APPLICATION OF DIFFERENTIAL SPLINES FOR RELIEF SIMULATION

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ABSTRACT

For automatic map-making comprising as partial processes the isoline map compiling, three-dimensional surface image construction as well the additional characteristics that are different from them, it is necessary to create a mathematical model of the relief.
Such modeling depends on the means of presenting the output information as well as the approximation methods and the surface interpolating.
The output information about the relief is usually presented in a discrete form with regular, semi-regular and irregular arrangement of points.
The most frequently used construction methods of DEM are known to be as follows: polynomial method, multiquadrics method, spline method, distance-weighting method on the basis of triangulation, Kriging method.
The result of our research is as follows: for effective approximation of relief a mathematical model is constructed on the basis of collocations when the locality surface is approximated by low degree polynomial with additional interpolating the differential spline.
The experimental research for various types of relief proves that a highly accurate reproduction of the surface, excluding places with a certain difference of inclinations with the approximation error is about 1/2 - 2/3 of relief cross-section.

1. INTRODUCTION

Digital Elevation Models (DEM) have an sugnificant importance for solving a series of the engineering type applied problems or as the basis for creating joint information model of terrain. DEM is applied in computer map-making, which consists of contour map compiling, three-dimensional surface model construction, profile lines, geting slope and aspect maps, erosion processes.
DEMs for huge teritories, even for whole states are created with data digitized from contours. In a large scale, applied problems, which accuracy depends on input map accuracy, solves with DEM.

2. THEORY OF DEM CONSTRUCTION

DEM building accuracy depends on such main factors as: input information accuracy and presenting methods and mathematical relief presentation methods. The actual question is accuracy of DEMs based on contours scanned from maps.
Maximal automatization is acheeved using map scanning. Relief is presented in discrete form with points arrangement at contours and specific places.
Input information presenting methods cause less opposite thoughts than mathematical relief modelling methods. The most frequently used methods for DEM construction are: polynomial method, spline method, multiquadrics method, distance-weighting method, triangulation method, Kriging method.
It was stated from our research that model based on mean square collocation [Moritz H., 1983] is efficient for relief modelling. The main equation of this model for relief approximation is:

$$Z = AX + s + n \,, \tag{1}$$

where Z - elevation value vector; s - relief component after trend removal; n - random value or noise dependent from measurement error. The expression

$$T = AX \tag{2}$$

presents systematic component or trend and approximated by polynomial of low power.
Since measurements are carried out not on terrain but on map, so when simplifying mathematical model we'll consider $n=0$.
In mean square collocations method the signal is presented by covariance function dependent from distance between points. The expression for signal

evaluation will be:

$$\hat{s}_k = C_{ki} \cdot C_{ii}^{-1} \cdot \Delta Z_i \tag{3}$$

where i= 1,2,... ,n - number of input points. Or, in expanded record

$$\hat{s}_k = \begin{bmatrix} C_{k1} \\ C_{k2} \\ \vdots \\ C_{kn} \end{bmatrix}^T \cdot \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \cdot \begin{bmatrix} \Delta Z_1 \\ \Delta Z_2 \\ \vdots \\ \Delta Z_n \end{bmatrix}, \tag{4}$$

where $\hat{s}_k$ - estimate of signal at k point, which elevation is definad; C_{ki} - vector of covariances between defined and input points; C_{ii} - matrix of covariances between input points; ΔZ_i – vector of differences at input points between measured elevations and the approximated expression Eq. 2.

$$\Delta Z_i = Z_i - T_i \,. \tag{5}$$

This model was realised in Kriging method with such the difference: covariances were changed by semi-variances and covariance function was changed by variogram [Burshtynska Kh., Tumska O., Lelukh D., 2000]. The expression Eq. 4 by Kriging will be:

$$\hat{s}_k = \omega_1 \Delta Z_1 + \omega_2 \Delta Z_2 + \ldots + \omega_n \Delta Z_n \,, \tag{6}$$

where ω_i – weights, found from solution of such system of equations:

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} & 1 \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \\ \mu \end{bmatrix} = \begin{bmatrix} \gamma_{k1} \\ \gamma_{k2} \\ \vdots \\ \gamma_{kn} \\ 1 \end{bmatrix}. \tag{7}$$

In those equation values γ_{ij} – semi-variance, find from variogram [Davis J.C., 1977]

$$\gamma_{ij} = C_{ii} - C(R_{ij}), \tag{8}$$

where R_{ij} – distance between points.

