CALCULATING TOPOLOGICAL SPATIAL RELATIONS BASED ON REGION ALGEBRA: THE THINKING IN ADJACENCY RELATIONSHIP

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ABSTRACT:

Spatial relation calculation based on Delaunay structure is dual hot problem, which gather the theories and applications of GIS. Firstly, it is the important supplement of the theory of spatial relation calculation. Secondly, it tones up the application popularity properties of Delaunay Triangulation and Voronoi diagram, which are representative of the mixed space partition data structure. In part of foundational theory, (i) proving that $\mathcal{CDT}$ is simplicial complex in $\mathbb{E}^2$. (ii) importing chain structure in $\mathcal{CDT}$ and educe including & approximating theorem and reduced including & approximating theorem, which are based on vector half-plane and σ's edge neighbor and are used for estimating the left, middle and right side properties of σ. (iii) defining the region in $\mathcal{CDT}$ and establishing region algebra (RA), which use the set of region as computational space and use the intersection operator as binary operation. (iv) describing basic forms of node and chain which are contained within complex σ. In part of dynamic spatial relation calculation, (i) describing the spatial object's three entries, i.e. exterior ($^-$), boundary (∂) and interior ($^\circ$), with left, middle and right of σ and their combination. (ii) establishing the spatial relation calculation model-region nine intersection(R9I), which is used the intersection operation(∩) and form operation(τ) as basic operations and is based on the generic intersection model. (iii) calculating thirty-two spatial relations of simple objects with R9I.

1. INTRUDUCTION

Data models in spatial information systems are usually based either on boundary representations consisting of points, line segments and polygons, or on regular, rectangular subdivisions of space in which each cell is associated with an identity or classification of the phenomena represented (Jones and Ware 1998). Constrained Delaunay triangulation (CDT) which fuses the graph theory and geometry question solution as a body belongs to the type of field-based spatial data model in essence. It is the common observing way for the vector and raster data models (Wu 2000, Chen 2002). CDT is a kind of irregular spatial division structure and it is a variation of triangulated irregular network (TIN). In the 2-Dimentation Euclidean space, the CDT has two characteristics as its restraint condition: one is the discrete data has many "directional broken line"; the other is the discrete data has many "close polygon ring" (Wu and Shi 2003). In CDT, the triangulations spread all over the whole region without superposition and slot. The spatial relations between entities are expressed by the connection of triangulations. Sibson (1978) proposed that there is only one triangulation division for the limited discrete point set, i.e. Delaunay triangulation. The Delaunay triangulation is the dual graph of Dirichlet, Voronoi or Thiesson spatial partition structure. This is an important concept in the geographical information science, because the Thiesson polygon can be defined as the influence region of arbitrary spatial entities (McCullagh and Ross 1980, Chen 2000). Because of its proper characteristics of "the empty circumcircle criterion" and "the local max-min angle criterion" (Preparata and Shamos 1985), the CDT is regarded as the powerful tool to express the adjacency spatial relationship which is defined by Voronoi diagram (Ai 2000). In the past decade, the idea of adjacency has been apply in many fields of GIS.

In the domain of map generalization, Delaunay triangulation and its dual Voronoi diagram are widely used in the spatial conflict detection of object and calculation of adjacency relationships. In these researches, many data models were proposed. Jones and Ware (1995, 1997, 1998) established simplex data structure (SDS) model, which constructed the CDT of roads and building outlines and then used the adjacency information to modify and move the buildings. Peng (1995) proposed the EFDS model by amending the FDS model of Molenaar in order to extract the "safe area" and "unsafe area" which are used in expressing the object generating space in CDT.

In the domain of spatial query language, Chen and Cui (1997) extended query abilities of adjacency and lateral adjacency relationships with CDT in MapInfo. Chen and Zhao (2004) proposed the concept of *k*-order adjacency according to Voronoi adjacency, and built contour tree with it, their method given an availability way for automating evaluation of contour. In order to append adjacency query ability in spatial database, Li *et al*. (2006) proposed the unitsDelaunay structure which built the bridge between adjacency objects and current spatial index methods. Their method for discrete areal objects is that, firstly, built up the CDT of objects set; secondly, classified triangles in three types; thirdly, aggregated triangle which represented same adjacency relationship as a unit; finally, approximated the scope of units with the minimal boundary rectangle (MBR) and integrated the MBR with grid file spatial index.

De Floriani (1987) stated that the topology of a triangular subdivision is completely and unambiguously represented by any suitably selected subset of nine adjacency relations (NAR) between entities (vertices, edges, triangles). As shown in figure 1(a), for point's entities, adjacency relationship definition which is based on the TIN is clear. However, as shown in figure 1(b) and (c), for linear and areal entities, we can't directly get the adjacency relationship by triangle which in the CDT, because we don't know that every triangle describing topological information. Therefore, while get adjacency relationship with CDT, we have to do some preparing work by dint of some topological relations computing method. For the situations of figure 1(b) and (c), firstly guaranteeing the ***p*** and ***c*** are disjoint, and then we can get ***p*** has the adjacency relationship to ***c*** according to the triangle σ.

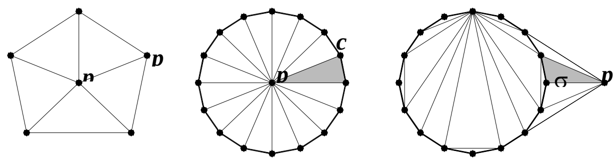


Figure 1. (a) Points and adjacency relationships between them in a TIN. (b) One point and one polygon in a CDT, the point ***p*** locates in the ***c***'s interior. (c) One point and one polygon in a CDT, the point ***p*** locates in the ***c***'s exterior.

In this paper we focus on the problem is that can the general topological relations be got from CDT. If we can do this, it meaning the other way for spatial relation computing and availability using for CDT. This problem includes below several contents:

- What are the basic characters of CDT? And how to representing spatial objects with triangles in CDT?
- What is the mathematic structure which supporting geometry computing on CDT?
- How to describing the basic topological forms with triangle or composing of triangles? And how many kinds of topological relations can be got from CDT?

The paper is organized as follows. In section 2, basic concepts about CDT and the region algebra structure are introduced. Section 3 presents the description, basic operators of spatial feature based on region algebra and a new model of spatial relation calculation (R9I). We also present relations calculation of simple features with R9I in the section. In section 4, we give a multi-rule topological examination for vector data with our method. Finally, conclusions and future works are given in the last section.

2. THE REGION ALGEBRA BASED ON CDT

2.1 Description of Triangle and Chain Based on Topology

DEFINITION 2.1.1.(simplex) Given v_0, $v_1, \cdots$, v_k are $k+1$ substantive points in $\mathbb{E}^2$, $\mathbf{S}_k$ is the minimal convex set composed of these points which is called as the k-dimension simplex. The $\mathbf{S}_k$ can be expressed as linear combination $\mathbf{S}_k = \{v \mid v = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k\}$, λ is a non-negative real number and satisfied to $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$.

By definition 2.1.1, a k-simplex is a convex body. Specially, if $k = 0$, 0-simplex $\mathbf{S}_0$ is a substantive point; if $k = 1$, 1-simplex $\mathbf{S}_1$ is a line segment; 2-simplex is a triangle. In the domain of geometry, a 2-simplex $\mathbf{S}_2$ can be expressed as full information set, i.e.

$$\mathbf{S}_2 = \{v_1v_2v_3,\ v_1v_2,\ v_2v_3,\ v_3v_1,\ v_1,\ v_2,\ v_3\}, \qquad (2\text{-}1\text{-}1)$$

where v_1, v_2, v_3 are the 0-simplex vertexes; v_1v_2, v_2v_3, v_3v_1 are 1-simplex line segments; $v_1v_2v_3$ is a 2-simplex triangle. The formula (2-1-1) is equal to be expressed as

$$\mathbf{S}_2 = \bigcup_{i=0}^{2} \mathbf{S}_i(\cdot), \qquad (2\text{-}1\text{-}2)$$

where $\mathbf{S}_i(\cdot)$ is the set composed of all i-simplexes.

DEFINITION 2.1.2. There are two 2-simplexes $\mathbf{S}_2$ and $\mathbf{S}'_2$, $v(x, y) \in \mathbf{S}_2$, $v'(x', y') \in \mathbf{S}'_2$. If $x = x'$ and $y = y'$, then v is equal to v', i.e. $v = v'$.

DEFINITION 2.1.3.(chain) A chain is a set, which composed of many elements of $\mathbf{S}_1$. The formula is

$$\boldsymbol{C} = \{\mathbf{S}_1^1, \mathbf{S}_2^1, \cdots\cdots, \mathbf{S}_n^1\},\ |\mathbf{S}_i^1 \cap \mathbf{S}_{i+1}^1| = 1, n > 1,\ i < n, n, i \in \mathbf{N}, \qquad (2\text{-}1\text{-}3)$$

if exists $\mathbf{S}_1^1 = \mathbf{S}_n^1$, then the chain is called as the cycle chain. The linear features of chain are showed by the continuation and order of elements. So, the chain can be described as a vector,

$$\boldsymbol{C} = (\mathbf{S}_1^1, \mathbf{S}_2^1, \cdots\cdots, \mathbf{S}_n^1), \quad n > 1, n \in \mathbf{N}. \qquad (2\text{-}1\text{-}4)$$

DEFINITION 2.1.4.(child chain) The chain which is composed of n ($n \geq 1$) elements of chain $\boldsymbol{C}$ and remains the continuation of $\boldsymbol{C}$ is called as the child chain of $\boldsymbol{C}$, denoted as $\boldsymbol{C}'$, if $n = 1$, the chain is called as the minimal child chain of $\boldsymbol{C}$, denoted as $\boldsymbol{C}''$.

As showed in figure 2, chain (23 , 24) is the child chain of $\boldsymbol{C}$; chain (67) is composed of one element of $\boldsymbol{C}$, so it is one of the minimal child chains of $\boldsymbol{C}$; chain (24) and chain (48) aren't the minimal child chains of $\boldsymbol{C}$, so chain (24, 48) aren't the child chain of $\boldsymbol{C}$.

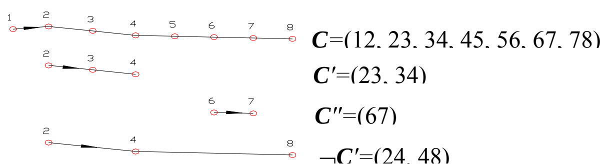


Figure 2. Chain, child chain and the minimal child chain

DEFINITION 2.1.5.(triangle neighbor) Given two 2-simplex $\mathbf{S}_2$ and $\mathbf{S}'_2$, if there is

$$|\mathbf{S}_0(\cdot) \cap \mathbf{S}'_0(\cdot)| = 2,\ \mathbf{S}_0(\cdot) \subset \mathbf{S}_2 \wedge \mathbf{S}'_0(\cdot) \subset \mathbf{S}'_2, \qquad (2\text{-}1\text{-}5)$$

then $\mathbf{S}_2$ has neighbor relationship with $\mathbf{S}'_2$.

Definition of triangle neighbor is easy to show, in general, that two 2-simplexs share a common 1-simplex, which is composed by two 0-simplexs.

In the vector algebra, a quantity completely specified by a magnitude and a direction as a vector, the 1-simplex (v_1v_2) is a simple vector. By the method of vector algebra, the (2-1-1) formula $\mathbf{S}_2$ is expressed as following form,

$$\mathbf{S}_2 = \{\{(v_1v_2), (v_2v_3), (v_3v_1)\}, \{(v_1v_2), (v_2v_3), (v_3v_1)\}, \{v_1, v_2, v_3\}\}. \quad (2\text{-}1\text{-}6)$$

For the direction of simplex, there are following descriptions: the 0-simplex has arbitrary direction; the direction of 1-simplex (v_1v_2) is from v_1 to v_2 or from v_2 to v_1, but these directions are not equivalent. The direction of 2-simplex is defined as anticlockwise or clockwise. Every simplex can be confirmed by its vertexes, and then a 2-simplex can be equivalently expressed as the following algebra form,

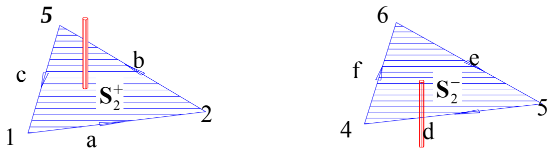
$$\mathbf{S}_2 = \langle v_1v_2v_3\rangle \equiv \langle v_2v_3v_1\rangle \equiv \langle v_3v_1v_2\rangle. \quad (2\text{-}1\text{-}7)$$

Definition 2.1.6.(vector product in triangle) In $\mathbf{S}_2$, the vector product of vertex i ($i = 1, 2, 3$) is

$$\times_i = (v_{i-1}v_i) \times (v_iv_{i+1}) = |(v_{i-1}v_i)|\,|(v_iv_{i+1})|\sin\theta,\ \ \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (2\text{-}1\text{-}8)$$

where θ is the vector included angle of $(v_{i-1}v_i)$ and (v_iv_{i+1}), $\times_i$ is a vector value, which is perpendicular to the plane composed of $(v_{i-1}v_i)$ and (v_iv_{i+1}), and $(v_{i-1}v_i)$, (v_iv_{i+1}) with $\times_i$ forms the right-handed system.

