THE BASIC TOPOLOGY ISSUES ON SPHERICAL SURFACE QUTERNARY TRANGULAR MESH

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ABSTRACT:

SGDM (Sphere Grid Data Model) is an efficient method to deal with the global data because of the advantages of multi-resolution and hierarchy. However, SGDM has no distinct descriptions and lack of round mathematical basis for various applications. What's more, most of mathematical model about global application have been based on the continuous methods. Although several researchers have considered the digital topology in 2-dimension and 3-dimension Euclidean Plane, complete theoretical foundation of proper theory for SQTM is still missing. In fact, it's more convenient and efficient to compute spatial relationship based on SQTM.

Firstly, this paper constructs SQTM based on manifold, *i.e.* the digitization of the spherical surface as a common spatial framework just as planar. Secondly, the concept of spherical surface digital topology will be given in the context of adjacency. Then topological properties and paradox of discrete objects will be discussed in SQTM. In the end, we give some potential applications about spherical surface digital topology.

1 INTRODUCTION

The surface of the earth is an important spatial domain, which is of course topologically equivalent to the surface of a spherical surface, ellipse and geoid. In fact, it's not topologically equivalent to any subset of the Cartesian plane (Sahr 1996; White 1992, 1998). So it's unpractical to analyze the spherical surface with the planar methods. Three typical conceptual modeling approaches are used to apply and adapt spatial analysis techniques to the spherical surface as follows (Raskin 1994).

- Map projection: The spherical surface is projected onto a plane using a conventional map projection. The projection approach is used implicitly in conventional studies that ignore the curvature of the earth; however, no one projection can keep both distance and area. What's more, the map projection transforms the spherical surface manifold to planar Euclidean space, therefore, the distance, orientation and area in large field are not accurate at all.

- Embeddings: The spherical surface is considered a constrained subset of the three-dimensional space R^3. Tree-dimensional spatial analysis is performed, with a constraint imposed to limit solutions to the spherical surface. The most typical one is direction cosine, which avoids the singularity of the pole. Although direction cosine has a perfect mathematics base, so it belongs to the vector method and does not accord with the discrete properties of real word in essential.

- Intrinsic: The spherical surface is considered an intrinsic space in its own right, with analysis performed in non-Euclidean space S^2. Longitude and latitude coordinate and SGDM are most typical two. SGDM (Sphere Grid Data Model) is research topic of this paper just because of the advantages of multi-resolution and hierarchy.

This paper is concerned with spherical surface digital topology. Digital topology provides a sound mathematical basis for various image-processing applications including surface detection, border tracking, and thinning in 2D Euclidean space (Kong 1986; Rosenfield 1975). We often use voxel representation to describe objects on a computer. Specifically, SQTM is partitioned into unit triangles. In this representation, an object in spherical surface is described by an array of bits. In this way, a spherical surface digital object can be defined as an array augmented by a neighborhood structure. The emphasis of this paper is on the differences between planar and spherical surface digital topology. It is specific to the basic topology model on the surface of an earth, and thus, the ellipsoidal nature of the earth and its vertical dimension are not considered.

The paper is organized as follows. Next section presents the definitions of SQTM based on manifold. In Section 3, the basic topology model of SQTM is discussed. In Section 4, the discussions and the future works end the paper.

2 THE DEFINITION OF SQTM BASED ON MANIFOLD

Regular grid sampling structures in the plane are a common spatial framework for many applications. Constructing grids with desirable properties such as equality of area and shape is more difficult on a sphere (White et al. 1998). To deal with the problems on the Earth conveniently, it is necessary to construct a similar regular mesh structure as a common spatial framework for spherical surface just as planar. Such similar regular mesh system is named as SQTM, which is the digitization of the spherical surface. That is, spherical surface can be described with discrete point sample in SQTM. Therefore, it is necessary to subdivide the spherical surface according to its characteristics. There are three steps to get the sphere digital space just as follows.

2.1 Initial Partition of the Spherical Surface

The Platonic solids are reasonable starting points for a spherical subdivision (shown in Figure 1). Three of the five polyhedrons have triangular faces, such as the tetrahedron (four faces), the octahedron (eight faces), and the icosahedron (20 faces). The other Platonic solids are the cube (six faces) and the pentagonal dodecahedron (12 faces). The icosahedron has the greatest number of initial faces, and would therefore show the least distortion in the subdivision. However, the larger number of faces makes it somewhat harder to deal with the problems through the borders of the initial faces. In a word, the spherical surface is more easily covered by triangles, and the triangles of the initial partition need not be equilateral. Distortion could be decreased considerably by dividing each equilateral triangular side of an initial Platonic figure into equivalent scalene triangles (White et al. 1998).

