ASPECTS OF ERROR PROPAGATION IN MODERN GEODETIC NETWORKS

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ABSTRACT:

Spatial information Web services are beginning to be used for disseminating general purpose (mapping-grade) geographic information. They allow integrated use of geolocated data from varied sources by first transforming them to a common reference frame, typically referred to a "WGS84".
Applying this paradigm in a geodetic context meets two problems. Firstly, one must, as for mapping-grade geographic information, bring traditional co-ordinate information from various local sources onto a unified geocentric GPS datum, typically a locally canonical realization of WGS84. Secondly, on this precision level, the concept WGS84 is no longer uniquely defined. Instead, different realizations, all geocentric on the cm level, are in use in different locales. They must be brought from the locally canonical realization onto a common globally valid geocentric datum.
In geodesy, traditionally the propagation of co-ordinate precision has been carefully managed by designing networks hierarchically, creating a well-behaved spatial variance structure in which the inter-point precision of points located close together is never unduly large in relation to their distance. Then, "criterion matrices" are used to formally describe this precision behaviour.
In this paper, we tentatively develop, with a view to geodetic co-ordinate Web services, methods for bringing geodetic co-ordinate data onto a common geocentric reference frame through a two-step datum transformation procedure addressing these two problems. We design simple criterion variance structures to describe the spatial precision behaviour of co-ordinates in both steps.

1 INTRODUCTION

In geodesy we produce and manage highly precise co-ordinate data. Traditionally we do this by successive, controlled propagation of precise measurements down a *hierarchy* of progressively more localized and detailed network densifications: working "from the large to the small".

Compared to the market for geographic information used for mapping applications, where precision is less critical and often in the range $\pm 0.1 - 1\,\mathrm{m}$, geodetically precision-controlled co-ordinate data forms a much smaller field of application. However, this field is vitally important, including the precise cadastral, urban planning and construction surveys that make modern society possible. Bringing this area of activity within the scope of geographic information services would require adapting these to the management of the spatial precision *structures* found in these network hierarchies, codifying traditional geodetic practice.

One of us (KK) has studied in detail the technical aspects of co-ordinate Web services for geodesy (Kollo, 2004).

In geodesy, the complexity of describing the precision of point sets is often handled by defining simple *criterion functions* that model the point co-ordinates' overall variance behaviour as a function of relative point location, without having to specify a detailed covariance matrix.

Next we shall first briefly present the current state of spatial information services for the World Wide Web, including co-ordinate transformation services. Then we discuss geodetic networks, network hierarchies, error propagation and criterion matrices.

We propose to bring sets of geodetic co-ordinate data upon a globally unique realization of WGS84 by a two-step procedure:

1. Perform an overdetermined tie of the given geodetic network (which may be a traditionally measured, pre-GPS, one) by a triangle-wise affine (bi-linear) transformation to a given set of GPS-positioned points. This technique is currently in use in Finland, cf. (Anon., 2003) appendix 5; after this, the network will be in the national realization EUREF-FIN of WGS84, i.e. the locally canonical realization for the territory of Finland.
2. Perform a three-dimensional Helmert transformation of the result to a single, globally unique WGS84 realization. In this operation the given set of GPS points, which could be considered errorless in the EUREF-FIN datum, will acquire a non-zero variance structure again.

We derive criterion functions modelling the variance propagation behaviour of both steps.

2 SPATIAL DATA WEB SERVICES

Geographic information services as existing today supply spatial information over the World-Wide Web. They are commonly based upon standards established by the Open Geospatial Consortium (OGC), an international non-profit geospatial information standards group. Using these standards, one may extract geographic data from a variety of conforming data sources, which may all be in different datums or co-ordinate reference systems.

Services of this kind can be classified as Web Map Services (WMS, (OGC, 2001)), Web Feature Services (WFS, (OGC, 2002)), and many others.

Web standards are based on the XML (Extensible Modeling Language) description language; OGC has defined the GML (Geographical Mark-up Language) for this (OGC, 2003). The language provides for specifying position precisions of points and point sets, either as individual point position precisions, or as between-points relative position precisions. Additionally it allows specification of a full variance-covariance matrix. The standard speaks of "data quality" (`dataQuality.xsd`).