Weights are found under condition $\sum_{i=1}^{n} \omega_i = 1$.

Covariance function or variogram depends on relief type and defines its statistic characteristics. The most Kriging problem is variogram definition. It is defined a priori but during DEM construction variogram parameters changes with approximation field type.

As it is shown in [Moritz H., 1983] covariance function or variogram may be presented by different analitic basis functions to simplify the solution. Differential splines shows good approximation properties. For DEM construction we used such differential spline:

$$F = \sum_{i=1}^{n} \lambda_i R_i + a + bx_i + cy_i \,, \tag{9}$$

where

$$\begin{aligned} &R_i(x, y) = [(x - x_i)^2 + (y - y_i)^2] \times \\ &\times \ln[(x - x_i)^2 + (y - y_i)^2] \end{aligned};$$

$$R_i(x_i, y_i) = 0 \,. \tag{10}$$

There are such conditions for coefficients definition λ_i, a, b, c:

$$\begin{gathered} F_i = Z_i, \qquad \sum_{i=1}^{n} \lambda_i = 0, \\ \sum_{i=1}^{n} \lambda_i x_i = 0, \quad \sum_{i=1}^{n} \lambda_i y_i = 0. \end{gathered} \tag{11}$$

Coefficients λ_i, a, b, c are defined from solution of equation system:

$$\left[\begin{array}{c|c} G & M \\ \hline M^T & \theta \end{array} \right] \left[\begin{array}{c} \lambda \\ \hline \tau \end{array} \right] = \left[\begin{array}{c} l \\ \hline O \end{array} \right], \tag{12}$$

where G – symmetric matrix of $n \times n$ size with elements $R_i(x, y)$;
M - matrix of $n \times 3$ size, which rows are filled with numbers $1, x_i, y_i$;
θ - zero matrix of 3×3 size;
λ/τ - vector of defined coefficients λ_i, a, b, c;
I - vector of input data, $F_i = Z_i$;
O - zero vector with three rows.

To compare approximation properties excepts input function Eq. 9 we had issued modified function:

$$F' = \sum_{i=1}^{n} \lambda_i R_i' + a + bx_i + cy_i \,, \tag{13}$$

where

$$\begin{aligned} &R_i'(x, y) = \sqrt{(x - x_i)^2 + (y - y_i)^2} \times \\ &\times \ln \sqrt{(x - x_i)^2 + (y - y_i)^2} \end{aligned}. \tag{14}$$

3. EXPERIMENTS AND RESULTS

We choose the part of topography plan with scale 1:5000 cross-section 1 m, inclination angles from 0,1° to 10°, with different relief types (plains, soft slopes, slopes with sharp inclination jumps) for comparative analisys of presented relief approximation methods. We didn't took any information from the lower section because of the sharp bluff there.

Input information was received by map scanning and data digitized from contours becides specific points elevations record. Map scanning was done with 300 *dpi* resolution.

Contours digitizing was done manual with mean step of 1 mm (step changes with contour curvature).

The programs for Krining method of relief modelling realization were build on *Object Pascal* language in visual programming environment *Borland Delphi 5.0.* We used wandering surface method to build DEM with the presented methods. We realized point search programm by radius and by direction (quadrant, octant).

During approximation methods programm realization for DEM building we determine that unknown find problem is unstable. This make impossible finding of equations inverse matrix by classic method. For solution stabilization we used regularization method by Tikhonow [Zhurkin I., Neyman J., 1988.]. This method main point is that we obtain inverse matrix of equations coefficients from relationship:

$$A^{-1} = (A + \alpha I)^{-1}, \tag{15}$$

where α – regularization parameter, calculated as local field variance;

I – identity matrix.

Digital Elevation Model build with 25 m step, input points for approximation methods were defined uniformly: eight by octant.