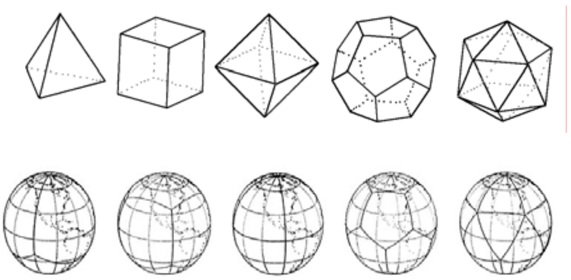


Figure 1. Platonic solids and their spherical surface subdivision (White et al. 1992)

The octahedron has more distortion, but it has the advantage that its faces and vertices map to the important global features: meridians, the equator, and the poles (Goodchild and Shiren 1992). Therefore, in this paper octahedron is selected as common initial partition in which eight base triangles are produced.

2.2 Subdivision of Triangular Cells

There are several ways to hierarchically subdivide an equilateral triangle such as quaternary subdivision and binary subdivision. All of these are subject to distortion when transferred to the spherical surface. Different decisions will have different effects on the uniformity of shape and size of cells within a given level of the hierarchy, as well as on the ease of calculation. Here, the quaternary subdivision is selected, in which a triangle is subdivided by joining the midpoints of each side with a new edge, to create four sub-triangles.

The quaternary subdivision is a good compromise. It is relatively easy to work with, and non-distorting on the plane: a planar equilateral triangle is divided into four equilateral triangles. But a spherical base triangle may be divided into four equivalent triangles. The result of subdivision based on octahedron with quaternary subdivision is as follows in Figure 2 (Dutton 1996).

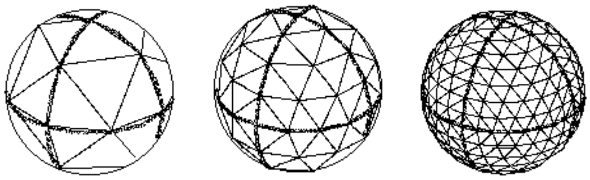


Figure 2. The result of subdivision based on octahedron (Dutton 1996)

2.3 The Definition of SQTM Based on Manifold

Manifold is the extension of Euclidean just because every point in manifold has a homeomorphism of an open set in Euclidean. So local coordinates system can be set up for every point in manifold. It seems that manifold is a result plastered with many Euclidean spaces. It can be proved that spherical surface is a 2-dimension smooth manifold (Evidence omitted).

If the spherical surface is divided by quaternary subdivision based on octahedron, the SQTM is $8\times 4^N\,(N=\{0,1,...,n-1\})$ regular mesh based on finite discrete space, expressed as T^2. In the first level, spherical surface has the 8 base triangles, which are local coordinates systems of manifold. The relationship between 8 local coordinate systems can be described by spherical surface spacefilling curves (shown as Figure 3), which is a continuous mapping from a one-dimensional interval, to the points on the spherical surface. Continuous ordering based on spacefilling curves have been proven useful in heuristics related to a number of spatial, combinatorial, and logistical problems (Bartholdi and Goldsman 2001)

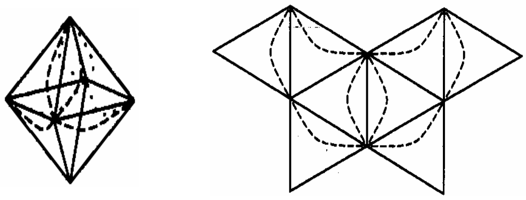
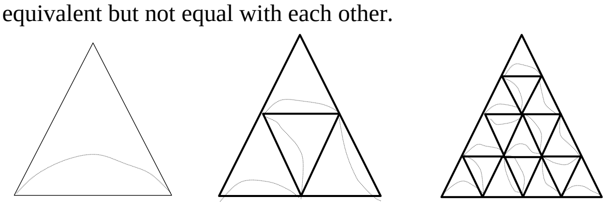


Figure 3. Spherical spacefilling curves based on octahedron with quaternary subdivision

To every base triangle, quaternary spherical surface spacefilling curve still can be used to express the relationship between every sub-triangle. In quaternary subdivision, the relationship between sub-triangles can be depicted with quaternary spherical surface spacefilling curve (shown as Figure 4). In given resolution, SQTM can be continuously indexed by quaternary spherical spacefilling curve (Details in Bartholdi 2001). Comparing with the other model (Dutton 1991), SQTM has the advantage of continuous ordering. It makes us to index the sphere digital space continuously to allow quick and efficient search at multi-scale. At the same time, SQTM has the intrinsic disadvantage that the triangle is equivalent but not equal with each other.