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More recent work on data quality is going on in ISO, the International Standards Organization: e.g.. ISO 19113 “Quality Principles” and ISO 19138 “Data Quality Measures” (A. Jakobsson, personal comm.). Clearly co-ordinate precision is only a small part of what the concept “quality” covers when applied to spatial information. Conversely, however, there is much more to co-ordinate quality than is often understood, about which more later.

In practical implementations such as GeoServer (Anon., 2005d) or MapServer (Anon., 2005b) we tend to see a limited set of predefined datums (e.g., the European Petroleum Survey Group (EPSG) set, cf. (European Petroleum Survey Group, 2005)) and projections and transformations (e.g., the PROJ.4 set, cf. (Anon., 2005c)) being included. A more scalable approach is using a co-ordinate service specification for the Web. Both standardization and implementation work in this direction is now being done in a number of places. There exist a WCTS (Web Co-ordinate Transformation Server) specification and experimental implementations (see, e.g., (Anon., 2005a)).

Spatial data Web services as currently designed are aimed at the large, complex market of users of various map products for a broad range of applications. These products are often of limited resolution and precision, co-ordinate precision not being their focus. To some extent this is also a cultural difference, cf., e.g., (Jones, Winter 2005/2006).

3 CHOOSING A “ROSETTA FRAME”

A well known spatial data application like PROJ.4, e.g., does not distinguish between the various realizations of WGS84, such as the different international ITRF and European ETRF frames (F. Warmerdam, email). As long as we work within a domain where there is only one canonical realization, like EUREF-FIN in Finland, this is a valid procedure. PROJ.4 uses WGS84 as the common “exchange datum” to which all other datums are transformed, typically by applying (after, if necessary, transformation to 3D Cartesian using a reference ellipsoid model) either a three parameter shift, or a seven parameter Helmert transformation.

For geodetic use, it is not enough to consider the various realizations of WGS84 as representing the same datum. The differences between the various regional and national “canonical realizations” – as well as between the successively produced international realizations of ITRS/ETRS – are on the several-centimetre level. To illustrate this, we mention a recent report (Jivall et al., 2005) which derives the transformation parameters between the various Nordic national realizations of ETRS 89, and a common, truly geocentric system referred to as ITRF2000 epoch 2003.75. This allows the combination of co-ordinate data from these countries in an unambiguous way.

4 NETWORK HIERARCHY IN THE GPS AGE

Some claim that in the GPS age the notion of network hierarchy has become obsolete. We can measure point positions anywhere on Earth, using the satellite constellation directly, without referring to higher order terrestrial reference points. In reality, if again robustly achieving the highest possible precision is the aim, this isn’t quite true.

Measurements using the satellite constellation directly violate the “from the large to the small” principle. If we measure, e.g., independently absolute positions in a terrestrial GPS network on an area of 1000×1000 km using satellites at least 20000 km away, we will not obtain the best possible *relative* positions between these terrestrial points. Rather, one should measure *vectors* between the terrestrial points, processing measurements made simultaneously from these points to the same satellites, to obtain co-ordinate *differences* between the points. This, *relative GPS measurement*, is the standard for precise geodetic GPS.

In relative GPS positioning within a small area, one point may be kept fixed to its conventionally known co-ordinates, defining a local datum. From this datum point outward, precision deteriorates due to the various error contributions of geodetic GPS. For covering a larger area, one should keep more than one point fixed. These points are typically taken from a globally adjusted point network, like the well known ITRF or ETRF solutions. In Finland, e.g., one uses points in the EUREF-FIN datum, a national realization of WGS84 providing a field of fixed points covering Finland. To bring a geodetic network into the EUREF-FIN datum, it must be attached to a number of these points, which formally, in the EUREF-FIN datum, are “errorless”.

5 VARIANCE BEHAVIOUR UNDER DATUM TRANSFORMATION

Any realistic description of a geodetic network’s precision should capture its *spatial structure*, the fact that inter-point position precision between adjacent points is the better, the closer together the two points are. For points far apart, precision may be poorer, but that will be of no practical consequence. What matters is the *relative* precision, e.g., expressed in ppm of the inter-point distance.