Theorem 2.1.1. Given $\mathbf{S}_2$, the order of vertexes is anticlockwise or clockwise, if the value of $\exists\times_i$ ($i = 1, 2, 3$) is more than 0 (> 0), the order of vertexes is anticlockwise, denoted as $\mathbf{S}^+$; if the value of $\exists\times_i$ ($i = 1, 2, 3$) is less than 0 (< 0), the order of vertexes is clockwise, denoted as $\mathbf{S}^-$. (omitted)



(a) Positive vector product and vertices of triangulation anticlockwise ordering (b) Negative vector product and vertices of triangulation clockwise ordering
Figure 3. Relation of triangle's vector product and vertices ordering

2.2 Definition of CDT and Its Basic Characteristics

Definition 2.2.1.(plane Voronoi diagram) Given $\mathbb{E}^2$, point set $\boldsymbol{P}$♣ $= \{p_1, p_2, \ldots, p_n\}$, $(3 \le n < \infty, p_i \ne p_j, i \ne j, i, j \in \mathbb{N})$,

$$V(p_i) = \{p \mid \boldsymbol{d}(p, p_i) \le \boldsymbol{d}(p, p_j),\ i \ne j, i, j \in \mathbb{N}\}, \quad (2\text{-}2\text{-}1)$$

the region defined by the formula (2-2-1) is called as Voronoi polygon of point p_i. The Voronoi polygon set of all point $p_1, p_2, \ldots, p_n$ in point set $\boldsymbol{P}$ is

$$\mathcal{V}(\boldsymbol{P}) = \{V(p_1), V(p_2), \ldots, V(p_n)\}, \quad (2\text{-}2\text{-}2)$$

the Voronoi polygon set make up of the plane Voronoi diagram of $\boldsymbol{P}$.

Definition 2.2.2.($\mathcal{DT}$) Given $\mathbb{E}^2$, point set $\boldsymbol{P}$ and $\mathcal{V}(\boldsymbol{P})$, the neighbor grid which results from the connection of points in $\boldsymbol{P}$ in condition of share Voronoi boundary is called Delaunay Triangulation (DT) of set $\boldsymbol{P}$, denoted as $\mathcal{D}(\boldsymbol{P})$.

If the Delaunay triangulation is applied to the chain structure set ($\boldsymbol{C}$) and the Delaunay Triangulation remains the linear features of $\boldsymbol{C}$, then this kind of Delaunay Triangulation is called as constrained Delaunay triangulation ($\mathcal{CDT}$).

Theorem 2.2.1. $\mathcal{D}(\boldsymbol{P})$ is simplicial complex.

Proof. (1) If $\sigma_n \in \mathcal{D}(\boldsymbol{P})$, then any face of $\sigma_n \in \mathcal{D}(\boldsymbol{P})$.

Given $\sigma_n \in \mathcal{D}(\boldsymbol{P})$, according to formula (2-1-2), $\sigma_n = \{\mathbf{S}_0(\cdot), \mathbf{S}_1(\cdot), \mathbf{S}_2(\cdot)\}$. $\because \sigma_n \in \mathcal{D}(\boldsymbol{P})$, apparently there is the following formula, i.e. $\forall \mathbf{S}_i(\cdot) \subset \sigma_n \Rightarrow \mathbf{S}_i(\cdot) \in \mathcal{D}(\boldsymbol{P})$, $i = 1, 2, 3$. $\therefore$ if $\sigma_n \in \mathcal{D}(\boldsymbol{P})$, there exist any face of $\sigma_n \in \mathcal{D}(\boldsymbol{P})$, (1) is tenable.

(2) If $\sigma_n, \sigma_k \in \mathcal{D}(\boldsymbol{P})$, then $\sigma_n \cap \sigma_k$ is $\varnothing$ or common face. Exclusive method, assuming $\sigma_n, \sigma_k \in \mathcal{D}(\boldsymbol{P})$, then $\sigma_n \cap \sigma_k = \neg\varnothing$ $(\sigma_n \ne \sigma_k)$ has five cases, as showed in figure 3.

① σ_n, σ_k intersects at non-common face-point, as showed in figure 4(c). According to the definition 2.1.8, there is

$$(\mathbf{23}) \Rightarrow (\exists p, \boldsymbol{d}(\mathbf{2}, p) = \boldsymbol{d}(p, \mathbf{3})) \wedge (p = \lambda_0\mathbf{2} + \lambda_1\mathbf{3}, \lambda_0, \lambda_1 \ge 0, \lambda_0 + \lambda_1 = 1),$$

$\because$ (**243**) is collinearity, $\therefore$ exist $\boldsymbol{d}(\mathbf{2}, p) = \boldsymbol{d}(p, \mathbf{3}) < \boldsymbol{d}(\mathbf{4}, p) \Leftrightarrow p \subset V(\mathbf{2}) \wedge p \subset V(\mathbf{3}) \wedge p \subset V(\mathbf{4})$, i.e. p is the equal distance point of **2** and **3** and p is inside $V(4)$, this case is incompatible with definition 2.2.1. $\therefore$ ① is not tenable.

② σ_n, σ_k intersects at non-common face-line segment, as showed in figure 4(d). The case of ② is not tenable. The prove method is same to ①.

③ σ_n, σ_k intersects at non-common face-face, as showed in figure 4(e). (**23**) intersects with (**64**) at point p. According to definition 2.2.2, for the (**23**) and (**64**), there are individually

$$(23) \Rightarrow \begin{cases} \boldsymbol{d}(2,p) < \boldsymbol{d}(p,3) \vee \\ \boldsymbol{d}(2,p) = \boldsymbol{d}(p,3) \vee \\ \boldsymbol{d}(2,p) > \boldsymbol{d}(p,3) \end{cases} \qquad (64) \Rightarrow \begin{cases} \boldsymbol{d}(6,p) < \boldsymbol{d}(p,4) \vee, \\ \boldsymbol{d}(6,p) = \boldsymbol{d}(p,4) \vee \\ \boldsymbol{d}(6,p) > \boldsymbol{d}(p,4) \end{cases}$$

there are $C_3^1C_3^1 = 9$ cases can be divided into three catalogs, i.e.

$\{(<, <), (<, >), (>, <), (>, >)\} \Leftrightarrow p$ exists inside of two different Voronoi polygons at the same time;

$\{(<, =), (>, =), (=, <), (=, >)\} \Leftrightarrow p$ exists inside and boundary of two different Voronoi polygons at the same time;

$\{(=, =)\} \Leftrightarrow$ necessarily exist (26), (24), (36) and (34).

$\because$ the conclusions of ① and ② are incompatible with definition 2.2.1. The conclusion of ③ is incompatible with (1), $\therefore$③ is not tenable. $\because$①, ② and ③ are all not tenable, $\therefore$(2) is tenable. (1) and (2) are tenable, theorem 2.2.1 is tenable. □

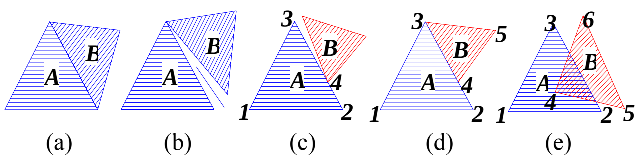


Figure 4. Regular togetherness of triangles (a), (b) and the irregular togetherness of triangles (c), (d), (e)

♣In this paper is an assumption that $\boldsymbol{P}$ is non-collinearity point set.

Definition 2.2.3.($\mathcal{CDT}$) Given point set $\boldsymbol{P}$, Delaunay triangulation $\mathcal{D}(\boldsymbol{P})$ and chain set $\boldsymbol{C}$, if the circumcircle of one random triangle in $\mathcal{D}(\boldsymbol{P})$ doesn't contain the point which is all visibility with the three vertexes of the triangle at the same time, the visibility is only on the condition that the chain p_ip_j of the nodes p_i, p_j ($p_i, p_j \in \sigma$) doesn't intersect with any segments in $\boldsymbol{C}$ (extreme end points excluded), then the $\mathcal{D}(\boldsymbol{P})$ is called as the constrained Delaunay triangulation of $\boldsymbol{C}$, denoted as $\mathcal{CD}(\boldsymbol{P})$.

Corollary 2.2.1. $\mathcal{CD}(\boldsymbol{P})$ is simplicial complex. (omitted)

$\boldsymbol{C}$ is described as the set of many chains, i.e.

$$\boldsymbol{C} = \{\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_n\},\ n \geq 1,\ n \in \mathbb{N}. \tag{2-2-3}$$

Theorem 2.2.2. Given $\boldsymbol{C}_1, \boldsymbol{C}_2 \subset \boldsymbol{C}$, $\boldsymbol{C}''_i(\boldsymbol{ab}) \in \boldsymbol{C}_i$, $\boldsymbol{C}''_j(\boldsymbol{cd}) \in \boldsymbol{C}_j$, $\exists p$ satisfies $\begin{cases} p = \lambda_0\boldsymbol{a} + \lambda_1\boldsymbol{b}, \lambda_0, \lambda_1 \geq 0, \lambda_0 + \lambda_1 = 1 \wedge \\ p = \lambda'_0\boldsymbol{c} + \lambda'_1\boldsymbol{d}, \lambda'_0, \lambda'_1 \geq 0, \lambda'_0 + \lambda'_1 = 1 \end{cases}$, p doesn't change the continuation property of chain $\boldsymbol{C}''_i$, $\boldsymbol{C}''_j$.

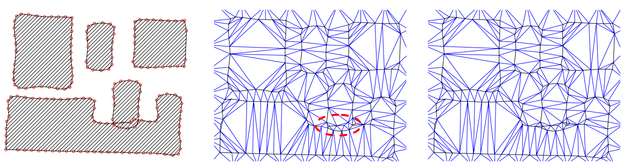
Proof. Only prove that $(\boldsymbol{ab}) \Leftrightarrow (\boldsymbol{a}p\boldsymbol{b})$ and $(\boldsymbol{cd}) \Leftrightarrow (\boldsymbol{c}p\boldsymbol{d})$ are all tenable. By the vector algebra method, $\because p$ satisfies the condition: $\therefore$ there is $(\boldsymbol{ab}) = (\boldsymbol{a}p) + (p\boldsymbol{b}) = (\boldsymbol{a}p\boldsymbol{b})$, $\therefore (\boldsymbol{ab}) \Leftrightarrow (\boldsymbol{a}p\boldsymbol{b})$ is tenable. By the same way, $(\boldsymbol{cd}) \Leftrightarrow (\boldsymbol{c}p\boldsymbol{d})$ is tenable. The theorem 2.2.2 is tenable. □

Corollary 2.2.2. $\boldsymbol{C}'' \in \boldsymbol{C}_i \subseteq \boldsymbol{C}$, $\boldsymbol{C}'' = (\mathbf{S}_1^1, \mathbf{S}_2^1, \cdots\cdots, \mathbf{S}_n^1)$, $n \in \mathbb{N}$, $n \geq 1$, there is $\forall \mathbf{S}_i^1 \in \mathcal{CD}(\boldsymbol{P})$.

Proof. (1) $\boldsymbol{C}''$ doesn't intersect self, i.e. $\boldsymbol{C}'' = \mathbf{S}_1^1$. $\because \mathcal{CD}(\boldsymbol{P})$ is simplicial complex, $\therefore$ there is $\boldsymbol{C}'' = \mathbf{S}_1^1 \in \mathcal{CD}(\boldsymbol{P})$.