DEM creation accuracy depends on variation curve form. See Fig. 1 for specified relief types variograms at sections (see Fig. 2): a) plain relief (section D); b) soft slope (section E), c) slope with sharp inclination jump (section A).

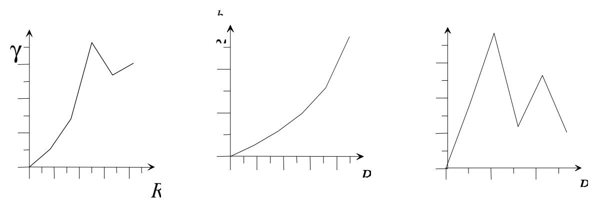


Figure 1. Variogram forms

Form of variograms *a* and *b* shows trend presence, at whole variograms have the soft character. Surface modelling by linear law is good after trend removal. Variogram *c* with sharp curve jumps indicates complicated relief character. We get the best approximation results from DEM building by Kriging method using such an exponential variogram:

$$\gamma(R) = kR^{0.5} \tag{16}$$

where k – scale coefficient, defined through local field semivariance.

DEM construction accuracy evaluation was done: by divergence of contour input points elevations; by control points; visually, by divergence input contours and reproducted by DEM contours.

For accuracy estimation we used about 3500 input points, which elevations were defined by 4 nearest DEM points so far as section boundaries. Control points (points, which elevations are subscribed at topographical plan) are evenly distributed across the section. Their total number is 55. See Table 1 for results of accuracy estimation.

Table 1

DEM construction accuracy estimation

Method	By contour points, m			By control points, m		
	1	2	3	1	2	3
Differential spline	0.56	-0.85	0.09	0.28	-0.28	0.12
Modified differential spline	0.60	-0.89	0.09	0.27	-0.24	0.11
Kriging* (linear variogram)	0.68	-0.54	0.08	0.26	-0.23	0.12
Kriging** (linear variogram)	0.70	-0.68	0.10	0.26	-0.24	0.12
Kriging*** (linear variogram)	0.68	-0.54	0.08	0.26	-0.24	0.12
Kriging* (exponential variogram)	0.70	-0.68	0.10	0.32	-0.23	0.12
Kriging** (exponential variogram)	0.68	-0.54	0.08	0.32	-0.23	0.11
Kriging*** (exponential variogram)	0.70	-0.68	0.10	0.32	-0.22	0.12

* local trend is approximated by mean value;

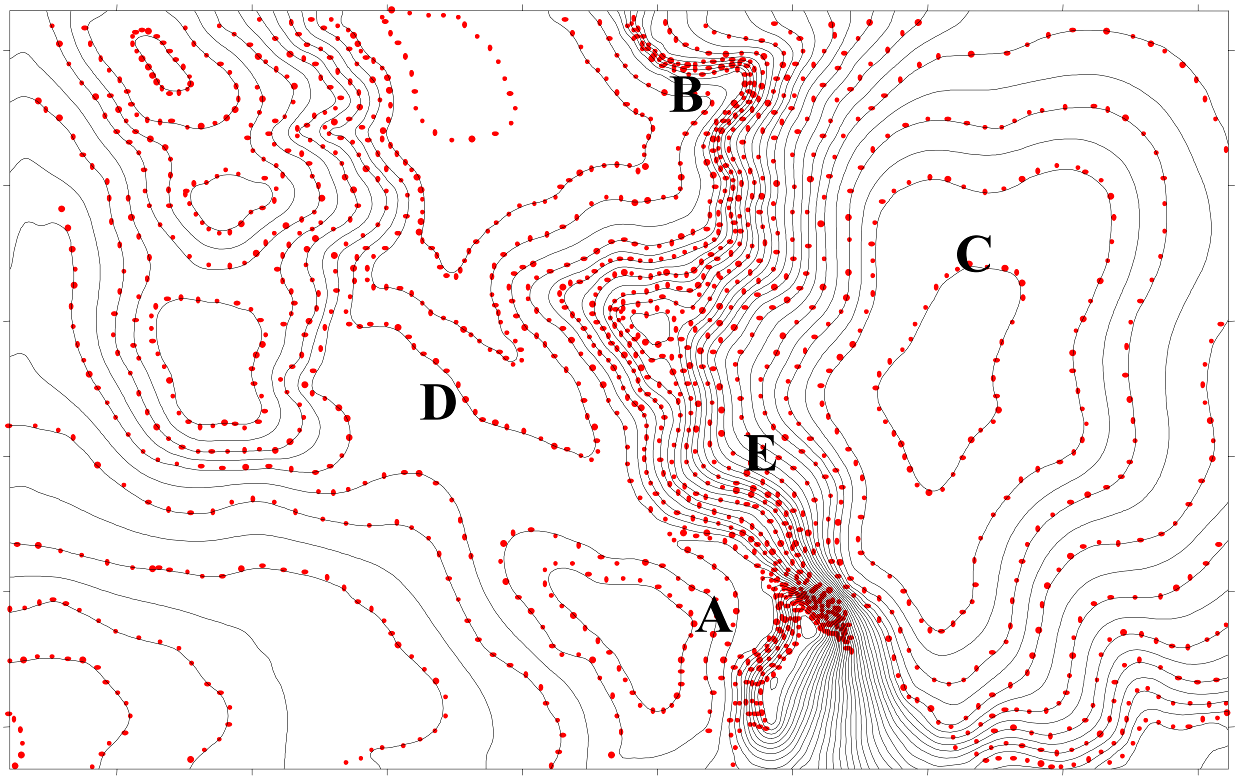
** local trend is approximated by plane;