The precision structure of a network depends on its *datum*, the set of conventionally adopted reference points that are used to calculate the network points’ co-ordinates. E.g., in the plane, two fixed points may be used to define a co-ordinate datum; the co-ordinates of those points, being conventionally agreed, will be errorless. Plotting the uncertainty ellipses describing the co-ordinate imprecision of the other points, we will see them grow outward from the datum points in all directions.

Choosing a different set of datum points will produce a different-looking pattern of ellipses: zero now on, and growing in all directions outward from, these new datum points. Yet, the precision *structure* described is the same, and well defined transformations exist between the two patterns: *datum transformations*, also called S-transformations.

6 CRITERION FUNCTIONS

We refer to the work of (Baarda, 1973) for the notion of criterion matrices, as well as the related notion of S- or datum transformations. The precision of a set of network points can be described collectively by a *variance-covariance matrix*, giving the variances and covariances of network point co-ordinates. If all point positions are approximately known, as well as the precision of all geodetic measurements made between them, this variance matrix is obtained as a result of the least squares adjustment of the network.

In a three-dimensional network of n points there will be $9n^2$ elements to the variance matrix – or $\frac{3}{2}n\,(3n+1)$essentially different ones –, so this precision representation doesn’t scale very favourably. Also, the original measurements and their precisions may be uncertain or not readily available. For this reason, geodesists have been looking for ways of describing the precision structure of a geodetic network – realistically, if only approximately – using a small number of defining parameters. Such synthetic variance matrices are called *criterion matrices* and their generating functions *criterion functions*.

Criterion functions are an attractive and parsimonious way to describe the precision structure of geodetic point sets or corpora of spatial information. They offer a more complete description than point or inter-point co-ordinate precision, yet take less space than full variance matrices, while in practice being likely just as good.

A formal requirement to be placed upon criterion matrices is, that they transform under datum transformations in the same way as real variance-covariance matrices would do. As this is known geodetic theory, we will not elaborate further.

7 GEOCENTRIC VARIANCE STRUCTURE OF A GPS NETWORK

Let us first derive a rough but plausible, geocentric expression for the variance-covariance structure of a typical geodetic network. The true error propagation of GPS measurements is an extremely complex subject. Here, we try to represent the bulk co-ordinate precision behaviour in a simple but plausible way.

Also the full theory of criterion matrices and datum transformations is complicated (Baarda, 1973, Vermeer et al., 2004). Here we shall cut some corners. We assume that the inter-point position variance between two network points A and B, co-ordinates (X_A, Y_A, Z_A) and (X_B, Y_B, Z_B), is of the form

$$\begin{aligned} &\text{Var}\left(\mathbf{r}_B - \mathbf{r}_A\right) = \\ &= Q_0 \left((X_B - X_A)^2 + (Y_B - Y_A)^2 + (Z_B - Z_A)^2 \right)^{\frac{k}{2}} \\ &= Q_0 d_{AB}^k, \end{aligned} \quad (1)$$

with k and Q_0 as the free parameters (assumed constant for now), and $d_{AB} = \|\mathbf{r}_B - \mathbf{r}_A\|$ the $A - B$ inter-point distance.

For this to be meaningful, we must know what is meant by the variance or covariance of vectors. In three dimensions, we interpret this as:

$$\text{Cov}\left(\mathbf{r}_A, \mathbf{r}_B\right) = \text{Cov}\left(\begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix}, \begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} \right) =$$

$$= \begin{bmatrix} Cov\left(X_A, X_B\right) & & \\ & \ddots & \\ & & Cov\left(Z_A, Z_B\right) \end{bmatrix},$$

i.e., a 3×3 elements tensorial function. Also Q_0 is in this case a 3×3 tensor. The approach is not restricted to three dimensions, however.

Eq. 1 is fairly realistic for a broad range of geodetic networks: for (one-dimensional) levelling networks we know that $k = 1$ gives good results. In this case $\sqrt{Q_0} = \sigma_0$, a scalar called the *kilometre precision* is expressed in $\mathrm{mm}/\sqrt{\mathrm{km}}$. For two-dimensional networks on the Earth's surface, we have due to isotropy $Q_0 = \sigma_0^2 I_2$, with I_2 the 2×2 unit matrix. This is valid in a small enough area for the Earth's curvature to be negligible, so that map projection co-ordinates (x, y) can be used.