(2) $\boldsymbol{C}''$ intersects. According to theorem 2.2.2, there is $\boldsymbol{C}'' = (\mathbf{S}_1^1, \mathbf{S}_2^1, \cdots\cdots, \mathbf{S}_n^1)$, $n \in \mathbb{N}$, $n \geq 1$, and $\boldsymbol{C}''$ remains its continuation property. $\because \forall \mathbf{S}_i^1 \in \boldsymbol{C}''$ all belong to the case (1), $\therefore$ there is $\forall \mathbf{S}_i^1 \in \mathcal{CD}(\boldsymbol{P})$. By summary, corollary 2.2.2 is tenable. □

The constrained chain set $\boldsymbol{C}$ and node set $\boldsymbol{P}$ is showed in figure 5(a), $\mathcal{D}(\boldsymbol{P})$ and $\mathcal{CD}(\boldsymbol{P})$ are showed in figure 5(b), 5(c). The $\mathcal{CD}(\boldsymbol{P})$ completely remains the linear features of chain set $\boldsymbol{C}$, but the $\mathcal{D}(\boldsymbol{P})$ partially lost the linear features of $\boldsymbol{C}$, as showed as the dashed line region in figure 5(b).



(a)Constrained chain set $\boldsymbol{C}$ and node set $\boldsymbol{P}$ (b) $\mathcal{D}(\boldsymbol{P})$ (c) $\mathcal{CD}(\boldsymbol{P})$

Figure 5. Constrained effect to the Delaunay triangulation of chain structures

2.3 Reasoning of σ'S Classification and σ'S Lateral Characteristic In $\mathcal{CDT}$

Definition 2.3.1.(1-simplex's type) The $\boldsymbol{C}$ is the set of chains, $\boldsymbol{P}$ is the set of boundary nodes in $\boldsymbol{C}$. $\boldsymbol{C}_m$, $\boldsymbol{C}_n$ is the chain ($\boldsymbol{C}_m, \boldsymbol{C}_n \subseteq \boldsymbol{C}$); $\mathbf{p}_m$, $\mathbf{p}_n$ are individually the set of nodes in $\boldsymbol{C}_m$, $\boldsymbol{C}_n$. $\forall\sigma \in \mathcal{CD}(\boldsymbol{P})$, p_1, p_2, p_3 are the three vertexes of σ, if p_i, p_j ($i \neq j$, i, j = 1, 2, 3) $\wedge\ p_i \in \mathbf{p}_m \wedge p_j \in \mathbf{p}_n$ is tenable, then $\boldsymbol{f}(p_i, p_j) = 0$; if p_i, p_j ($i \neq j$, i, j = 1, 2, 3) $\wedge$ ($p_i, p_j \in \mathbf{p}_m$) is tenable, then $\boldsymbol{f}(p_i, p_j) = 1$; specially, if p_i, p_j ($i = j$, i, j = 1, 2, 3) is tenable, then $\boldsymbol{f}(p_i, p_j) = 2$.

If the different types of the three edges are considered, the triangles can be divided into three categories. $\forall\sigma \in \mathcal{CD}(\boldsymbol{P})$, the absolute value of formula (2-2-4) can be used to distinguish these three kinds of triangle,

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{f}(p_1,p_1) & \boldsymbol{f}(p_1,p_2) & \boldsymbol{f}(p_1,p_3) \\ \boldsymbol{f}(p_2,p_1) & \boldsymbol{f}(p_2,p_2) & \boldsymbol{f}(p_2,p_3) \\ \boldsymbol{f}(p_3,p_1) & \boldsymbol{f}(p_3,p_2) & \boldsymbol{f}(p_3,p_3) \end{bmatrix}. \tag{2-2-4}$$

By formula (2-2-4), can take out the following the 5 cases:

$$T_1 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad T_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad T_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$T_4 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad T_5 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Definition 2.3.2.(σ's type) $\forall\sigma \in \mathcal{CD}(\boldsymbol{P})$, if the value of $|\det\boldsymbol{T}|$ is 4, the three edges of the triangle come from the same chain structure (type α, denoted as σ_α); if the value of $|\det\boldsymbol{T}|$ is 6, the two edges of the three edges of the triangle come from the two different chain structure and the other edge comes from the same chain structure (type β, denoted as σ_β); if the value of $|\det\boldsymbol{T}|$ is 8, the three edges of the triangle come from the three different chain structure (type γ, denoted as σ_γ).

In $\mathcal{CD}(\boldsymbol{P})$, σ is relative to $\boldsymbol{C}$. The vertices of σ come from $\boldsymbol{C}$, more particularity, the vertices come from the chain $\boldsymbol{C}_i \subseteq \boldsymbol{C}$. Thus, for some chain $\boldsymbol{C}_i \subseteq \boldsymbol{C}$, the type of σ is exactly described as the constrained type by $\boldsymbol{C}_i$.

Definition 2.3.3.(decomposition of σ's type and its algebra formula) If consider the constrained type of σ based on the 0-simplex, the σ, constrained chain $\boldsymbol{C}_i$ and constrained type of σ have the following relationships,

$$\sigma_{\alpha\leftarrow \boldsymbol{C}i} \Leftrightarrow \sigma_{\gamma\leftarrow \boldsymbol{C}i} \wedge \sigma_{\gamma\leftarrow \boldsymbol{C}i} \wedge \sigma_{\gamma\leftarrow \boldsymbol{C}i} \Leftrightarrow \alpha = \gamma + \gamma + \gamma, \tag{2-2-5}$$

$$\sigma_{\beta\leftarrow \boldsymbol{C}i} \Leftrightarrow \sigma_{\gamma\leftarrow \boldsymbol{C}i} \wedge \sigma_{\gamma\leftarrow \boldsymbol{C}i} \wedge \sigma_{\gamma\leftarrow \boldsymbol{C}j} \Leftrightarrow \beta = \gamma + \gamma + \gamma', \tag{2-2-6}$$

$$\sigma_{\beta\leftarrow \boldsymbol{C}i} \Leftrightarrow \sigma_{\beta\leftarrow \boldsymbol{C}i} \wedge \sigma_{\gamma\leftarrow \boldsymbol{C}j} \Leftrightarrow \beta = \beta + \gamma', \tag{2-2-7}$$

expressly, σ didn't make up of the chain $\boldsymbol{C}_i$ ($\boldsymbol{C}_i \subseteq \boldsymbol{C}$), denoted as $\sigma_{\perp \boldsymbol{C}i}$.

Definition 2.3.4.(σ's direction) $\forall\sigma_\alpha, \sigma_\beta \in \mathcal{CD}(\boldsymbol{P})$, p_i, p_j ($i \neq j$, i, j = 1, 2, 3) and $\boldsymbol{f}(p_i, p_j) = 0$, the arrangement direction of σ's vertexes is constrained by the direction of chain (p_ip_j), denoted as $\sigma\big|_{(p_ip_j)}$. The anticlockwise direction of σ is denoted as $\sigma^{\otimes}$ and the clockwise direction of σ is denoted as $\sigma^{\odot}$, the formula is described as

$$\boldsymbol{dir}(\sigma) = \sigma^*,\ * \in \{\otimes, \odot\}. \tag{2-2-8}$$

Theorem 2.3.1 $\forall\sigma_\alpha, \sigma_\beta \in \mathcal{CD}(\boldsymbol{P})$, its vertexes are p_i, p_j, p_k ($i \neq j \neq k$, i, j, k = 1, 2, 3) and $\boldsymbol{f}(p_i, p_j) = 1$, if the arrangement of vertexes is anticlockwise, the p_k locates at the left side of chain

(p_ip_j); if the arrangement of vertexes is clockwise, the p_k locates at the right side of chain (p_ip_j).

Proof. The anticlockwise arrangement of vertexes, according to theorem 2.1.1, there is $\times_i = (p_ip_j) \times (p_ip_k) > 0$, i.e. there is the included angle θ formed by vector (p_ip_j) and vector (p_ip_k), which is between 0 and π. $\because \boldsymbol{sin}\theta > 0$ $(0 < \theta < \pi)$, $\therefore \times_i > 0$, so p_k locates at the left side of (p_ip_j).

By the same way, while arrangement of σ's vertexes is clockwise, there are $\pi < \theta < 2\pi$ and $\times_i < 0$, p_k locates at the right side of vector (p_ip_j). Theorem 2.3.1 is tenable. □

DEFINITION 2.3.5(***f*** extension of σ's adjacency) $\exists\sigma_m, \sigma_n \in \mathcal{CD}(\boldsymbol{P}) \wedge \sigma_m \cap \sigma_n = \{p, p'\}$. If $\boldsymbol{f}(p, p') = 0$, there is $\boldsymbol{f}^0$ adjacency relationship between σ_m, σ_n, denoted as $\sigma_m\, \boldsymbol{f}^0\sigma_n$; if $\boldsymbol{f}(p, p') = 1$, there is $\boldsymbol{f}^1$ adjacency relationship between σ_m, σ_n, denoted as $\sigma_m\, \boldsymbol{f}^1\sigma_n$.

DEFINITION 2.3.6(symmetry of σ's adjacency) The $\boldsymbol{f}^0$ adjacency relationship and $\boldsymbol{f}^1$ adjacency relationship are symmetrical with each other, i.e. $\sigma_m\, \boldsymbol{f}^i\sigma_n \leftrightarrow \sigma_n\, \boldsymbol{f}^i\sigma_m$ $(i = 0, 1)$.

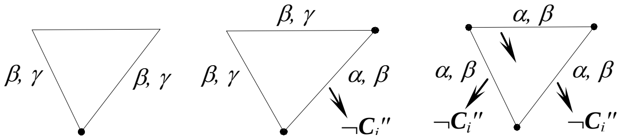
DEFINITION 2.3.7(degree of σ's adjacency) $\forall\sigma \in \mathcal{D}(\boldsymbol{P})$, the degree of σ's adjacency is the number of edge in σ when the value of ***f*** is 0 or 1, the formula is

$$\boldsymbol{deg}^i(\sigma) = \sigma\Big|_{f^i}, \quad i = 0,1\,. \tag{2-2-9}$$

DEFINITION 2.3.8(propagation characteristic of σ's adjacency) $\exists\sigma_k, \sigma_m, \sigma_n \in \mathcal{CD}(\boldsymbol{P})$, if there is $(\sigma_k\, \boldsymbol{f}^i\sigma_m) \wedge (\sigma_m\, \boldsymbol{f}^i\sigma_n)$ $(i = 0, 1)$, there exists propagation $\boldsymbol{f}^i$ $(i = 0, 1)$ adjacency relationship between σ_k and σ_n, denoted as $\sigma_k\, \boldsymbol{f}^i_{\rightarrow}\sigma_n$.

DEFINITION 2.3.9(semi-plane of chain) The left semi-plane of vector (v_iv_j) is $\boldsymbol{l}(v_iv_j) = \{p \mid (v_iv_j) \times (v_ip) > 0, p \in \mathbb{E}^2\}$, the right semi-plane of vector (v_iv_j) is $\boldsymbol{r}(v_iv_j) = \{p \mid (v_iv_j) \times (v_ip) < 0, p \in \mathbb{E}^2\}$.

For $\sigma_{\beta\leftarrow Ci}$, $\sigma_{\alpha\leftarrow Ci}$, if its arbitrary edge v_iv_j $(v_iv_j \in \sigma)$ is the minimal child chain $\boldsymbol{C}_i''$ of $\boldsymbol{C}_i$, whether the location of σ relative to chain $\boldsymbol{C}_i$ is left or right can be directly confirmed by $\boldsymbol{dir}(\sigma)$. If $\sigma_{\leftarrow Ci}$ can't be directly confirmed, it doesn't contain the minimal child chain $\boldsymbol{C}_i''$ of $\boldsymbol{C}_i$. The reasoning left or right locations of $\sigma_{\leftarrow Ci}$ according to the above two kinds of σ combining with the adjacency and propagation characteristic ***f*** adjacency are the important base for the description of spatial objects in $\mathcal{CDT}$, as showed in figure 6, there are three kinds of $\sigma_{\leftarrow Ci}$ which need inference of its left or right property.



