\mathbf{Q}_{Pi}$ diagonal and $\mathbf{Q}_{Ki} = \mathbf{I}$, $\otimes$ is the Kronecker product. This represents the case where the generic tie-point of $\mathbf{A}_i^{\mathrm{p}}$ has its own proper dispersion, that could differ among the $\mathbf{A}_i$ (i = 1 … m) tie-point coordinate matrices and respect to the rest of the p tie-points of the same configuration.

Of course this error model is still isotropic at the generic point level, since it considers the x, y, z coordinate components having the same error dispersion.

The next step is to consider $\mathbf{Q}_{Ki} \neq \mathbf{I}$ but diagonal in order to account for the different accuracy of the tie-point coordinates components.

Assuming $\mathbf{P}_i = \mathbf{Q}_{Pi}^{-1}$ (7)

and $\mathbf{K}_i = \mathbf{Q}_{Ki}^{-1}$ (8)

defining the product matrix $\mathbf{D}_i$ as:

$$\mathbf{D}_i = \mathbf{M}_i \mathbf{P}_i = \mathbf{P}_i \mathbf{M}_i \tag{9}$$

where $\mathbf{M}_i$ is the Boolean matrix previously defined, the GP problem assumes the following expression:

$$\mathrm{S} = \sum_i \mathrm{tr}\left(\mathbf{A}_i^{\mathrm{p}} - \mathbf{Z}\right)^{\mathrm{T}} \mathbf{D}_i \left(\mathbf{A}_i^{\mathrm{p}} - \mathbf{Z}\right)\mathbf{K}_i = \min \tag{10}$$

In this case, the least squares estimate $\hat{\mathbf{Z}}$ of the unknown "consensus" matrix $\mathbf{Z}$ under anisotropic error conditions is given by the following expression:

$$\mathrm{vec}\left(\hat{\mathbf{Z}}\right) = \left(\sum_{i=1}^{\mathrm{m}} \mathbf{K}_{\mathrm{i}} \otimes \mathbf{D}_i\right)^{-1} \sum_{i=1}^{\mathrm{m}} \mathbf{K}_{\mathrm{i}} \otimes \mathbf{D}_i \,\mathrm{vec}\left(\mathbf{A}_i^{\mathrm{p}}\right) \tag{11}$$

which represent a generalisation of the well known model reported in Equation (5) for the centroid estimation.

Proof

Imposing the minimum to the weighted norm expressed by Equation (10), corresponds to the satisfaction of the following condition:

$$\frac{\partial \sum_{i=1}^{\mathrm{m}} \mathrm{tr}\left(\mathbf{A}_i^{\mathrm{p}} - \mathbf{Z}\right)^{\mathrm{T}} \mathbf{D}_i \left(\mathbf{A}_i^{\mathrm{p}} - \mathbf{Z}\right)\mathbf{K}_i}{\partial \mathbf{Z}} = 0 \tag{12}$$

that is:

$$\frac{\partial \sum_{i=1}^{\mathrm{m}} \left(\mathrm{tr}\,\mathbf{A}_i^{\mathrm{p}\,\mathrm{T}} \mathbf{D}_i \mathbf{A}_i^{\mathrm{p}} \mathbf{K}_i - \mathrm{tr}\,\mathbf{Z}^{\mathrm{T}} \mathbf{D}_i \mathbf{A}_i^{\mathrm{p}} \mathbf{K}_i - \mathrm{tr}\,\mathbf{A}_i^{\mathrm{p}\,\mathrm{T}} \mathbf{D}_i \mathbf{Z} \mathbf{K}_i + \mathrm{tr}\,\mathbf{Z}^{\mathrm{T}} \mathbf{D}_i \mathbf{Z} \mathbf{K}_i\right)}{\partial \mathbf{Z}} = 0$$

$$\sum_{i=1}^{\mathrm{m}} \left(-\mathbf{D}_i \mathbf{A}_i^{\mathrm{p}} \mathbf{K}_i - \mathbf{D}_i^{\mathrm{T}} \mathbf{A}_i^{\mathrm{p}} \mathbf{K}_i + \mathbf{D}_i^{\mathrm{T}} \mathbf{Z} \mathbf{K}_i^{\mathrm{T}} + \mathbf{D}_i \mathbf{Z} \mathbf{K}_i\right) = 0$$

In the case in which: $\mathbf{D}_i^{\mathrm{T}} = \mathbf{D}_i$ and $\mathbf{K}_i^{\mathrm{T}} = \mathbf{K}_i$ it follows:

$$\sum_{i=1}^{\mathrm{m}} \left(-\not{2}\mathbf{D}_i \mathbf{A}_i^{\mathrm{p}} \mathbf{K}_i + \not{2}\mathbf{D}_i \mathbf{Z} \mathbf{K}_i\right) = 0$$

$$\sum_{i=1}^{\mathrm{m}} \left(-\mathbf{K}_i \otimes \mathbf{D}_i \,\mathrm{vec}\left(\mathbf{A}_i^{\mathrm{p}}\right) + \mathbf{K}_i \otimes \mathbf{D}_i \,\mathrm{vec}(\mathbf{Z})\right) = 0$$

$$\sum_{i=1}^{\mathrm{m}} \mathbf{K}_i \otimes \mathbf{D}_i \,\mathrm{vec}(\mathbf{Z}) - \sum_i \mathbf{K}_i \otimes \mathbf{D}_i \,\mathrm{vec}\left(\mathbf{A}_i^{\mathrm{p}}\right) = 0$$

that leads to the expression given in Equation (11).