Also for GPS networks an exponent of $k = 1$ has been found appropriate (e.g., (Beutler et al., 1989)). The 3×3 matrix Q_0 contains the component variances and will, in a local horizon system (x, y, H) in a small enough area, typically be diagonal:

$$Q_{0,hor} = \begin{bmatrix} \sigma_h^2 & & \\ & \sigma_h^2 & \\ & & \sigma_v^2 \end{bmatrix},$$

where σ_h^2 and σ_v^2 are the separate horizontal and vertical standard variances. In a geocentric system we get then the location-dependent expression

$$Q_0\left(\mathbf{r}\right) = R\left(\mathbf{r}\right) Q_{0,hor} R^T\left(\mathbf{r}\right),$$

with $R\left(\mathbf{r}\right)$the rotation matrix from geocentric to local horizon orientation for location $\mathbf{r}$.

Now if we choose the following expressions for the variance and covariance of absolute (geocentric) position vectors:

$$\begin{aligned} \text{Var}\left(\mathbf{r}_A\right) &= Q_0\left(\mathbf{r}_A\right) R^k, \\ \text{Var}\left(\mathbf{r}_B\right) &= Q_0\left(\mathbf{r}_B\right) R^k \\ \text{Cov}\left(\mathbf{r}_A, \mathbf{r}_B\right) &= \overline{Q}_{0,AB} \left[R^k - \frac{1}{2} d_{AB}^k \right], \end{aligned}$$

with R the Earth's mean radius, then we obtain the following, generalized expression for the difference vector:

$$\text{Var}\left(\mathbf{r}_B - \mathbf{r}_A\right) = \overline{Q}_{0,AB} d_{AB}^k,$$

with $\overline{Q}_{0,AB} \equiv \frac{1}{2}\left[Q_0\left(\mathbf{r}_A\right) + Q_0\left(\mathbf{r}_B\right)\right]$. This yields a consistent variance structure.

In practice, the transformation to a common geocentric frame will be done using known parameters found in the literature (Boucher and Altamimi, 12.04.2001) for a number of combinations ITRFxx/ETRFyy, where xx/yy are year numbers. Our concern here is only the precision of the co-ordinates thus obtained. We need to know this precision when combining GPS data sets from domains having different canonical WGS84 realizations, requiring their transformation to a suitable common frame.

8 AFFINE TRANSFORMATION ONTO SUPPORT POINTS

Often, one connects traditional local datums to a global datum by an overdetermined Helmert transformation with least-squares estimated parameters. While this will work well in a small area, it doesn't yield geodetic precision over larger national or continental domains.

The PROJ.4 software models such transformations more precisely by augmenting the Helmert transformation by a regular "shift grid" of sufficient density describing a residual deformation field between the two datums. Unfortunately this technique obfuscates how these shifts were originally determined, usually by using a field of irregularly located "common points" known in both global and local systems.

We may derive a plausible variance structure for the current Finnish practice documented in (Anon., 2003), of transforming existing old *kkj* network co-ordinates into the new EUREF-FIN datum by a per-triangle affine transformation applied to a Delaunay triangulation of the set of points common to both datums. The parameters of this transformation follow from the shift vectors in a triangle's corner points and produce an overall transformation continuous over triangle boundaries. We abstract from the actual process producing those local measurements and postulate a formal covariance structure.

Let a given network be transformed to a network of support points assumed exact, forming a (e.g., Delaunay-) triangulation. Let one triangle be ABC and the target point P inside it. The transformation takes the form

$$\begin{aligned} \mathbf{r}_P^{(ABC)} &= \mathbf{r}_P - p^A \left(\mathbf{r}_A - \mathbf{r}_A^{(ABC)} \right) - \\ &- p^B \left(\mathbf{r}_B - \mathbf{r}_B^{(ABC)} \right) - p^C \left(\mathbf{r}_C - \mathbf{r}_C^{(ABC)} \right), \end{aligned}$$

where p^A, p^B, p^C are point P's *barycentric co-ordinates* within triangle ABC (cf. (Vermeer et al., 2004) and Figure 1), with always $p^A + p^B + p^C = 1$. These are readily computable.