(a) $\sigma_{\gamma\leftarrow Ci}$ (b) $\sigma_{\beta\leftarrow Ci}$ $(v_iv_j = \neg\boldsymbol{C}_i'', v_iv_j \in \boldsymbol{C}_i \cap \sigma)$ (c) $\sigma_{\alpha\leftarrow Ci}$ $(\forall v_iv_j = \neg\boldsymbol{C}_i'', v_iv_j \in \boldsymbol{C}_i \cap \sigma)$

Figure 6. Three kinds of $\sigma_{\leftarrow Ci}$ whose left-right property can't be directly confirmed

For the σ_γ, because there is only one vertex coming from $\boldsymbol{C}_i$, the confirmation of the left-right property of σ_γ is feasible on the condition of the confirmation of the left-right property of the other two vertexes on the help of two $\boldsymbol{f}^0$ adjacencies.

COROLLARY 2.3.1.($\boldsymbol{f}^0$ including & approximating) For $\forall\sigma_{\gamma\leftarrow Ci} \in \mathcal{CD}(\boldsymbol{P})$, $\exists\sigma_{\beta\leftarrow Ci}, \sigma'_{\beta\leftarrow Ci} \in \mathcal{CD}(\boldsymbol{P})$, if $\sigma_{\gamma\leftarrow Ci}$ satisfies

$$\sigma_{\gamma\leftarrow C_i}\, \boldsymbol{f}^0\sigma^{\otimes}_{\beta\leftarrow Ci}\Big|_{(v_i,v_j)} \wedge \sigma_{\gamma\leftarrow C_i}\, \boldsymbol{f}^0\sigma'^{\otimes}_{\beta\leftarrow Ci}\Big|_{(v'_i,v'_j)} \wedge (v_j = v'_i \in \sigma_{\gamma\leftarrow C_i} \cap \sigma_{\beta\leftarrow C_i} \cap \sigma'_{\beta\leftarrow C_i}),$$

then the $\sigma_{\gamma\leftarrow Ci}$ locates at the left side of chain $\boldsymbol{C}_i$.

Proof. Considering there exist three states of angle $\theta = \angle v_iv_jv'_j$, i.e.

(1) If $0 < \theta < \pi$, the left side of vector chain $(v_iv_j, v_jv'_j)$ is $\boldsymbol{l}(v_iv_j) \cap \boldsymbol{l}(v_jv'_j)$.

$\because \theta < \pi$, $\therefore v'_j \in \boldsymbol{l}(v_iv_j)$. According to definition of semi-plane of chain, there is $v'_j \in \boldsymbol{l}(v_i'v'_j)$, $\therefore v'_j \in \boldsymbol{l}(v_iv_j) \cap \boldsymbol{l}(v_jv'_j)$. By the same way, there is $v_i \in \boldsymbol{r}(v'_jv'_i)$, so there is $v_i \in \boldsymbol{l}(v'_iv'_j)$. $\because v_i \in \boldsymbol{l}(v_iv_j)$, $\therefore v_i \in \boldsymbol{l}(v_iv_j) \cap \boldsymbol{l}(v_jv'_j)$. According to theorem 2.3.1, there are

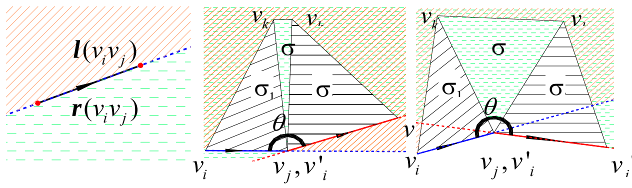
$$\sigma^{\otimes}_{\beta\leftarrow Ci}\Big|_{(v_i,v_j)} \Rightarrow v_k \in \boldsymbol{l}(v_iv_j),\ \sigma'^{\otimes}_{\beta\leftarrow Ci}\Big|_{(v'_i,v'_j)} \Rightarrow v'_k \in \boldsymbol{l}(v'_iv'_j).$$

And more according to theorem 2.2.1, σ, σ_1 and σ_2 regularly relate each other, ∴σ locates at the left side of $\boldsymbol{C}_i$. As shown in figure 7(b).

(2) If $\pi < \theta < 2\pi$, the left side of chain $(v_iv_j, v_jv'_j)$ is $\boldsymbol{l}(v_iv_j) \cup \boldsymbol{l}(v_jv'_j)$.

According to definition of semi-plane of chain there is $v_i \in \boldsymbol{l}(v_iv_j) \wedge v'_j \in \boldsymbol{l}(v'_iv'_j)$. According to theorem 2.3.1, $v_k \in \boldsymbol{l}(v_iv_j) \wedge v'_k \in \boldsymbol{l}(v'_iv'_j)$, and more according to theorem 2.2.1, σ, σ_1 and σ_2 regularly relate each other; ∴σ locates at the left side of $\boldsymbol{C}_i$. As shown in figure7(c).

(3) If $\theta = \pi$, the proof same to (1). Summarily, corollary 2.3.1 is tenable. □



(a) Left and right semi-plane of chain (b) $0<\theta<\pi$ (c) $\pi<\theta<2\pi$

Figure 7. $\sigma_{\gamma\leftarrow Ci}$ some side including & approximating corollary based on $\boldsymbol{f}^0$ adjacency

COROLLARY 2.3.2. For $\forall\sigma_{\gamma\leftarrow Ci} \in \mathcal{CD}(\boldsymbol{P})$, $\exists\sigma_{\beta\leftarrow Ci}, \sigma'_{\beta\leftarrow Ci} \in \mathcal{CD}(\boldsymbol{P})$, if $\sigma_{\gamma\leftarrow Ci}$ satisfies

$$\sigma_{\gamma\leftarrow C_i}\, \boldsymbol{f}^0\sigma^{\odot}_{\beta\leftarrow Ci}\Big|_{(v_i,v_j)} \wedge \sigma_{\gamma\leftarrow C_i}\, \boldsymbol{f}^0\sigma'^{\odot}_{\beta\leftarrow Ci}\Big|_{(v'_i,v'_j)} \wedge (v_j = v'_i \in \sigma_{\gamma\leftarrow C_i} \cap \sigma_{\beta\leftarrow C_i} \cap \sigma'_{\beta\leftarrow C_i}),$$

then the $\sigma_{\gamma\leftarrow Ci}$ locates at the right side of chain $\boldsymbol{C}_i$. (omitted)

COROLLARY 2.3.3.($\boldsymbol{f}^0_{\rightarrow}$ including & approximating) For $\forall\sigma_{\gamma\leftarrow Ci} \in \mathcal{CD}(\boldsymbol{P})$, $\exists\sigma_{\beta\leftarrow Ci}, \sigma'_{\beta\leftarrow Ci} \in \mathcal{CD}(\boldsymbol{P})$, if $\sigma_{\gamma\leftarrow Ci}$ satisfies

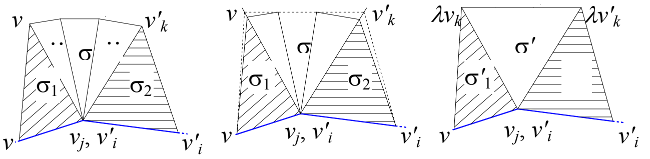
$$\sigma_{\gamma\leftarrow C_i}\, \boldsymbol{f}^0_{\rightarrow}\sigma^{\otimes}_{\beta\leftarrow Ci}\Big|_{(v_i,v_j)} \wedge \sigma_{\gamma\leftarrow C_i}\, \boldsymbol{f}^0_{\rightarrow}\sigma'^{\otimes}_{\beta\leftarrow Ci}\Big|_{(v'_i,v'_j)} \wedge (v_j = v'_i \in \sigma_{\gamma\leftarrow C_i} \cap \sigma_{\beta\leftarrow C_i} \cap \sigma'_{\beta\leftarrow C_i}),$$

then the $\sigma_{\gamma\leftarrow Ci}$ locates at the left side of $\boldsymbol{C}_i$.

Proof. The set of triangles which are propagated by the transfer from $\sigma_{\gamma\leftarrow Ci}$ to $\sigma_{\beta\leftarrow Ci}$ and from $\sigma_{\gamma\leftarrow Ci}$ to $\sigma'_{\beta\leftarrow Ci}$ is Σ. There exist two points λv_k, $\lambda' v'_k$ $(\lambda, \lambda' > 0)$ at the vector directions (v_jv_k) and $(v_jv'_k)$, as shown in figure 8(b), constructing assistant triangle $v_j\lambda' v'_k\lambda v_k$, which satisfies below formula

$$\forall p \in \Sigma \rightarrow p \in v_j\lambda' v'_k\lambda v_k,\ p \in \mathbb{E}^2,$$

construct assistant triangles again, as shown in figure 8(c), $\sigma'_1 = v_i v_j \lambda v_k$, $\sigma'_2 = v'_i v'_j \lambda' v'_k$, $\sigma' = v_j \lambda' v'_k \lambda v_k$, according to the corollary 2.3.1, σ' locates at the left side of $\boldsymbol{C}_i$. $\because \sigma \subset \sigma'$, $\therefore \sigma$ locates at the left side of $\boldsymbol{C}_i$. Corollary 2.3.3 is tenable. □



(a) (b) (c)

Figure 8. $\sigma_{\gamma\leftarrow C_i}$ some side including & approximating corollary based on $\boldsymbol{f}^0_{\rightarrow}$ adjacency

COROLLARY 2.3.4. For $\forall\sigma_{\gamma\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P})$, $\exists\sigma_{\beta\leftarrow \boldsymbol{C}i}$, $\sigma'_{\beta\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P})$, if $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ satisfies

$$\sigma_{\gamma\leftarrow \boldsymbol{C}_i} \boldsymbol{f}^0_{\rightarrow} \sigma^{\odot}_{\beta\leftarrow \boldsymbol{C}i}\Big|_{(v_i,v_j)} \wedge \sigma_{\gamma\leftarrow \boldsymbol{C}_i} \boldsymbol{f}^0_{\rightarrow} \sigma'^{\odot}_{\beta\leftarrow \boldsymbol{C}i}\Big|_{(v'_i,v'_j)} \wedge (v_j = v'_i \in \sigma_{\gamma\leftarrow \boldsymbol{C}_i} \cap \sigma_{\beta\leftarrow \boldsymbol{C}_i} \cap \sigma'_{\beta\leftarrow \boldsymbol{C}_i}),$$

then $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ locates at the right side of $\boldsymbol{C}_i$.(omitted)

COROLLARY 2.3.5. For $\forall\sigma_{\gamma\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P})$, $\exists\sigma_{\beta\leftarrow \boldsymbol{C}i}$, $\sigma'_{\beta\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P})$, if $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ satisfies

$$\sigma_{\gamma\leftarrow \boldsymbol{C}_i} \boldsymbol{f}^0 \sigma^{\otimes}_{\beta\leftarrow \boldsymbol{C}i}\Big|_{(v_i,v_j)} \wedge \sigma_{\gamma\leftarrow \boldsymbol{C}_i} \boldsymbol{f}^0_{\rightarrow} \sigma'^{\otimes}_{\beta\leftarrow \boldsymbol{C}i}\Big|_{(v'_i,v'_j)} \wedge (v_j = v'_i \in \sigma_{\gamma\leftarrow \boldsymbol{C}_i} \cap \sigma_{\beta\leftarrow \boldsymbol{C}_i} \cap \sigma'_{\beta\leftarrow \boldsymbol{C}_i}),$$

then $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ locates at the left side of $\boldsymbol{C}_i$.