(a) level 1 (b) level2 (c) level 3

Figure 4. The quaternary spherical surface spacefilling curve

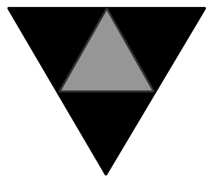
Planar digital space is a simple Euclidean space, but SQTM is a more complex manifold. So SQTM is not the simple copy of planar digital space. It has some special properties just as follows. SQTM is not a Euclidean space, that is to say, it is no homomorphous to planar and no single coordinates system can be set up to express every point in spherical surface. Although cells of SQTM are approximately equivalent, it still has a multi-scale and continuous ordering advantages (Bartholdi and Goodsman 2001).

3 THE BASIC TOPOLOGY ISSUES ON SQTM

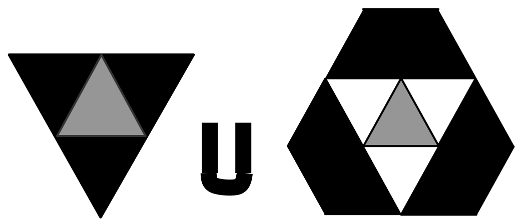
From the definition of SQTM, T^2 is the result of partitioning the connected spherical surface into small triangular pieces that cover the whole spherical surface space. Each spherical triangle is viewed as an element, called "spel" (short for spatial element). All the spels in the spherical surface can form a new set, which can be named as grid set T^2. The set T^2 can then be regarded as the hardware of the SQTM. The transitive closure δ of the adjacency relation between the two spels in T^2 can be considered as software. This system can be expressed as $\langle T^2, \delta \rangle$, where δ is the binary relations. This binary relation determines the connectedness between the spels in T^2. $\langle T^2, \delta \rangle$ is also referred to as "spherical surface digital topology". S^2 is a connected space, but the T^2 is not connected space. In T^2, this implicit assumption of connectedness in S^2 no longer works.

3.1 General Definitions and Notations

Points of T^2 associated with triangles that have value 1 are called black points, and those associated with triangles with value 0 are called white points. The set of black points normally corresponds to an object in the digital image. First, we consider objects as subsets of the SQTM T^2. Elements of T^2 are called "spels" (short for spatial element). The set of spels which do not belong to an object O is included in T^2 constitute the complement of the object and is denoted by $\overline{O}$. Any spel can be seen as a unit triangle centered at a point with integer coordinates. Now, we can define some binary symmetric antireflexive relations between spels. Two spels are considered as 3-adjacency if they share an edge and 12-adjacent if they share a vertex. For topological considerations, we must always use two different adjacency relations for an object and its complement (shown as Figure 5). We sum this up by the use of a couple (n, n') with $(n, n') = \{3;12\}$, the n-adjacency being used for the object and the n'-adjacency for its complement. By transitive closure of these adjacency relations, we can define another one: connectivity between spels. We define an n-path π with a length k from spel a to spel b in O included in T^2 as a sequence of $\mathrm{voxels}\,(\mathrm{i})\ \mathrm{i} = 0; \ldots, \mathrm{k}$, such that for $0 \le \mathrm{i} \le \mathrm{k}$, the spel v_i is n-adjacent or equal to $\mathrm{v}_{\mathrm{i}+1}$, with $\mathrm{v}_0 = \mathrm{a}$ and $\mathrm{v}_\mathrm{k} = \mathrm{b}$. Now we define connectivity: two voxels a and b are called n-connected in an object O if there exists an n-path π from a to b in O. This is an equivalence relation between spels of O, and the n-connected components of an object O are equivalence classes of spels according to this relation. Using this equivalence relation on the complement of an object we can define a background component of O as an n'-connected component of O'.