*** local trend is approximated by bilinear surface.

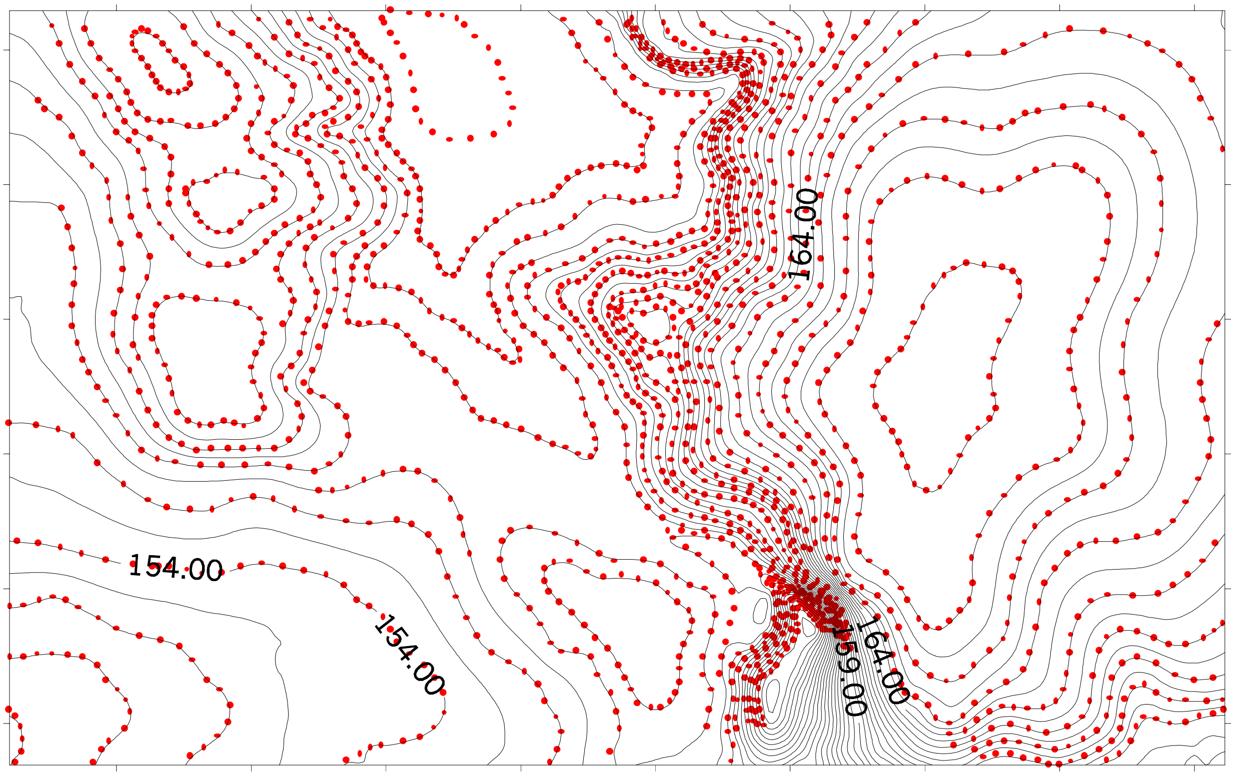
There are maximum and minimum elevations of points divergences in the columns numbered 1, 2 correspondingly, and mean square errors in the columns numbered 3.