Proof. Given $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ arrives to $\sigma^{\otimes}_{\beta\leftarrow \boldsymbol{C}_i}\Big|_{(v'_i,v'_j)}$ by n ($n > 1$) times of propagation; $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ arrives to σ by one time of propagation on direction of $\sigma^{\otimes}_{\beta\leftarrow \boldsymbol{C}_i}\Big|_{(v'_i,v'_j)}$. According to corollary 2.3.3, σ arrives to $\sigma^{\otimes}_{\beta\leftarrow \boldsymbol{C}_i}\Big|_{(v_i,v_j)}$ by 2 times of propagation; σ arrives to $\sigma^{\otimes}_{\beta\leftarrow \boldsymbol{C}_i}\Big|_{(v'_i,v'_j)}$ by n-1 times of propagation, i.e. σ locates at the left side of $\boldsymbol{C}_i$. According to corollary 2.3.1, $\sigma^{\otimes}_{\beta\leftarrow \boldsymbol{C}_i}\Big|_{(v_i,v_j)}$, $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ and σ satisfy the $\boldsymbol{f}^0$ including & approximating, $\because \sigma$ locates at the left side of $\boldsymbol{C}_i$. $\therefore \sigma_{\gamma\leftarrow \boldsymbol{C}i}$ locates at the left side of $\boldsymbol{C}_i$. The corollary 2.3.5 is tenable. □

THEOREM 2.3.2.(including & approximating theorem) For $\forall\sigma_{\gamma\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P})$, if $\exists\sigma'_{\leftarrow \boldsymbol{C}i}$, $\sigma''_{\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P})$, they all locate the left (or right) side of chain $\boldsymbol{C}_i$ at the same time, if $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ satisfies

$$\sigma_{\gamma\leftarrow \boldsymbol{C}i} \boldsymbol{f} \sigma'_{\leftarrow \boldsymbol{C}i} \wedge \sigma_{\gamma\leftarrow \boldsymbol{C}i} \boldsymbol{f} \sigma''_{\leftarrow \boldsymbol{C}i}, \ \boldsymbol{f} \in \{\boldsymbol{f}^0, \boldsymbol{f}^0_{\rightarrow}\},$$

then the $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ locates the left (or right) side of $\boldsymbol{C}_i$. (omitted)

For $\sigma_{\beta\leftarrow \boldsymbol{C}i}(v_i v_j = \neg\boldsymbol{C}''_i$, v_i, $v_j \in \boldsymbol{C}_i \cap \sigma_{\beta\leftarrow \boldsymbol{C}i})$, according to formula (2-2-6), it is divided as $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ and $\sigma_{\gamma'\leftarrow \boldsymbol{C}i}$. So, its left-right properties can be got by the same way as $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$.

COROLLARY 2.3.6. For $\forall\sigma_{\beta\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P}) \wedge (v_i v_j = \neg\boldsymbol{C}''_i, v_i, v_j \in \boldsymbol{C}_i \cap \sigma_{\beta\leftarrow \boldsymbol{C}i})$, if $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ or $\sigma_{\gamma'\leftarrow \boldsymbol{C}i}$ locates at the left (or right) side of chain $\boldsymbol{C}_i$ and $\sigma_{\gamma\leftarrow \boldsymbol{C}i} \wedge \sigma_{\gamma'\leftarrow \boldsymbol{C}i} \rightarrow \sigma_{\beta\leftarrow \boldsymbol{C}i}$ is tenable, then the $\sigma_{\beta\leftarrow \boldsymbol{C}i}$ locates at the left (or right) side of $\boldsymbol{C}_i$. (omitted)

For the $\sigma_{\alpha\leftarrow \boldsymbol{C}i}$ $(\forall v_i v_j = \neg\boldsymbol{C}''_i$, v_i, $v_j \in \boldsymbol{C}_i \cap \sigma_{\alpha\leftarrow \boldsymbol{C}i})$, it can be transformed as judge the left-right properties of $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ by the type decomposition method.

COROLLARY 2.3.7. For $\forall\sigma_{\alpha\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P}) \wedge (\forall v_i v_j = \neg\boldsymbol{C}''_i, v_i, v_j \in \boldsymbol{C}_i \cap \sigma_{\alpha\leftarrow \boldsymbol{C}i})$, if $\forall\sigma_\gamma \in \boldsymbol{\sigma}$ ($\wedge_{i=1}^{|\boldsymbol{\sigma}|}\sigma_i \rightarrow \sigma_{\alpha\leftarrow \boldsymbol{C}i}$, $\sigma_i \in \boldsymbol{\sigma}$) locates at the left (or right) side of chain $\boldsymbol{C}_i$, then the $\sigma_{\alpha\leftarrow \boldsymbol{C}i}$ locates at the left (or right) side of chain $\boldsymbol{C}_i$. (omitted)

COROLLARY 2.3.8. $\forall\sigma_{\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P}) \wedge \forall\, \boldsymbol{C}''_i \notin \sigma_{\leftarrow \boldsymbol{C}i}$, there is a set $\boldsymbol{\sigma} = \{\sigma \mid \sigma_{\gamma\leftarrow \boldsymbol{C}i}\}$ after type decomposition of $\sigma_{\leftarrow \boldsymbol{C}i}$, if satisfies

$$(\forall\sigma_{\gamma\leftarrow \boldsymbol{C}i} \in \boldsymbol{\sigma})\ (\sigma_{\gamma\leftarrow \boldsymbol{C}i} \boldsymbol{f} \sigma_{\beta\leftarrow \boldsymbol{C}i} \wedge \sigma_{\gamma\leftarrow \boldsymbol{C}i} \boldsymbol{f} \sigma'_{\beta\leftarrow \boldsymbol{C}i} \wedge \sigma_{\beta\leftarrow \boldsymbol{C}i}, \sigma'_{\beta\leftarrow \boldsymbol{C}i} \text{ locate different side of chain } \boldsymbol{C}_i \wedge \sigma_{\beta\leftarrow \boldsymbol{C}i} \boldsymbol{f}^1 \sigma'_{\beta\leftarrow \boldsymbol{C}i}),$$

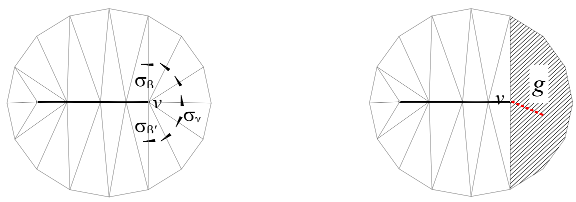
$$\boldsymbol{f} \in \{\boldsymbol{f}^0, \boldsymbol{f}^0_{\rightarrow}\},$$

then the result of $\sigma_{\gamma\leftarrow \boldsymbol{C}i} \cap \boldsymbol{C}_i$ is the start node or end node of chain $\boldsymbol{C}_i$.

Proof. (proof by contradiction) As shown in figure 9. Given $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ arrives to $\sigma_{\beta\leftarrow \boldsymbol{C}i}$ by m ($m > 0$, $m \in \mathbb{N}$) times of $\boldsymbol{f}$ adjacency propagation and arrives to $\sigma'_{\beta\leftarrow \boldsymbol{C}i}$ by n ($n > 0$, $n \in \mathbb{N}$) times of $\boldsymbol{f}$ adjacency propagation, then the set of triangles affected by the propagation constructs a region $\boldsymbol{g}$, every triangle in the set has node v and region $\boldsymbol{g}$ contains node i.e. $v \in \sigma_{\gamma\leftarrow \boldsymbol{C}i} \cap \boldsymbol{C}_i$. According to the known conditions, there is $\sigma_{\beta\leftarrow \boldsymbol{C}i} \boldsymbol{f}^1 \sigma'_{\beta\leftarrow \boldsymbol{C}i}$, i.e. $v \in \sigma_{\beta\leftarrow \boldsymbol{C}i} \cap \sigma'_{\beta\leftarrow \boldsymbol{C}i}$. Assuming that v is not the start node or end node of $\boldsymbol{C}_i$, there is a minimal child chain (v is a node of minimal child chain) of $\boldsymbol{C}_i$ certain cross region $\boldsymbol{g}$. This conclusion is incompatible with the theorem 2.2.1 that triangles are regular togetherness in $\mathcal{CDT}$, so the assumption is not tenable. Corollary 2.3.8 is tenable. □

If the start node and the end node of the chain is the same node, the chain is a cycle. Cycle is the special state of chain structure. The following sections prove the left-right relationship between σ and $\boldsymbol{C}_i$ on the case that $\boldsymbol{C}_i \subseteq \boldsymbol{C}$ is directed cycle, given the ordering of cycle vertices is anticlockwise. Explicitly, the above definitions, theorems and corollaries are tenable on the case that $\boldsymbol{C}_i$ is directed cycle.

As a special structure, the directed cycle partitions the 2-dimension space into two clear regions, i.e. the left side region of cycle and the right side region of cycle, and the left side region is a closure region. So, the including & approximating theorem for chain structure is reduced on the known conditions.



(a) The $\sigma_{\gamma\leftarrow \boldsymbol{C}i}$ arrives to $\sigma_{\beta\leftarrow \boldsymbol{C}i}$ and $\sigma'_{\beta\leftarrow \boldsymbol{C}i}$. (b) The condition of assumption

Figure 9.

THEOREM 2.3.3.(reduced including & approximating theorem) Given cycle $\boldsymbol{C}_i \subseteq \boldsymbol{C}$, $\exists\sigma_{\leftarrow \boldsymbol{C}i}$ locates at the left (right) side of $\boldsymbol{C}_i$. For $\forall\sigma'_{\leftarrow \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P})$, if it satisfies

$$\sigma'_{\leftarrow \boldsymbol{C}i} \boldsymbol{f}^0 \sigma_{\leftarrow \boldsymbol{C}i},$$

then $\sigma'_{\leftarrow \boldsymbol{C}i}$ and $\sigma_{\leftarrow \boldsymbol{C}i}$ locate at the same side of cycle $\boldsymbol{C}_i$. (omitted)

COROLLARY 2.3.9. Given cycle $\boldsymbol{C}_i \subseteq \boldsymbol{C}$, $\exists\sigma_{\gamma\leftarrow \boldsymbol{C}i}$, $\exists\sigma'_{\gamma\leftarrow \boldsymbol{C}i}$ all locate at the left side of cycle $\boldsymbol{C}_i$, for $\forall\sigma_{\lrcorner \boldsymbol{C}i} \in \mathcal{CD}(\boldsymbol{P})$, if it satisfies

$$\sigma_{\lrcorner \boldsymbol{C}i} \boldsymbol{f}^0 \sigma_{\gamma\leftarrow \boldsymbol{C}i} \wedge \sigma_{\lrcorner \boldsymbol{C}i} \boldsymbol{f}^0 \sigma'_{\gamma\leftarrow \boldsymbol{C}i},$$

then $\sigma_{\lrcorner \boldsymbol{C}i}$ locates at the left side of cycle $\boldsymbol{C}_i$.

Proof. Given $\sigma_{\gamma\leftarrow C_i} = \{v_1, v_2, v_3\}$, $\sigma'_{\gamma\leftarrow C_i} = \{v'_1, v'_2, v'_3\}$, $v_1, v'_1 \in \boldsymbol{C}_i$, $\because \sigma_{\lrcorner C_i}\, \boldsymbol{f}^{\,0} \sigma_{\gamma\leftarrow C_i} \wedge \sigma_{\lrcorner C_i}\, \boldsymbol{f}^{\,0} \sigma'_{\gamma\leftarrow C_i}$, $\therefore \sigma_{\lrcorner C_i} = \{v_2, v_3, v'_3\} \vee \sigma_{\lrcorner C_i} = \{v_2, v_3, v'_2\}$. According to including & approximating theorem, for $\forall v \in \{v_2, v_3, v'_2, v'_3\}$, it locates at the left side of cycle $\boldsymbol{C}_i$, $\therefore \sigma_{\lrcorner C_i}$ locates at the left side of closure cycle $\boldsymbol{C}_i$. Corollary 2.3.9 is tenable. □

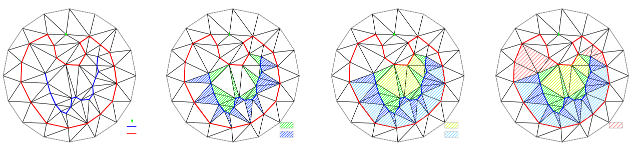
COROLLARY 2.3.10. Given cycle $\boldsymbol{C}_i \subseteq \boldsymbol{C}$, $\exists \sigma_{\lrcorner C_i} \in \mathcal{CD}(\boldsymbol{P})$, it locates at the left side of cycle $\boldsymbol{C}_i$. For $\forall \sigma'_{\lrcorner C_i} \in \mathcal{CD}(\boldsymbol{P})$, if it satisfies

$$\sigma'_{\lrcorner C_i}\, \boldsymbol{f}^{\,i} \sigma_{\lrcorner C_i},\ i = 0, 1,$$

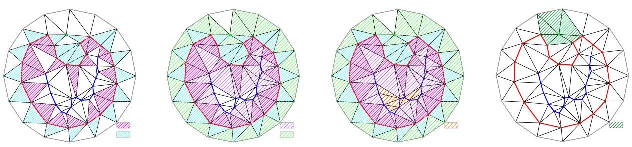
then $\sigma'_{\lrcorner C_i}$ locates at the left side of cycle $\boldsymbol{C}_i$.