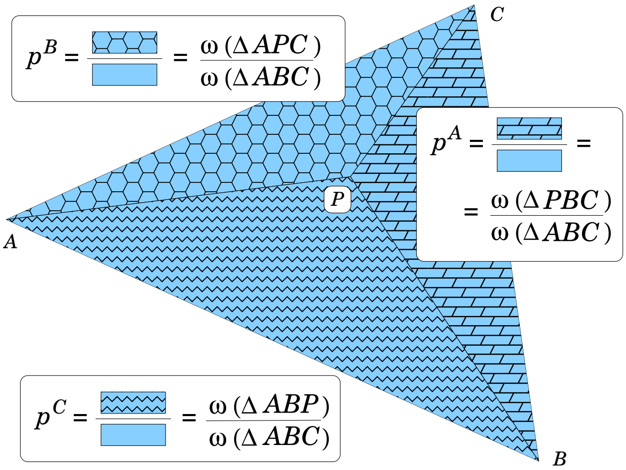

Figure 1: Barycentric co-ordinates illustrated. Every barycentric co-ordinate is the quotient of two triangle surface areas ω: e.g., p^C is the area of triangle ABP divided by the total surface area of ABC.

Then, if we postulate the *a priori* covariance function to be of form

$$\text{Cov}\left(\mathbf{r}_P, \mathbf{r}_Q\right) = g\left(\mathbf{r}_Q - \mathbf{r}_P\right) = g\left(d_{PQ}\right),$$

with $d_{AB} \equiv \|\mathbf{r}_Q - \mathbf{r}_P\|$ the $P - Q$ inter-point distance, and assume the "given co-ordinates" $\mathbf{r}_A^{(ABC)}, \mathbf{r}_B^{(ABC)}, \mathbf{r}_C^{(ABC)}$ to be error free, we get, by propagation of variances, the *a posteriori* variance at point P as

$$\text{Var}\left(\mathbf{r}_P^{(ABC)}\right) = \left[\begin{array}{cccc} 1 & -p^A & -p^B & -p^C \end{array}\right] \cdot$$

$$\cdot \left[\begin{array}{cccc} g(0) & g(d_{PA}) & g(d_{PB}) & g(d_{PC}) \\ g(d_{PA}) & g(0) & g(d_{AB}) & g(d_{AC}) \\ g(d_{PB}) & g(d_{AB}) & g(0) & g(d_{BC}) \\ g(d_{PC}) & g(d_{AC}) & f(d_{BC}) & g(0) \end{array}\right] \left[\begin{array}{c} 1 \\ -p^A \\ -p^B \\ -p^C \end{array}\right].$$

If we further postulate, implicitly defining f:

$$\begin{array}{ccccc} \text{Var}(\mathbf{r}_P) = \text{Var}(\mathbf{r}_Q) & = & g(0) & = & \alpha^2, \\ \text{Cov}(\mathbf{r}_P, \mathbf{r}_Q) & = & g(d_{PQ}) & = & \alpha^2 - \frac{1}{2} f(d_{PQ}), \end{array}$$

then substituting this into the above yields

$$\text{Var}\left(\mathbf{r}_P^{(ABC)}\right) = -\frac{1}{2}\left[\begin{array}{cccc} 1 & -p^A & -p^B & -p^C \end{array}\right] \cdot$$

$$\cdot \left[\begin{array}{cccc} 0 & f(d_{PA}) & f(d_{PB}) & f(d_{PC}) \\ f(d_{PA}) & 0 & f(d_{AB}) & f(d_{AC}) \\ f(d_{PB}) & f(d_{AB}) & 0 & f(d_{BC}) \\ f(d_{PC}) & f(d_{AC}) & f(d_{BC}) & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ -p^A \\ -p^B \\ -p^C \end{array}\right], \tag{2}$$

where the arbitrary α^2 (assumed only to make the variance positive over the area of study) has vanished. A plausible form for the function f, which describes the *inter-point* (*a priori*) variance behaviour, i.e., that of the point difference vector $\mathbf{r}_Q - \mathbf{r}_P$, would be

$$\text{Var}\left(\mathbf{r}_Q - \mathbf{r}_P\right) = f\left(d_{PQ}\right) = Q_0 d_{PQ}^k, \tag{3}$$

with k and Q_0 as the free parameters.

