METHOD OF BOUNDARY EXTRACTION BASED ON SCHRÖDINGER EQUATION

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ABSTRACT:

To overcome the main drawbacks of global minimal for active contour models (L. D. Cohen and Ron Kimmel) that the contour is only extracted partially for low SNR images, Method of boundary extraction based on Schrödinger Equation is proposed. Our Method is based on computing the numerical solutions of initial value problem for second order nonlinear Schrödinger equation by using discrete Fourier Transformation. Schrödinger transformation of image is first given. We compute the probability $P(b,a)$ that a particle moves from a point a to another point b according to I-Type Schrödinger transformation of image and obtain boundary of object by using quantum contour model..

1. INTRODUCTION

Deformable models based on particle motion in classical mechanics, also called snakes, or active contour models, were first proposed by Kass and Terzopoulos in 1987. Since then, a variety of improvements have been made such as balloon models (L.D.Cohen,1991), geometric model (V.Caseles, F.Catte,T.Coll, and F.Dibos, 1993), as well as the topology adaptive deformable model (T.McInerney and D.Terzopoulos, 1999). In 1997,Cohen and Kimmel described a method for integrating object boundaries by searching the path of a minimal active deformable model's energy between two points.

Lou and Ding used point tracking by estimating the probability of a particle moving from one point to another in quantum mechanics, and did not impose any smoothness constraints to ensure the extraction of the details of a concave contour (Liantang Lou and Mingyue Ding, 2007). Feynman and Hibbs had used path integration method to count the kernel of the particle In this article, the probability of a particle moving from one point to another is directly computed according to the relation between the kernel $K(b,a)$ and image gradient $G(\mathbf{x})$ (see Figure 1) by using discrete Fourier Transformation.
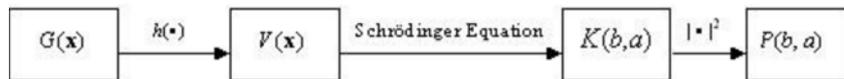
Figure 1. The relation between the probability $P(b,a)$ and image gradient $G(\mathbf{x})$

2. RELATION BETWEEN THE PROBABILITY AND IMAGE GRADIENT

The active contour model or Snake model had their profound physical background. If the parameter s in the deformable contour curve $\mathbf{x}(s) = (x(s), y(s))$ could be understood as time t, object contour curve $\mathbf{x}(t)$ could be considered as the path of the particle in plane motion.

Suppose a particle moves from the position a at the time t_a to the position at the time t_b ,e.g., $a = x(t_a)$, $b = x(t_b)$. According to the theory of quantum mechanics, the probability of a particle moving from the position a to b at t_b , denoted by $P(b,a)$, is dependent on the kernel $K(b,a)$, which is the sum of all paths contribution between $\mathbf{x}_a$ and $\mathbf{x}_b$, i.e.,

$$K(b,a) = \sum_{R(a,b)} \phi\,(x(t)) , \qquad (1)$$

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where $R(a,b)$ is the set of all paths between $\mathbf{x}_a$ and $\mathbf{x}_b$. $\phi(\mathbf{x}(t))$ is the contribution of a path $\mathbf{x}(t)$ with a phase proportional to its energy $E(\mathbf{x}(t))$, i.e.,

$$\phi(\mathbf{x}(t)) = Ce^{(2\pi i/h)E(\mathbf{x}(t))} , \qquad (2)$$

where h is the Planck's Constant, C is a constant, and $E = \int_{t_a}^{t_b}\left(\frac{m}{2}|\mathbf{x}'(t)|^2 - V(\mathbf{x},t)\right)dt$ is the energy functional of path. The probability from point $\mathbf{x}_a$ at time t_a to point $\mathbf{x}_b$ at time t_b is the square of absolute value of kernel $K(b,a)$ from $\mathbf{x}_a$ to $\mathbf{x}_b$, that is,