Proof. (proof by contradiction) According to known condition $\sigma'_{\lrcorner C_i}\, \boldsymbol{f}^{\,i} \sigma_{\lrcorner C_i}$, there is that two vertexes of $\sigma'_{\lrcorner C_i}$ locate at the left side of $\boldsymbol{C}_i$. Assuming that the third vertex of $\sigma'_{\lrcorner C_i}$ locates at the right side of cycle $\boldsymbol{C}_i$, and then $\sigma'_{\lrcorner C_i}$ intersects with $\boldsymbol{C}_i$ certain. This conclusion is incompatible with that the $\mathcal{CD}(\boldsymbol{P})$ is simplicial complex, so the assumption is not tenable. $\therefore$ the third vertex of $\sigma'_{\lrcorner C_i}$ locates at the left side of $\boldsymbol{C}_i$, i.e. $\sigma'_{\lrcorner C_i}$ locates at the left side of cycle $\boldsymbol{C}_i$. The corollary 2.3.10 is tenable. □

The reasoning results of node structure, vector chain structure and directed cycle structure in $\mathcal{CDT}$ are showed in figure 10.



(a) $\mathcal{CDT}$ building on node structure, vector chain structure and directed cycle structure (b) Direct taking out the left and right side of vector chain structure (c) Taken left and right side of chain by the including & approximating theorem (d) Triangles associating with start and end node of chain



(e) Direct taking out the left and right side of directed cycle structure (f) Taken left and right side of directed cycle by the reduced including & approximating theorem (g) Ensuring left side of cycle $\neg\sigma_{\leftarrow C_i}$ (h) Node and it associating with triangle set
Figure 10. Reasoning and propagation of $\sigma_{\leftarrow C_i}$ by triangle neighbour

2.4 0-Simplex, 1-Simplex in $\mathcal{CDT}$ and the Basic Morphology of Σ

$\mathcal{CDT}$ is simplicial complex, in which can get structure relations base form σ's properties. Following we give three type of compositive relationships, i.e. node-node, node-chain and chain-chain.

2.5 Structure Description and Region Algebra Structure in $\mathcal{CDT}$

In $\mathcal{CDT}$, the triangle is the least unit for describing the chain structure, cycle structure and node structure. Every arbitrary chain, ring and node structure can be denoted as a set of triangle. According to the definition 2.3.4, structure can be described with the following formula,

$$\boldsymbol{\sigma}_{str} = \{\boldsymbol{\sigma}_{\alpha\leftarrow str} \cup \boldsymbol{\sigma}_{\beta\leftarrow str} \cup \boldsymbol{\sigma}_{\gamma\leftarrow str}\}. \qquad (2\text{-}5\text{-}1)$$

However, formula (2-5-1) can't exact describe the position and shape of *str*. Considering the own direction of *str* (0 dimension-arbitrary direction, 1 dimension-direction from start node to end node, 2 dimension-clockwise or anticlockwise direction), the direction of $\forall\sigma\in\boldsymbol{\sigma}_{str}$ are affected by its own direction, i.e. the result of **dir**(σ) is clockwise or anticlockwise direction. According to the above definitions, theorems and corollaries, the chain structure $\boldsymbol{\sigma}_{chain}$ can be divided into three parts, i.e. the left side of chain, the right side of chain and the middle side of chain satisfied to corollary 2.3.8,

$$\boldsymbol{\sigma}_{chain} = \{\boldsymbol{\sigma}_{\boldsymbol{l}(chain)} \cup \boldsymbol{\sigma}_{\boldsymbol{r}(chain)} \cup \boldsymbol{\sigma}_{\boldsymbol{m}(chain)}\}. \qquad (2\text{-}5\text{-}2)$$

Because the ring structure can't construct the middle triangle, formula (2-5-2) is reduced as the following formula,

$$\boldsymbol{\sigma}_{cycle} = \{\boldsymbol{\sigma}_{\boldsymbol{l}(cycle)} \cup \boldsymbol{\sigma}_{\boldsymbol{r}(cycle)}\}. \qquad (2\text{-}5\text{-}3)$$

For the independence vertex constructing the chain and cycle, formula (2-5-3) is transformed as formula (2-5-4),

$$\boldsymbol{\sigma}_{vertex} = \{\boldsymbol{\sigma}_{\boldsymbol{l}(vertex)}\} \vee \{\boldsymbol{\sigma}_{\boldsymbol{r}(vertex)}\} \vee \{\boldsymbol{\sigma}_{\boldsymbol{m}(vertex)}\}. \quad (2\text{-}5\text{-}4)$$

THEOREM 2.5.1. $\boldsymbol{\sigma}_{str}$ set associating with finite vertex structure in $\mathcal{CDT}$ consequentially is a closure region in $\mathbb{E}^2$.

Proof. $\boldsymbol{\sigma}_{str}$ of independence vertex is composed of $\boldsymbol{f}^{\,0}$ adjacency. The structure of chain and cycle takes out its left and right side based on σ_β which contains minimal child chain and the including & approximating theorem to stipulate 1 dimension segment as an adjacency condition, which is progress of σ_β extension with segment sharing. $\because$ σ is the simplex in 2 dimension space and $\because$ node is limited and ordered, $\therefore\boldsymbol{\sigma}_{str}$ set is a closure region in $\mathbb{E}^2$. □

DEFINITION 2.5.1. $\boldsymbol{\sigma}_{str}$ set associating with a structure and representing closure region in $\mathbb{E}^2$ is named structure region.

Structure set (**SS**) is transformed as structure region set (**SRS**) in $\mathcal{CDT}$ and there is the following relationship between **SRS** and $\mathcal{CDT}$,

$$\mathcal{CDT} = \bigcup_{i=1}^{n} \boldsymbol{\sigma}_i, \quad n = \mathrm{card}(\mathbf{SRS}), \quad \boldsymbol{\sigma}_i \in \mathbf{SRS}\ , (2\text{-}5\text{-}4)$$

considering family of sets $\Xi = 2^{\mathcal{CDT}}$, $\forall\xi \in \Xi \wedge \xi \neq \{\varnothing\}$, if ξ satisfied for the following condition,

$$\mathrm{card}(\xi) = 1 \vee \forall\xi \in \xi(\forall\xi' \in \xi(\xi\, \boldsymbol{f}^{\,i}_{\rightarrow}\xi' \ \textit{is not tenable}) \vee \textit{at least}\ \exists!\xi' \in \xi, \xi\, \boldsymbol{f}^{\,i}\xi' \ \textit{is tenable},\ i = 0,1)$$

then the ξ is called the subject region of $\mathcal{CDT}$, denoted as $\xi^s_{\mathcal{CDT}}$, the set of all subject region of $\mathcal{CDT}$ is denoted as $\Xi^s_{\mathcal{CDT}}$. If ξ satisfied into the following condition,

$$\mathrm{card}(\xi) = 1 \vee \forall \xi \in \xi (at\ least\ \exists! \xi' \in \xi, \xi f^i \xi'\ is\ tenable,\ i = 0,1),$$

then the ξ is called as the closure subject region of $\mathcal{CDT}$ (closure subject region). Clearly, every structure in $\mathcal{CDT}$ is a closure subject region. Thus, for the object structure, the entireness and part of object structure can be $\mathcal{CDT}$ described by use of $\Xi^{s}_{\sigma_{str}}$ .Constructing the following set,

$$\mathbf{F} = \Xi^{s}_{\mathcal{CDT}}. \quad (2\text{-}5\text{-}5)$$

Additionally, the **F** is the whole computing space. Thus the intersection operator (∩) can be used as the binary operator of **F**. Constructing the following mathematic structure,

$$\mathrm{RA} = \langle \mathbf{F}, \cap, \mathcal{CDT}, \{\varnothing\} \rangle. \quad (2\text{-}5\text{-}6)$$

THEOREM 2.5.2. RA is algebra structure.

Proof. (1) $\forall \omega, \omega' \in \mathbf{F}$, $\omega \cap \omega'$ is significative. ω and ω′ express the whole or partial structure in $\mathcal{CDT}$, the intersection operator (∩) for ω, ω′ is valid.

(2) $\forall \omega, \omega' \in \mathbf{F}$, $\omega \cap \omega' \in \mathbf{F}$. If $\omega \cap \omega' = \varnothing$, $\because \varnothing \in \mathbf{F}$, $\therefore \omega \cap \omega' \in \mathbf{F}$. If $\omega \cap \omega' = \neg\varnothing$, the result of intersection operator sure of the common subject region of ω and ω′, given the result is ω^*, there is $\omega^* \in \Xi^{s}_{\omega} \wedge \omega^* \in \Xi^{s}_{\omega'}$, $\because \Xi^{s}_{\omega}, \Xi^{s}_{\omega'} \in \mathbf{F}$, $\therefore$ there is surly $\omega^* \in \mathbf{F}$.

$\because \forall \omega \in \mathbf{F}$, there is $\mathcal{CDT} \cap \omega = \omega \wedge \omega \cap \mathcal{CDT} = \omega$, $\therefore \mathcal{CDT}$ is the unit element, and more $\because \varnothing \cap \omega = \varnothing \wedge \omega \cap \varnothing = \varnothing$, $\therefore \varnothing$ is the zero element. So, RA is algebra structure. □

F denotes the region set of structure in $\mathcal{CDT}$. So the above algebra is called as region algebra (RA). The forms of Region (subject region) are various and are equal with topology and algebra. The region is correlation. The region is not only the combination of topology unit and modal unit but also has spatial and algebraic characteristics.

3. SPATIAL RELATIONS CALCULATING BASED ON RA

The spatial description method of geographic objects and the spatial relationship calculation method of spatial features are the spirit of GIS. The former is the base of the later that behaves as the computation processes in different mathematics space.

The spatial relationship calculation can be analyzed based on the spatial description method of geographic features. The most popularity used model is the 9I model in the domain of spatial relationship calculation (Egenhofer and Herring 1991). According to the theory of unit structure (Corbett 1985), the 2-dimension feature is boundary by the 1-dimension and the 1-dimension feature is boundary by the 0-dimension in the 2-dimension Euclidean space $\mathbb{E}^2$. So, the boundary of spatial object is composed of 0-dimension points and 1-dimension segments. The boundary describes the position and shape of spatial object. In the 9I model, every entity is defined as three entities, an interior (°), a boundary (∂) and an exterior (⁻). Except the region held by the entity self and its boundary, the exterior of entity is filling all universe. When the universe is $\mathbb{E}^n$ ($n \geq 2$), the description of entity has continuous restriction, which behaves that the description of single entity (i.e. interior, boundary or exterior) must depend on the other two entities. Zhilin Li (2000) has found that these three entities have linear relative relationship. In order to eliminate the continuous bondage in entity description, many researchers work hard in this domain. Chen Jun (2000) deflated the exterior as a limited area relative and defined the area as dependent entity by replacing the exterior with the Voronoi region of feature, but the interior and boundary still depend on each other; Zhilin Li (2000) described the entity in grid space $\mathbb{Z}^2$ not in the Euclidean space $\mathbb{E}^2$, thus the three entities can be individually described as the set of grid unit. But from the math base for entity description, $\mathbb{Z}^2$ can't be replaced by $\mathbb{E}^2$ after all.

The spatial relationship calculation can be analyzed from the geometric operation. The most important operation is intersection in order to distinguish the feature relationships, that is build up line intersection-based. Line intersection is a kind of graphics quantitative operation which needs complex computation and needs complex structure as the support. Therefore, researchers (Gold 1992, Chen 1998) proposed two problems: Whether the complex graphics operation (such as line intersection) can be replaced by comparison of simple characteristics in computing graphics relationship or not? Whether other spatial relationship can be got by using the support structure required by graphics operation or not? The two problems can be summarized as the qualitative and reasoning computation. They are the core ideas of the dynamic spatial relationship calculation. In the qualitative computation domain, many researches have been undertaken in order to resolve the adjacency spatial relationship computation by use of Voronoi diagram (Gold 1992, Okabe, Boots and Sugihara 1992, 1994 Chen 1998, Gahegan 2000). The adjacency spatial relationship computation is realized by judging whether the Voronoi feature's regions exist the common boundaries or not. Because the Voronoi diagram is complete division structure to space, the share operation of Voronoi boundary is a typical geometry qualitative operation; in the domain of reasoning computation, Li and Chen (1998) proposed a 4 adjacency reasoning model based on Voronoi region which is an effective extend to resolve the question of adjacency spatial relationship computation by Voronoi diagram. The method used in the model is that the spatial relationship among objects is got by judging the relationship among Voronoi regions of object on the precondition of Voronoi region of object existence.