Symbolically we can describe the above as

$$\text{Var}\left(\mathbf{r}_P^{(ABC)}\right) = \mathbf{p}_{P(ABC)} \mathbf{Q}_{P(ABC)}^{P(ABC)} \mathbf{p}_{P(ABC)}^T,$$

where

$$\mathbf{p}_{P(ABC)} \equiv \left[\begin{array}{cccc} 1 & -p_A & -p_B & -p_Q \end{array}\right]$$

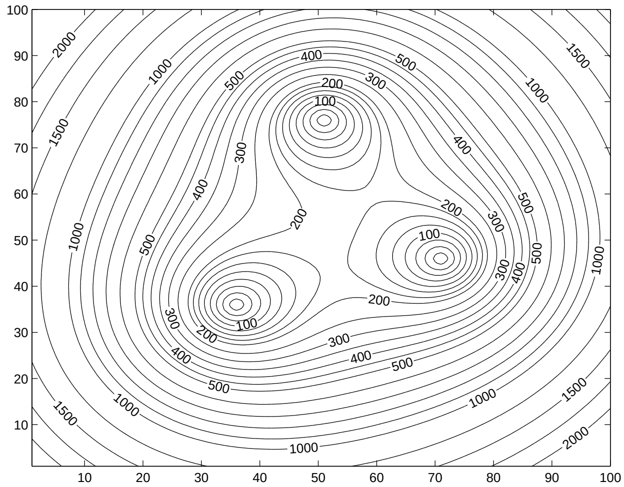

Figure 2: Example plot of point variance after transformation to support points (assumed errorless) within a single triangle. MatLab simulation, arbitrary units.

and

$$\mathbf{Q}_{P(ABC)}^{P(ABC)} =$$

$$= -\frac{1}{2}\left[\begin{array}{cccc} 0 & f(d_{PA}) & f(d_{PB}) & f(d_{PC}) \\ f(d_{PA}) & 0 & f(d_{AB}) & f(d_{AC}) \\ f(d_{PB}) & f(d_{AB}) & 0 & f(d_{BC}) \\ f(d_{PC}) & f(d_{AC}) & f(d_{BC}) & 0 \end{array}\right].$$

In Figure 2 we give for illustration one example of the point variance behaviour after tying to the three corner points of a triangle. Cf. (Vermeer et al., 2004).

Including the uncertainty of the given points, we can write:

$$\text{Var}\left(\mathbf{r}_P^{(ABC)}\right) = \mathbf{p}_{P(ABC)}\left[\mathbf{Q}_{P(ABC)}^{P(ABC)} + \mathbf{Q}_{ABC}^{ABC}\right] \mathbf{p}_{P(ABC)}^T,$$

where we have denoted the *a priori* variance matrix of the given points by

$$\mathbf{Q}_{ABC}^{ABC} \equiv \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & Q_{AA} & Q_{AB} & Q_{AC} \\ 0 & Q_{AB} & Q_{BB} & Q_{BC} \\ 0 & Q_{AC} & Q_{BC} & Q_{CC} \end{array}\right].$$

This represents the given points' variance-covariance information, computed geocentrically as described earlier, i.e.: $Q_{AA} = \text{Var}(\mathbf{r}_A) = Q_0(\mathbf{r_A}) R^k, Q_{AB} = \text{Cov}(\mathbf{r}_A, \mathbf{r}_B) = \left[\frac{1}{2}Q_0(\mathbf{r}_A + Q_0(\mathbf{r}_B))\right]\left[R^k - \frac{1}{2}d_{AB}^2\right]$, etcetera. As a result, we will obtain the *total* point variances and covariances in a *geocentric, unified* datum.