Mean square errors pictures general view at approximation accuracy without picturing accuracy at the local sections. So, all methods look like uniformly by mean square errors value, but visual comparing of input contours and contours produced by DEM shows sugnificant divergences at the local sections.

Fig. 2 pictures the map of digitaled contours (pictured with points) and contours produced by DEM, constructed with Kriging method (solid lines); a) using exponential variogram (exponent power 0.5); b) using linear variogram. Divergence analisys shows inexact surface approximation when using linear variogram at sections with sharp jumps of inclination angles (sections A and B), where surface reproduction is more accurate when using exponential variogram.

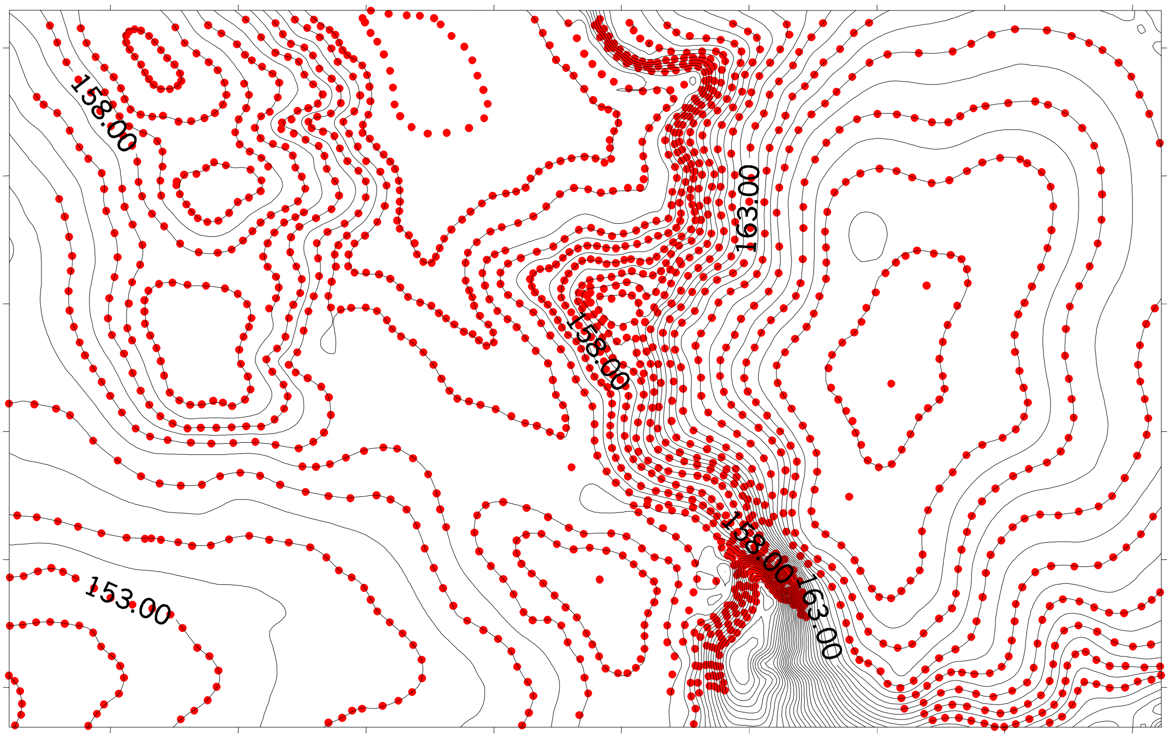


a)