(a) 3-adjacent



(b) 12-adjacent

Figure 5. The definition of 3-adjacent and 12-adjacent

In 2D SQTM, we consider spherical surface triangle mesh to express spherical surface digital image. In this paper, points refer to grid points in SQTM unless stated otherwise. Two nonempty sets of points S_1 and S_2 are said to be 3-adjacent or 12-adjacent if at least one point of S_1 is 3-adjacent or 12-adjacent to at least one point of S_2. The adjacency definition is important not only in the computation of raster distance between two spels but also in topological analysis (LI et al. 2000). Let S be a nonempty set of points. An 3-path between two points p, q in S means a sequence of distinct points $p = p_0, p_1, ..., p_n = q$ of S such that p_i is 3-adjacent to p_{i+1}, $0 \le i < n$. Two points $p, q \in S$ are 3-connected in S if there exists an 3-path from p to q in S. An 3-component of S is a maximal subset of S where each pair of points is 3-connected.

A 2D spherical surface digital object E can be defined as the set of black points that is spatial entity in spherical digital space. Samely, $T^2 - E$ is the set of white points, which is called the background of E. 3-adjacency or 12-adjacency are the adjacencies used for finding 3-components and

12-components in E and $I - E$ respectively. In this paper, we use 12-adjacency for black points and 3-adjacency for white points and call 12-components of E black components and 3-components of $I - E$ white component. The basic topological components of a spatial entity in SQTM are still interior, boundary and exterior. A point $p \in E$ is called an interior point of E if $N(p) \subset E$, otherwise p is called a border point of E. The set of all interior points of E is called the interior of E and is denoted as E°. The set of all border points of E is called the border of E and is denoted as ∂E. The closure of E is denoted as $\overline{E}$. The relationship between interior, closure and boundary is as follows:

$$\mathrm{E}^o \cap \partial \mathrm{E} = \Phi$$

$$\mathrm{E}^o \cup \partial \mathrm{E} = \overline{\mathrm{E}}$$

$$\partial \mathrm{E} = \overline{\mathrm{E}} \cap (\overline{\mathrm{E}})^-$$

3.2 The Properties of SQTM

Property I. In the SQTM, if the length of a piece of 3-connected simple closed curve exceeds 12, its interior and exterior are non-empty(Just as Figure 6).

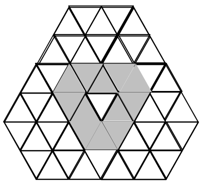


Figure 6. Property I of SQTM

Property II. In the QTM-spherical raster space, if the length of a piece of 12-connected simple closed curve exceeds 3, its interior and exterior are non-empty (Just as Figure 7).

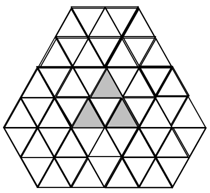


Figure 7. Property II of SQTM

Property III. Let C denotes a piece of 12-connected simple closed curve whose length exceeds 3, L a piece of curve connecting any interior point and any exterior point of C, then C and L are interconnected (Just as Figure 8) .

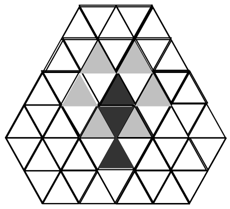


Figure 8. Property III of SQTM

Property IV. Let C denotes a piece of 3-connected simple closed curve whose length exceeds 12, L a piece of curve connecting any interior point and any exterior point of C, then C and L are interconnected (Just as Figure 9).

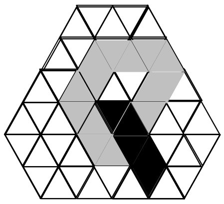


Figure 9. Property IV of SQTM

3.3 Topological Paradox Associated with Definition of Adjacency in SQTM

Topological paradox in SQTM just as Figure 10, there are six black spels, one gray spel and some white spels. The gray spel is surrounded by the six black spels. If 12-adjacency is defined, the black spels are connected and should form a closed line; however, this black line cannot separate the central gray spel from the white spels. If 3-adjacency is defined, the black spels do separate the central gray spel from the white spels; however, these black spels are totally disconnected and thus no closed line has been formed by the black spels in this case. So this leads to the topological paradox in raster space T^2. To deal with this paradox, the white spels are defined as being 3-connected and black spels 12-connected, vice versa. In SQTM, background and object have the different connectedness. That is to say, the spatial entity in spherical surface is defined as being 12-connected, but the background is defined as being 3-connected. So, the six black spels defined as 12-connected should be connected. However, gray spel and black spels just as background should be not connected if the background is defined as being 3-connected. So the continuous curve (connected path in T^2) separate the spherical surface two parts.