3.1 Description of Integrating Discrete and Continuous Spatial Object

The point, line and polygon features in $\mathbb{E}^2$ can be described by use of independent node, chain and ring structure in $\mathcal{CDT}$ with RA. Thus, the feature can be described as three independent entities, i.e. left (ω^l), middle (ω^m) and right (ω^r).

THEOREM 3.1.1. The left set (ω^l), middle set (ω^m) and right set (ω^r) of spatial objects (point^R, line^R and $\mathrm{polygon}^R$) are all countable sets.

Proof. If the n(n is countable, $n \in \mathbb{N}$) nodes are considered as the generators, the Voronoi diagram of this generators is a set of n Voronoi regions. The set is a countable set and is a kind of division of space. ∵ Voronoi diagram is dual with Delaunay triangulation each other, ∴ Delaunay triangulation is sure of a countable set composed of many triangles, i.e. $\mathcal{CDT}$. And more $\because \forall \sigma \in \omega^l \Rightarrow \sigma \in \mathcal{CDT}$, $\therefore \omega^l \subset \mathcal{CDT}$. By the same way there is

$\omega^m \subset \mathcal{CDT}$ and $\omega^r \subset \mathcal{CDT}$. $\because \mathcal{CDT}$ is a countable set, $\therefore \omega^l$, ω^m and ω^r are all countable sets. Theorem 3.1.1 is tenable. □

COROLLARY 3.1.1. The left (ω^l), middle (ω^m) and right (ω^r) of spatial objects (point[R], line[R] and polygon[R]) are all partial relative regions.

Proof. According to theorem 3.1.1, the left, middle and right set are all countable sets and there is $\omega^l \cup \omega^m \cup \omega^r \subseteq \mathcal{CDT}$, i.e. $\omega^l \cup \omega^m \cup \omega^r$ is the subset of $\mathcal{CDT}$. And more $\forall \sigma \in \omega^l \cup \omega^m \cup \omega^r$, σ is all relative to the spatial objects in $\mathbb{E}^2$, i.e. the nodes of σ result from the spatial object. In summary, the left (ω^l), middle (ω^m) and right (ω^r) of spatial objects are all partial relative regions. □

THEOREM 3.3.2. The left set (ω^l), middle set (ω^m) and right set (ω^r) of spatial objects (point[R], line[R] and polygon[R]) are reverted as the corresponding structures (point, line and polygon) in $\mathbb{E}^2$.

Proof. Polygon object is $\omega^l \cup \omega^r$, given operation $\boldsymbol{seg}(\sigma^l, \sigma^r)(\sigma^l \in \omega^l, \sigma^r \in \omega^r)$ is the segment used for seeking the $\boldsymbol{f}^1$ adjacency between σ^l and σ^r, thus the segment is sure of the part of boundary of polygon object and its two extreme points are sure of the boundary node of the polygon object. Given the segment set sought by ***seg*** operation is **S**, then **S** can be sought by the following formula,

$$\mathbf{S} = \bigcup_{i=1}^{|\omega^l|} \bigcup_{j=1}^{|\omega^r|} \boldsymbol{seg}(\sigma^l, \sigma^r), \quad \sigma^l \in \omega^l, \sigma^r \in \omega^r, \quad (3\text{-}1\text{-}1)$$

$\because \mathcal{CDT}$ is a kind of spatial division, $\therefore$ the boundary of the area object can be got by linking the segments in **S** on the condition of the same node order, and i.e. this is the cycle structure in $\mathbb{E}^2$. By the same way, the case of line object and point object can be proved. Theorem 3.3.2 is tenable. □

According to the theorem 3.3.2, the discrete definition of spatial object completely remains its continuation property in $\mathbb{E}^2$, that is to say, it remains its basic position and shape of in $\mathbb{E}^2$. $\mathcal{CDT}$ is the common way to observe the raster and vector characteristics of spatial objects.

DEFINITION 3.1.1.(basic definition of spatial object in $\mathcal{CDT}$) the spatial objects (point, line and polygon) in $\mathbb{E}^2$ can be denoted as the point[R], line[R] and polygon[R] in $\mathcal{CDT}$ and there is the following corresponding relationship,

$$polygon^{\mathrm{R}} \overset{def}{\Leftrightarrow} poly^{-} + poly^{\partial} + poly^{\circ} \Leftrightarrow \omega^{-}(\sigma_{r(C)}) + \omega^{\partial}(\sigma_{l(C)} \cup \sigma_{r(C)}) + \omega^{\circ}(\sigma_{l(C)}), \quad (3\text{-}1\text{-}2)$$

$$line^{\mathrm{R}} \overset{def}{\Leftrightarrow} line^{-} + line^{\partial} + line^{\circ} \Leftrightarrow \omega^{-}(\sigma_{l(C)} \cup \sigma_{r(C)} \cup \sigma_{m(C)}) + \omega^{\partial}(\sigma_{m(C)}) + \omega^{\circ}(\sigma_{l(C)} \cup \sigma_{r(C)}), \quad (3\text{-}1\text{-}3)$$

$$point^{\mathrm{R}} \overset{def}{\Leftrightarrow} point^{-} + point^{\partial} + point^{\circ} \Leftrightarrow \omega^{-}(\sigma_{\leftarrow C}) + \omega^{\partial}(\sigma_{\leftarrow C}) + \omega^{\circ}(\sigma_{\leftarrow C}) \cdot \quad (3\text{-}1\text{-}4)$$

3.2 Basic Operators and the Computation Model of Spatial Relationship

Intersection operation "$\cap$" is the exclusive 2-tuple operator in the region algebra structure. Based on the intersection operator, the region modality operator (τ) applied to the spatial object can be got. The "$\cap$" and "τ" are the basic operators for relationship calculation.

Symbol	Definition	Computation result
$\cap(\omega^A, \omega^B)$	Given ω^A, ω^B are two subject regions, then there is the following formula, $\cap(\omega^A, \omega^B) = \{\sigma \mid \sigma \in \omega^A \wedge \sigma \in \omega^B\}$.	$\{\varnothing, \{\xi \mid \xi \in \Xi^s_{\omega_A} \wedge \xi \in \Xi^s_{\omega_B}\}\}$
$\tau(\omega)$	Compute the basic modality of subject region ω.	$\mathbf{re} = \{\varnothing$, nen,nan,ndn, nicl,nicr,nic,noc,nacl, nacr,ndc,chec,cec,cnc, cicl,cicr,cic,ctc,cac,cdc$\}$ $\tau(\omega) \subset 2^{\mathbf{re}}$

Table 3.2. Symbols, definitions and computing results of the basic operators

DEFINITION 3.2.1. The modality operation of subject region ($\cap^{\tau}$) is the composite operation of "$\cap$" and "τ", its formula is,

$$\cap^{\tau}(\omega^A, \omega^B) = \tau(\cap(\omega^A, \omega^B)) = \tau(\omega^A \cap \omega^B). \quad (3\text{-}2\text{-}1)$$

Adbelmoty (1995) proposed a generalized intersection model applied to describe the spatial relationship. Firstly, it decomposes the spatial object and region space as typical subsets. Secondly, describes the spatial relationships based on the intersection of these subsets. Given $\boldsymbol{\sigma}$ is a spatial object and $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is its subsets, then there is $\boldsymbol{\sigma} = \bigcup_{i=1}^{m} \boldsymbol{\sigma}_i$, the complement set of $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}^{-} = \bigcup_{j=m+1}^{n} \boldsymbol{\sigma}_j$, the spatial reverse located by spatial object is Σ, then there is $\Sigma = \boldsymbol{\sigma} \cup \boldsymbol{\sigma}^{-}$. By decomposing the spatial object based on union, the relationship between two spatial objects can be defined as the following formula,

$$R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \boldsymbol{\sigma} \cap \boldsymbol{\sigma}' = (\textstyle\bigcup_{i=1}^{m} \boldsymbol{\sigma}_i) \cap (\bigcup_{j=1}^{n} \boldsymbol{\sigma}'_j) = (\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}'_1, \boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}'_2, \cdots, \boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}'_n, \boldsymbol{\sigma}_2 \cap \boldsymbol{\sigma}'_1, \cdots, \boldsymbol{\sigma}_m \cap \boldsymbol{\sigma}'_n),$$

where $\boldsymbol{\sigma}_i \cap \boldsymbol{\sigma}'_j$ may fetch $\varnothing$ or $\neg\varnothing$, the intersection may be expressed with a $m \times n$ matrix. By different division strategy of spatial object, generalized intersection model can describe the spatial relationship among complex spatial objects.

According to the generalized intersection model and the basic definition of spatial object (definition 3.1.1), a spatial relationship calculation model is established on the following conditions: the modality operation of subject region is regarded as the basic operation and the result resulted from the intersection each other of the three entities (exterior, boundary and interior) of the two spatial objects. The formula of this model is denoted as a 3×3 matrix, as showed in formula 3-2-2, which named as region nine intersection model (R9I).

$$\mathrm{R9I}(A, B) = \begin{bmatrix} \omega_A^{-} \cap^{\tau} \omega_B^{-} & \omega_A^{-} \cap^{\tau} \omega_B^{\partial} & \omega_A^{-} \cap^{\tau} \omega_B^{\circ} \\ \omega_A^{\partial} \cap^{\tau} \omega_B^{-} & \omega_A^{\partial} \cap^{\tau} \omega_B^{\partial} & \omega_A^{\partial} \cap^{\tau} \omega_B^{\circ} \\ \omega_A^{\circ} \cap^{\tau} \omega_B^{-} & \omega_A^{\circ} \cap^{\tau} \omega_B^{\partial} & \omega_A^{\circ} \cap^{\tau} \omega_B^{\circ} \end{bmatrix}. \quad (3\text{-}2\text{-}2)$$

According to the result of subject region modality computation, the computation results of R9I model can be classified as follow: if every unit value of R9I is single type, i.e. $\mathbf{re} = \{\varnothing\} \vee |\mathbf{re}| = 2$, the type is single modality type; if $|\mathbf{re}| > 2$, the type is duplicate modality type.

3.3 Computing Spatial Relations among Simple Objects by R9I

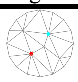
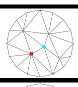
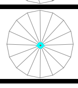
No	Legend	Value of R9I	Semantics
1		$\begin{bmatrix}\emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\end{bmatrix}$	disjoint
2		$\begin{bmatrix}\text{nan} & \text{nan} & \text{nan}\\ \text{nan} & \text{nan} & \text{nan}\\ \text{nan} & \text{nan} & \text{nan}\end{bmatrix}$	adjacency
3		$\begin{bmatrix}\text{nen} & \text{nen} & \text{nen}\\ \text{nen} & \text{nen} & \text{nen}\\ \text{nen} & \text{nen} & \text{nen}\end{bmatrix}$	equal

Table 3.3.1. Relationships of point^R&point^R

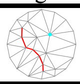
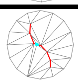
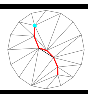
No	Legend	Value of R9I	Semantics
1		$\begin{bmatrix}\emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\end{bmatrix}$	disjoint
2		$\begin{bmatrix}\text{nic} & \emptyset & \text{nic}\\ \text{nic} & \emptyset & \text{nic}\\ \text{nic} & \emptyset & \text{nic}\end{bmatrix}$	contain
3		$\begin{bmatrix}\text{noc} & \text{nen} & \text{noc}\\ \text{noc} & \text{nen} & \text{noc}\\ \text{noc} & \text{nen} & \text{noc}\end{bmatrix}$	boundary