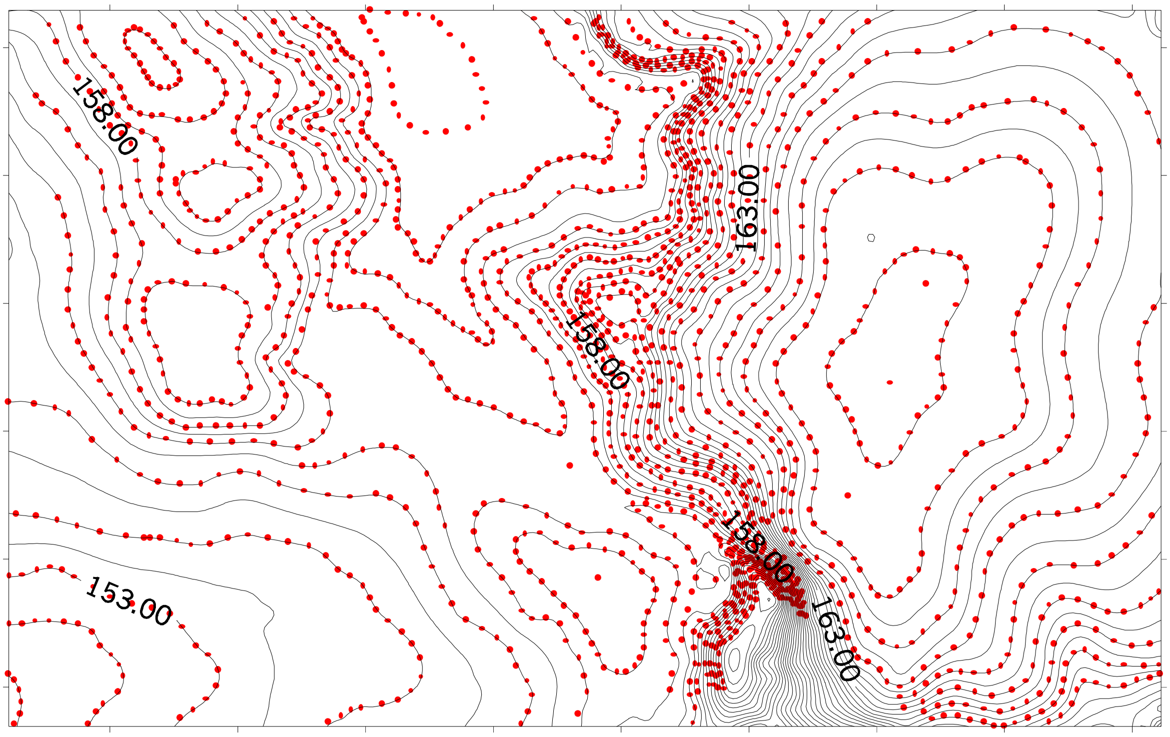


b)

Figure 2. The map of digitaled contours and contours created by DEM constructed with Kriging method.
a) from exponential variogram; b) from linear variogram



a)



b)

Figure 3. The map of digitaled contours and contours created by DEM constructed with differential splines. a) from Eq. 9; b) from Eq. 13

As we see from Fig. 3, differential splines of two types give us false contours; error of relief reproduction at section A achieves 3/4 of relief cross-section. There are false contours at section C when using differential spline (Fig. 2a). In general, modified differential spline interpolates the surface with less divergences.

4. CONCLUSIONS

From up taken researches we can make such a conclusion:
Differential spline give us high accuracy during DEM construction for soft surfaces. Spline approximation properties sugnificantly improves with additional information, such as extreme points.
Kriging method as statistic method gives us optimal results for DEM construction without additional information. The approximation function choice has sugnificant value for local empirical variograms. Local reproducting function can not be set for a whole section but must be defined for each local fragment when DEM construction high accuracy is demanded.
Results of accuracy estimation by mean square errors shows the global character of the estimation without account of approximation local peculiarities. Data digitized from contours for smooth surfaces with slope inclination up to 10° permits us to construct Digital Elevation Models with accuracy about 1/8 of relief cross-section and accuracy in the range 2/3 – 3/4 of relief cross-section for sections with sharp jump of slopes inclination.

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