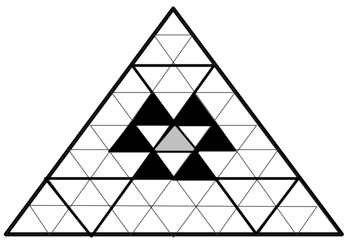


Figure 10. Topological paradox in SQTM

But why this topological paradox happens? We use the six spels in figure 11 to explain it. In reality, when one considers spels 1, 3 and 5 to be connected, one has already implicitly assumed that P belongs to the black line. On the other hand, when one considers spels 2, 4 and 6 to be connected, one has already implicitly assumed that P belongs to the white spels. That is, the point P belongs to two different things (LI et al. 2000). If the black spels represent spatial entities and the white spels represent the background, then point P belongs to both the background and the entity at the same time, thus having dual meanings. This of course leads to paradox–a kind of ambiguity. To solve the problem, one must eliminate the dual meanings of point P. One should only allow P to belong to either the entity or the background but not both. In this paper, the spels belonging to background are defined as 3-connected, however, the spels belonging to the object are defined as 12-connected.

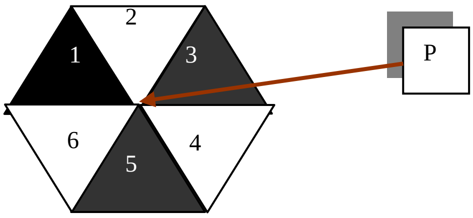


Figure 11. Topological paradox caused by the ambiguity at point P

3.3 Relationship between Topology between SQTM and Spherical Surface Continuous Space

The connectedness of raster space is based on the adjacency of two neighboring spels (LI et al. 2000). In SQTM, there is a common line (Figure 12a) between the two spels in the case of 3-connectedness. On the other hand, in the case of 12-connectedness, the common part could be either a line, a point, or both. In other words, there is at least a point in common if the two spels are to be connected. If an arbitrary (vector) point is selected from each spel, say ``a'' and ``b'', then the path from ``a'' to ``b'' intersects the common line at P. Points ``a'', ``b'' and P are points in vector space. Points ``a'' and P are connected in the left spel and points P and ``b'' are also connected in the right spel in vector space. As the connectedness is transitive, points ``a'' and ``b'' are therefore connected. As a result, any point in the left spel is connected to any point in the right spel. It means that the connectedness concept in vector space has been implicitly adopted when the connectedness concept in raster space is discussed.

4 CONCLUSION

SQTM is an efficient method to deal with the global data because of the advantages of multi-resolution and hierarchy. However, SQTM has no distinct descriptions and lack of round mathematical basis for various applications. This paper gave the definition of SQTM, which has the characters as follows:

- Similar regular grids based on spherical surface discrete space.
- Spherical spacefilling curves can be used to express the relationship between basic local coordination.
- No single coordination system can express every point in the spherical surface.
- Multi-scale and continuous ordering.

As another important part, this paper set up the basic topology model which include the topological structure of SQTM, the basic topological components of a spatial entity in SQTM, topological paradox associated with definition of adjacency in SQTM and so on. This paper is just an introduction to studying the characterization of 2D digital spherical manifold and the Jordan separation theorem, which are all round mathematic basis of spherical spatial computing and reasoning.

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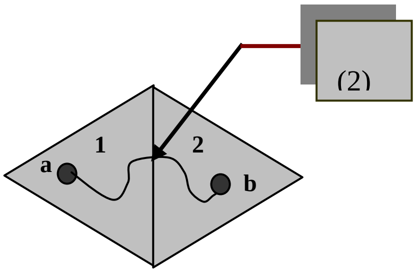
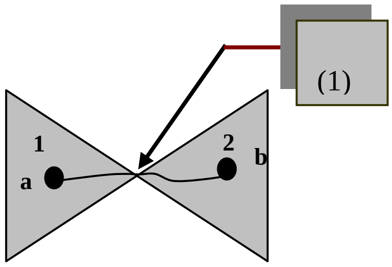


(a) in the case of 3-adjacency



(a) in the case of 12-adjacency

Figure 12. Implicit dependency of topological connectedness in T^2

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