9 INTER-POINT VARIANCES

It is straightforward if laborious to derive also expressions for the *a posteriori* inter-point variances:

$$\text{Var}\left(\mathbf{r}_Q^{(DEF)} - \mathbf{r}_P^{(ABC)}\right) = \text{Var}\left(\mathbf{r}_Q^{(DEF)}\right) +$$

$$+\text{Var}\left(\mathbf{r}_P^{(ABC)}\right) - 2\text{Cov}\left(\mathbf{r}_Q^{(DEF)}, \mathbf{r}_P^{(ABC)}\right), \tag{4}$$

by application of variance propagation like in (2); separately for the cases of P and Q within the same triangle, in different triangles, or in different but adjacent triangles sharing a node or a side. We obtain, for the general case of different triangles ABC and DEF:

$$\mathrm{Cov}\left(\mathbf{r}_Q^{(DEF)}, \mathbf{r}_P^{(ABC)}\right) =$$
$$= \mathbf{p}_{Q(DEF)}\left[\mathbf{Q}_{P(ABC)}^{Q(DEF)} + \mathbf{Q}_{ABC}^{DEF}\right]\mathbf{p}_{P(ABC)}^T,$$

where

$$\mathbf{Q}_{P(ABC)}^{Q(DEF)} =$$
$$= -\frac{1}{2}\begin{bmatrix} 0 & f(d_{DP}) & f(d_{EP}) & f(d_{FP}) \\ f(d_{QA}) & 0 & f(d_{EA}) & f(d_{FA}) \\ f(d_{QB}) & f(d_{DB}) & 0 & f(d_{FB}) \\ f(d_{QC}) & f(d_{DC}) & f(d_{EC}) & 0 \end{bmatrix}$$

and

$$\mathbf{Q}_{ABC}^{DEF} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_{DA} & Q_{DB} & Q_{DC} \\ 0 & Q_{EA} & Q_{EB} & Q_{EC} \\ 0 & Q_{FA} & Q_{FB} & Q_{FC} \end{bmatrix}.$$

From this, we obtain the general relative variance expression by substitution into Eq. 4. Note that for a datum of this type, the locations of the fixed points used become *part of the datum definition*, though for any single variance or covariance to be computed, only six point positions are needed at most.

When representing the spatial precision structure in this way, the representation chosen should also be *semantically valid*, in that it should be possible to extract both point and inter-point mean errors for specified points, and *use them*, .e.g., for detecting inconsistencies between different data sources by statistical testing. This is related to the topic of the Semantic Web and the use of ontologies for specifying integrity constraints (K. Virrantaus, personal comm., and (Mäs et al., 2005)).

10 THE CASE OF UNKNOWN POINT LOCATIONS

If the locations of the common fit points are not actually known, we may derive a *bulk covariance structure* not depending on them. Assume a mean point spacing D and a uniform triangle size. Formula (2) yields, with $P = A$:

$$\mathrm{Var}\left(\mathbf{r}_P^{(ABC)}\right) = -\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}\cdot$$
$$\cdot\begin{bmatrix} 0 & f(d_{AA}) & f(d_{AB}) & f(d_{AC}) \\ f(d_{AA}) & 0 & f(d_{AB}) & f(d_{AC}) \\ f(d_{AB}) & f(d_{AB}) & 0 & f(d_{BC}) \\ f(d_{AC}) & f(d_{AC}) & f(d_{BC}) & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} =$$
$$= -\frac{1}{2}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0,$$

and similarly for the other corner points. The *a posteriori* variance reaches its maximum in the centre of gravity of the triangle, where the barycentric weights are $p^A = p^B = p^C = \frac{1}{3}$. Assuming furthermore that the triangle is equiangular, i.e., $d_{AB} = d_{AC} = d_{BC} \equiv D$, we have also

$$d_{PA} = d_{PB} = d_{PC} = \frac{D}{\sqrt{3}},$$

and

$$\mathrm{Var}\left(\mathbf{r}_P^{(ABC)}\right) = -\frac{1}{2}\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}\cdot$$
$$\cdot\begin{bmatrix} 0 & f\left(\frac{D}{\sqrt{3}}\right) & f\left(\frac{D}{\sqrt{3}}\right) & f\left(\frac{D}{\sqrt{3}}\right) \\ f\left(\frac{D}{\sqrt{3}}\right) & 0 & f(D) & f(D) \\ f\left(\frac{D}{\sqrt{3}}\right) & f(D) & 0 & f(D) \\ f\left(\frac{D}{\sqrt{3}}\right) & f(D) & f(D) & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} =$$
$$= f\left(\frac{D}{\sqrt{3}}\right) - \frac{1}{3}f(D).$$