$$P(b,a) = |K(b,a)|^2 . \qquad (3)$$

3. SCHRÖDINGER TRANSFORMATION OF IMAGE

We must face problem counting the kernel $K(b,a)$ to introduce law of particle motion in quantum mechanics into image processing and analysis. For a system with a simple Lagrangian function, $K(b,a)$ can be calculated directly from the path integral (see 0) while for a system with a complex Lagrangian function, it is difficult and time-consuming to estimate the value of $P(b,a)$ from $K(b,a)$. In order to avoid such difficulty, Lou and Ding estimate the probability of a particle moving from point a to point b directly from specific particle models. In the paper, We'll compute the probability $P(b,a)$ that a particle moves from a point a to another point b using Schrödinger transformation of image.

Replacing the kernel $K(b,a)$ with the wave function $u(\mathbf{x},t)$ in the position $\mathbf{x}$ at the time t, then $u(\mathbf{x},t)$ satisfied the following Schrödinger equation:

$$\bar{h}i \cdot \frac{\partial u}{\partial t} = -\frac{\bar{h}^2}{2m}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + V(\mathbf{x},t)u(\mathbf{x},t) , \qquad (4)$$

where $\bar{h} = h/2\pi = 1.054 \times 10^{-27} crg \cdot s$.
We could rewrite Eq.(4) as the initial-value problem:

$$\begin{cases} i \cdot u_t + a\Delta u = v(\mathbf{x})u \\ u|_{t=0} = \varphi(\mathbf{x}) \end{cases} \qquad (5)$$

By applying Fourier transform to equation (5) and making use of the properties of Fourier transform, we have

$$\begin{cases} i \cdot \hat{u}_t - a|\mathbf{y}|^2\hat{u} = \hat{v}(\mathbf{y}) * \hat{u} \\ \hat{u}|_{t=0} = \hat{\varphi}(\mathbf{y}) \end{cases} , \qquad (6)$$

where the mark '$*$' denotes convolution of two functions, '$\wedge$' denotes Fourier transformation of function. When $v(\mathbf{x}) = 0$, both $u(\mathbf{x},t)$ and $\hat{u}(\mathbf{y},t)$ have the following analytic solutions (L. C. Evans, 1998):

$$\hat{u} = e^{-ait|\mathbf{y}|^2}\hat{\varphi}(\mathbf{y}) \qquad (7)$$

$$u(\mathbf{x},t) = \frac{1}{4\pi ait}\iint e^{\frac{i}{4at}|\mathbf{x}-\mathbf{y}|^2}\hat{\varphi}(\mathbf{y})d\mathbf{y}, \ \mathbf{x} \in R^2, t > 0 , \qquad (8)$$

When $v(\mathbf{x}) \neq 0$, $u(\mathbf{x},t)$ and $\hat{u}(\mathbf{y},t)$ also have analytic solutions (L. C. Evans, 1998), but they are quiet complex so that they could not be used to compute their numerical solutions. We give the following definition of Schrödinger transformation of image because of Eq. (5):
Schrödinger transformation of image $\varphi(\mathbf{x})$ based on $v(\mathbf{x})$ was defined as the solution of Eq. (5). And the transformation is called *I-Type Schrödinger transformation* when $v(\mathbf{x}) = 0$, otherwise the transformation is called *II-Type Schrödinger transformation*. Supposed both $\varphi(\mathbf{x})$ and $v(\mathbf{x})$ are $m \times n$ images, then two-dimensional *discrete Schrödinger transformation of image* $\varphi(\mathbf{x})$ based on $v(\mathbf{x})$ is expressed with the following differential equation which its Fourier transformation satisfies:

$$\begin{cases} i \cdot \vec{u}_t = (V + a|\mathbf{y}|^2)\vec{u} \\ \vec{u}|_{t=0} = \vec{\varphi} \end{cases} , \qquad (9)$$

where $\vec{u}_t$ is mn-dimensional column vector formed by concatenating all the rows of $m \times n$ matrix $\hat{u}_t$. $mn \times mn$ matrix $|\mathbf{y}|$ was diagonal matrix whose diagonal elements express distance. $mn \times mn$ matrix V is a block cyclic matrix, i.e.,