4. CASE STUDY

The mathematical model and the algorithm described in this paper can be suitably applied e.g. for the optimal registration of range images produced by aerial laser scanners at low-resolution (let's say less than 1 point per squared metre) or in general when the average laser footprint density is lower than the nominal RMSE of the measurements.

In these situations, the edges of the real objects commonly assumed as tie points, like roofs, walls and building corners, are not accurately defined. When one collimates one point on a range image, he is obliged to consider the coordinates of the measured point closest to the selected position because no three-dimensional local interpolations are always possible. The two point positions, the real and the measured ones, rarely coincide and consequently an unknown systematic error is frequently introduced.

With respect to the whole set of the tie-points selection, this error can be considered not isotropic in the 3-D components. In fact, for the geometrical reasons explained above (Figure 1), the position error does not affect the three tie-point coordinates in the same way. The component parallel to the laser beam, that in aerial laser scanning approximately corresponds to the z axis, is in fact less sensitive to the resulting laser footprint position than the other ones. In most applications, particularly when aiming to flat targets, the z coordinate component does not significantly change its value also for planar misplacements of a certain amount.

For this reason, performing a model block adjustment could be convenient to assign different weights to the tie-point coordinates, depending on the laser footprint density.

To test this hypothesis a numerical simulation has been set up in laboratory.
A virtual scenery with four distinct buildings has been created using Matlab™ (Fig. 3) and some synthetic laser range images have been produced by randomly sampling the surface points of the model at an average resolution of 0.5 points/m^2. Contemporary, to better represent the real conditions, a random error of ±10 cm has been introduced in the (laser scan) measurements.
In this way a set of four range images of a perfectly known - geometrically speaking - object was obtained, in order to compare the effects of the different weighting models.

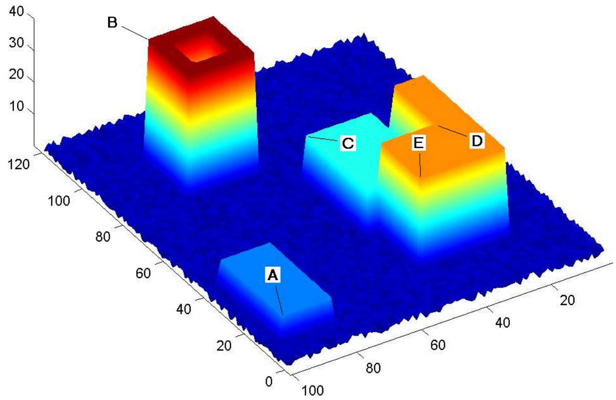


Figure 3: The virtual scenery created for the simulations and the tie-points (A, B, C, D, E) used for the global registration

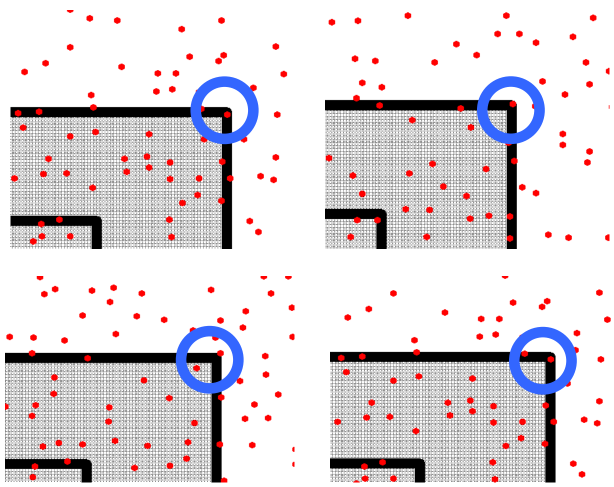


Figure 4. Particular of the distribution of the laser footprints around the corner of one building (tie-point B) in four different scans.

Figure 4 shows the distribution of the laser footprints all around one specific tie-feature outlined by the circle, resulted in four distinct range images. It represents a real situation: when collimating one specific tie-point, the operator has to select the laser footprint closest to this one and laying on the same plane.
The four 3-D range images of the virtual scenery have been simultaneously registered one to each other in two ways, assuming an isotropic and an anisotropic error condition of the tie-point coordinates respectively.
Comparing the results of the two adjustments with respect to the real geometry, the anisotropic error model effectively shows a reduction of the RMSE of the tie-point coordinate components (true *vs.* adjusted) evident along the z vertical axis (Table 1).

	Isotropic model			Anisotropic model		
Point	Δx	Δy	Δz	Δx	Δy	Δz
A	+0.117	+0.010	+0.071	+0.119	+0.011	+0.052
B	+0.355	+0.169	-0.278	+0.347	+0.170	-0.251
C	-0.301	+0.284	-0.291	-0.304	+0.287	-0.261
D	-0.078	-0.167	-0.098	-0.080	-0.170	-0.088
E	-0.041	+0.156	+0.076	-0.041	+0.159	+0.048
RMSE	0.218	0.180	0.191	0.217	0.182	0.170

Table 1. Adjusted solutions residuals (real minus adjusted coordinate components) [metres]

5. CONCLUSIONS

We propose in this paper a generalisation of the analytical model already implemented for the simultaneous global registration of 3-D range images (Beinat and Crosilla, 2001).
The method seems promising on suitable applications, consisting in the global registration of low-resolution aerial laser scan images.
The first synthetic experiments indicate, as expected, that the anisotropic model slight better approaches the optimal solution (the real Digital Surface Model) than the isotropic one. This result is mainly consequence of the increased vertical positional accuracy of the DSM, and it is obtained by properly weighting the more precise z coordinates of the tie-points during the range images global registration.
Further investigations are certainly needed to test the effective capabilities and the limits of the method proposed, particularly on the rigorous weights definition and around the factors influencing it.

6. REFERENCES

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