For power law (3) we obtain

$$\mathrm{Var}\left(\mathbf{r}_P^{(ABC)}\right) =$$
$$= \sigma_0^2 D^k 3^{-k/2} - \frac{1}{3}\sigma_0^2 D^k = \sigma_0^2 D^k\left(3^{-k/2} - 3^{-1}\right).$$

For $k = 1$ this becomes

$$\mathrm{Var}\left(\mathbf{r}_P^{(ABC)}\right) = \sigma_0^2 D \cdot \frac{1}{3}\left(\sqrt{3} - 1\right) \approx 0.244\sigma_0^2 D.$$

We can symbolically write

$$\Delta_k \equiv 3^{-k/2} - 3^{-1}.$$

Use the above derived upper bound for the single point variance and postulate the following replacement variance structure:

$$\begin{aligned} \mathrm{Var}\left(\mathbf{r}_P^{(\Delta)}\right) &= \Delta_k\sigma_0^2 D^k, \\ \mathrm{Cov}\left(\mathbf{r}_P^{(\Delta)}, \mathbf{r}_Q^{(\Delta)}\right) &= \Delta_k\sigma_0^2 D^k - \frac{1}{2}F(d_{PQ}). \end{aligned}$$

Note that *here*, the constant $\Delta_k\sigma_0^2 D^k$, unlike α^2 above, is no longer arbitrary. It does similarly vanish, however, when we derive the inter-point variance:

$$\mathrm{Var}\left(\mathbf{r}_Q^{(\Delta)} - \mathbf{r}_P^{(\Delta)}\right) = \mathrm{Var}\left(\mathbf{r}_P^{(\Delta)}\right) + \mathrm{Var}\left(\mathbf{r}_Q^{(\Delta)}\right) -$$
$$-2\mathrm{Cov}\left(\mathbf{r}_P^{(\Delta)}, \mathbf{r}_Q^{(\Delta)}\right) = F(d_{PQ}).$$

We wish to see a variance structure, in which these *a posteriori* inter-point variances behave in the following reasonable way:

1. For P and Q close together (and often within the same triangle), we want the relative variance to behave according to the k-power law;
2. For larger distances, and P and Q in different triangles, we want the relative variance to "level off" to a constant value. We know it can never exceed twice the posterior variance of a single point, which is $\Delta_k\sigma_0^2 D^k$ max (And never less than 0, which happens if both P and Q coincide with nodes of the triangulation).

Therefore we choose

$$\begin{aligned} F(d_{PQ}) &= \frac{1}{1/\sigma_0^2 d_{PQ}^k + 1/2\Delta_k\sigma_0^2 D^k} = \\ &= \sigma_0^2\frac{1}{1/d_{PQ}^k + 1/2\Delta_k D^k} = \sigma_0^2\frac{2\Delta_k d_{PQ}^k D^k}{d_{PQ}^k + 2\Delta_k D^k}, \end{aligned}$$

which behaves in this way, with a smooth transition between the two regimes.

11 FINAL REMARKS

We believe that geodesists and spatial information specialists should get better acquainted with each other's ideas. Precise geodetic information still commonly moves around as files of co-ordinates, processed by dedicated software to maintain the highest precision. Dissemination using standard Web services promises many practical benefits, but is not well known in geodetic circles and currently used only for mapping-grade geographic information.

Now also in geodesy, awareness is growing, e.g., in connection with the GGOS (Geodetic Global Observing System) initiative, cf. (Neilan, 2005), that precise geodetic co-ordinate information should be seen and integrated as part of our spatial data infrastructure. Care should then be taken to properly represent and manage its spatial precision structure.

12 CONCLUSIONS

We have derived criterion matrix expressions for modelling the variance-covariance behaviour of

1. the geocentric co-ordinates of a set of GPS-determined "fixed points";
2. co-ordinates in a local geodetic network that has been tied to a set of GPS-positioned points by a triangle-wise affine (bi-linear) transformation.

We were motivated to present these derivations by their possible use in co-ordinate Web services for geodesy. They will allow proper co-ordinate precision modelling when bringing geodetic co-ordinate material from heterogeneous sources on a single common geocentric datum.

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