Table 3.3.2. Relationships of point^R&line^R

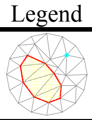
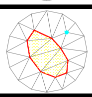
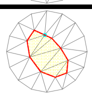
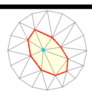
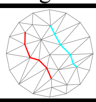
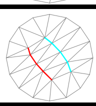
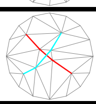
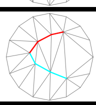
No	Legend	Value of R9I	Semantics
1		$\begin{bmatrix}\emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\end{bmatrix}$	disjoint
2		$\begin{bmatrix}\text{nacr} & \text{nacr} & \emptyset\\ \text{nacr} & \text{nacr} & \emptyset\\ \text{nacr} & \text{nacr} & \emptyset\end{bmatrix}$	adjacency
3		$\begin{bmatrix}\text{nicr} & \text{nic} & \text{nicl}\\ \text{nicr} & \text{nic} & \text{nicl}\\ \text{nicr} & \text{nic} & \text{nicl}\end{bmatrix}$	boundary
4		$\begin{bmatrix}\emptyset & \text{nacl} & \text{nacl}\\ \emptyset & \text{nacl} & \text{nacl}\\ \emptyset & \text{nacl} & \text{nacl}\end{bmatrix}$	contain

Table 3.3.3. Relationships point^R&polygon^R

No	Legend	Value of R9I	Semantics
1		$\begin{bmatrix}\emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\end{bmatrix}$	disjoint
2		$\begin{bmatrix}\text{cac} & \text{nac} & \text{cac}\\ \text{nac} & \text{nan} & \emptyset\\ \text{cac} & \emptyset & \text{cac}\end{bmatrix}$	adjacency
3		$\begin{bmatrix}\text{cic} & \emptyset & \text{cic}\\ \emptyset & \emptyset & \emptyset\\ \text{cic} & \emptyset & \text{cic}\end{bmatrix}$	intersection
4		$\begin{bmatrix}\text{cnc} & \text{noc} & \text{cnc}\\ \text{noc} & \text{nen} & \emptyset\\ \text{cnc} & \emptyset & \text{ccn}\end{bmatrix}$	meet at ends

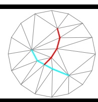
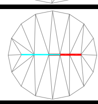
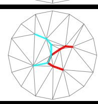
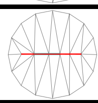
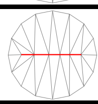
No	Legend	Value of R9I	Semantics
5		$\begin{bmatrix}\text{cicl(r)} & \text{nicl(r)} & \text{cicl(r)}\\ \text{nac} & \text{nan} & \text{nac}\\ \text{cicl(r)} & \text{nicr(l)} & \text{cicl(r)}\end{bmatrix}$	meet at one end and other middle
5		$\begin{bmatrix}\text{cec} & \text{nic} & \text{cec}\\ \text{nic} & \emptyset & \text{nic}\\ \text{cec} & \text{nic} & \text{cec}\end{bmatrix}$	overlap
6		$\begin{bmatrix}\text{cec} & \emptyset & \text{cec}\\ \emptyset & \emptyset & \emptyset\\ \text{cec} & \emptyset & \text{cec}\end{bmatrix}$	1D meet
7		$\begin{bmatrix}\text{cec} & \emptyset & \text{cec}\\ \text{noc} & \emptyset & \text{noc}\\ \text{cec} & \emptyset & \text{cec}\end{bmatrix}$	contain
8		$\begin{bmatrix}\text{cec} & \text{noc} & \text{cec}\\ \text{noc} & \text{nen} & \text{noc}\\ \text{cec} & \text{noc} & \text{cec}\end{bmatrix}$	equal

Table 3.3.4. Relationships of line^R&line^R

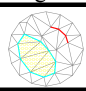
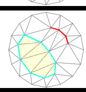
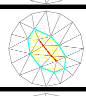
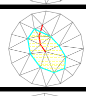
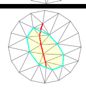
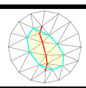
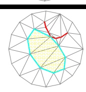
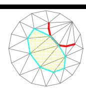
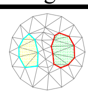
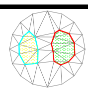
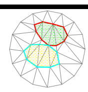
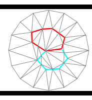
No	Legend	Value of R9I	Semantics
1		$\begin{bmatrix}\emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\end{bmatrix}$	disjoint
2		$\begin{bmatrix}\text{cac} & \text{nac} & \text{cac}\\ \text{cac} & \text{nac} & \text{cac}\\ \emptyset & \emptyset & \emptyset\end{bmatrix}$	adjacency
3		$\begin{bmatrix}\emptyset & \text{cac} & \text{cac}\\ \emptyset & \text{nacl} & \text{nacl}\\ \emptyset & \text{cac} & \text{cac}\end{bmatrix}$	contain
4		$\begin{bmatrix}\text{cicr} & \text{cic} & \text{cicl}\\ \emptyset & \text{nacl} & \text{nacl}\\ \text{cicr} & \text{cic} & \text{cicl}\end{bmatrix}$	entry/go out
5		$\begin{bmatrix}\text{cicr} & \text{cic} & \text{cicl}\\ \text{nicr} & \text{nicr} & \emptyset\\ \text{cicr} & \text{cic} & \text{cicl}\end{bmatrix}$	half-cross
6		$\begin{bmatrix}\text{cicr} & \text{cic} & \text{cicl}\\ \emptyset & \text{nacr} & \emptyset\\ \text{cicr} & \text{cic} & \text{cicl}\end{bmatrix}$	cross
7		$\begin{bmatrix}\text{ctc} & \text{ctc} & \text{nicl}\\ \emptyset & \emptyset & \emptyset\\ \text{ctc} & \text{ctc} & \text{nicl}\end{bmatrix}$	0-D meet
8		$\begin{bmatrix}\text{cec} & \text{cec} & \text{cec}\\ \emptyset & \emptyset & \emptyset\\ \text{cec} & \text{cec} & \text{cec}\end{bmatrix}$	1-D meet

Table3.3.5. Relationships of line^R&polygon^R

No	Legend	Value of R9I	Semantics
1		$\begin{bmatrix}\emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset\end{bmatrix}$	disjoint
2		$\begin{bmatrix}\text{cac} & \text{cac} & \emptyset\\ \text{cac} & \text{cac} & \emptyset\\ \emptyset & \emptyset & \emptyset\end{bmatrix}$	adjacency
3		$\begin{bmatrix}\text{ctc} & \text{ctc} & \text{nicl}\\ \text{ctc} & \text{ctc} & \text{nicl}\\ \text{nicl} & \text{nicl} & \emptyset\end{bmatrix}$	0-D meet
4		$\begin{bmatrix}\text{cnc} & \text{cecr} & \text{cecr}\\ \text{cnc} & \text{cec} & \text{cecr}\\ \text{cecl} & \text{cecl} & \emptyset\end{bmatrix}$	1-D meet

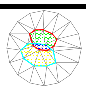
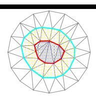
5		$\begin{bmatrix} cnc & cicr & cnc \\ cicr & cic & cicl \\ cnc & cicl & cnc \end{bmatrix}$	intersect
6		$\begin{bmatrix} \varnothing & \varnothing & \varnothing \\ cac & cac & \varnothing \\ cac & cac & cac \end{bmatrix}$	contain

Table 3.3.6. Relationships of polygon^R&polygon^R

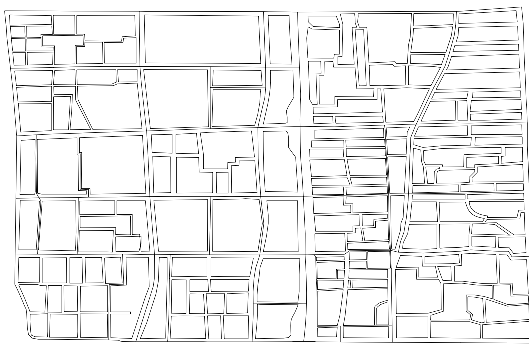
4. EXAMPLE

Multi-topological rules examination for vector data is a new requirement for GIS data product. In general, for different type of vector data which is independently organized and stored as a layer form, current GIS software provides the topological rules for checking errors included in one layer or between two layers. For example in Arc/Info, the "Must No Overlap" rule is used for checking a polygon feature must not overlap another polygon feature from the same layer, the "Boundary Must Be Covered By" rule is used for checking the boundaries of polygon feature from one layer must be covered by line features of another layer. However, every time only use one rule to the examination work.

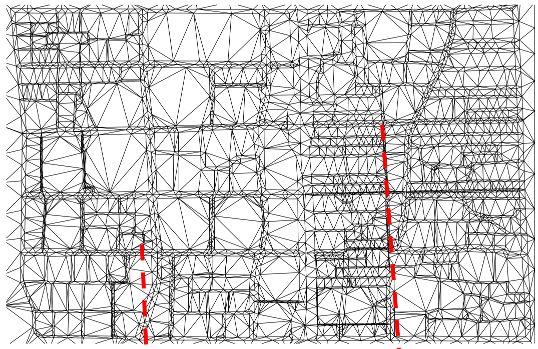
With the method proposed in this paper, we can use multi-rule. As shown in figure 11(a), (b), there are parts of block map and street map of Beijing and its CDT. The multi-rule includes three rules:

- The "Must No Overlap" rule for block map not self-intersecting.
- The "Must No Intersect" rule for the block map not intersects with the street map.
- The "Adjacency" rule for every street must adjacency at least one block.

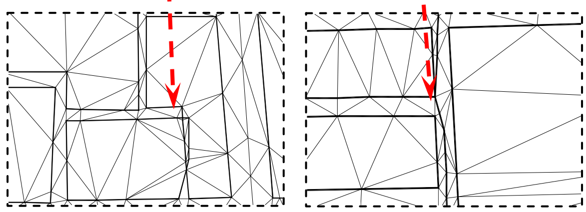
The topological errors by above multi-rule are shown in figure 11(c) and (d).



(a) Parts of block map and street map in Beijing



(b) The CDT of (a)



(c) Two blocks intersect (d) One street intersects with a block

Figure 11. Example data and it CDT

5. CONCLUSION AND FUTURE WORKS

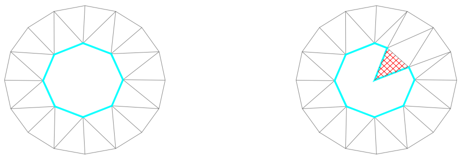
This work proposes a description method of 2D spatial object based on CDT which integrates continuous method and discrete method together. Based on the set theory, the subject region and computation space in CDT is defined. Additionally, the region algebra structure applied to compute spatial relationship is established on the base of intersection operation. By judging the modality of result set (subject region) of intersection among the three hefts of spatial object, the spatial relationship computation model-R9I is established. By R9I, the 32 kinds of spatial relationship among simple objects is distinguished each other.

The future works can be summarized as following aspects:

(1) The modality of surface features (spatial graphics) is one of important research subjects in the spatial relationship domain. The understanding to real world always starts from the individuals. However, the research on spatial relationship directly focuses on the relationship among individuals not the individual. For instance, "query which circular buildings are adjacent with a road" can't get the correct results by current spatial relationship theory. The constrained Delaunay triangulation division of spatial graphics is another important way to observe and analyze its modality. The figure 11 shows the knaggy characteristics of graphics, the exterior of convex polygon only contains β-triangle and γ-triangle. But the exterior of concave polygon also contains α-triangle.

(2) The computing of adjacency spatial relationship. The nature adjacency spatial relationship defined by Voronoi diagram of spatial objects is a special spatial relationship. It exits among the discrete spatial objects. However, the object exploration based on the spatial relationship always uses the intersection as the basic way. So, the expression method and index structure of adjacency region among spatial objects on the base that the Dalaunay triangulation is the base structure should be focused on. Then the adjacency spatial relationship can be detected by combining the line intersection operation in

the practical GIS application environment with the Dalaunay triangulation.



(a) Convex polygon and its $\sigma_{\beta,\gamma}$ at right side (b) Concave polygon and its σ_{α} at right side

Figure 11. Observing the knaggy characteristics of polygon by its σ

(3) The computing of spatial relationship among complex objects. The spatial object has the continuous (vector) and discrete (grid) characteristics at the same time in case that the spatial objects are described by the method based on CDT. The vector-grid characteristic of Delaunay triangulation should be further rooted in order to describe the complex spatial objects and compute the relationship among the complex objects by Delaunay triangulation.

(4) The thinking in λ. Whether $\sigma_n = \{\sum_{i=0}^{3}\lambda_i a^i \mid \lambda_i \geq 0, \sum_{i=0}^{3}\lambda_i = 1\}$ can be used to express the complex-tetrahedroid in $\mathbb{E}^3$ or not; whether the method in $\mathbb{E}^2$ can be generalized to $\mathbb{E}^3$ or not in order to solve the description of 3D spatial objects and compute their spatial relationship?

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