$$V = \begin{pmatrix} V_0 & V_{m-1} & \cdots & V_1 \\ V_1 & V_0 & \cdots & V_2 \\ \vdots & \vdots & \vdots & \vdots \\ V_{m-1} & V_{m-2} & \cdots & V_0 \end{pmatrix}, \qquad (10)$$

where V_i is a cyclic matrix,

$$V_i = \begin{pmatrix} v(i,0) & v(i,n-1) & \cdots & v(i,1) \\ v(i,1) & v(i,0) & \cdots & v(i,2) \\ \vdots & \vdots & \vdots & \vdots \\ v(i,n-1) & v(i,n-2) & \cdots & v(i,0) \end{pmatrix}, \qquad (11)$$

Obviously, the solution of Eq.(9) is

$$\hat{u}(\mathbf{y}) = e^{-it(V+a|\mathbf{y}|^2)}\hat{\varphi}(\mathbf{y}) . \qquad (12)$$

If the matrix $V + a|\mathbf{y}|^2$ can be diagonalized, that is, $V + a|\mathbf{y}|^2 = P^{-1}DP$, then

$$\hat{u}(\mathbf{y}) = P^{-1}Diag(e^{-itd_1}, e^{-itd_2}, \cdots, e^{-itd_{mn}})P\hat{\varphi}(\mathbf{y}), \qquad (13)$$

where $D = Diag(d_1, d_2, \cdots, d_{mn})$ was diagonal matrix. Eq.(13) degenerates into Eq.(7) when $v(\mathbf{x}) = 0$. The discrete Schrödinger transformation of image $\varphi(\mathbf{x})$ based on $v(\mathbf{x})$ can be obtained using Eq.(3).

4. BOUNDARY EXTRACTION BASED ON SCHRÖDINGER TRANSFORMATION OF IMAGE

Schrödinger transformation of image can be applied to image processing and analysis, such as, boundary extraction, edge enhancement, image inpainting, image restoration,etc. We extract boundary of object according to the approach given in （Liantang Lou and Mingyue Ding, 2007）. The steps are listed as follows:

(1) Compute discrete Schrödinger transformation of gradient image $G(\mathbf{x})$, denoted by $u(\mathbf{x}, t_0)$, where t_0 is a small positive constant.

(2) Compute the probability $P(\mathbf{x}) = u(\mathbf{x}, t_0)$.

(3) Estimate the expectation position of the particle at the next time $t_a + \Delta t$ by:

$$\mathbf{x}_e = \frac{\sum_{b \in \mathbf{S}} \mathbf{x}_b P(\mathbf{x}_b)}{\sum_{b \in \mathbf{S}} P(\mathbf{x}_b)} \qquad (14)$$

where the set $\mathbf{S}$ are possible positions after the interval time Δt when a particle moves from a starting position a at the time t_a. From Eq. (14), the next boundary point position is estimated. The whole boundary is tracked by repeating the expectation position calculation procedure iteratively.

5. EXPERIMENTAL RESULT

The following experiments (see Figure 2) show the meaning and function of Schrödinger transformation of image, that is, Schrödinger transformation of image can be seen as the result of original image shrinking inside and spreading outward, like as interference wave. The bigger *at* in Eq.(7) is, the more obvious the interference is. If we estimate the probability of a particle appearing in some point using Schrödinger transformation of gradient image, we can obtain the same conclusion given in the article (Liantang Lou and Mingyue Ding, 2007, Figure 1). The quantum contour model produces a contour around the true object boundary with the jagged particle trajectory while the deformable model produces a smooth but biased contour. Though quantum contour is zigzag and the contour extracted using deformable model was quite smooth, quantum contour has smaller system deviation.

The quantum contours of object are given in Figure 2. Figure 2 show that the contour with Schrödinger transformations is smoother than the contour without using Schrödinger transformations.

(a)   (b)   (c)
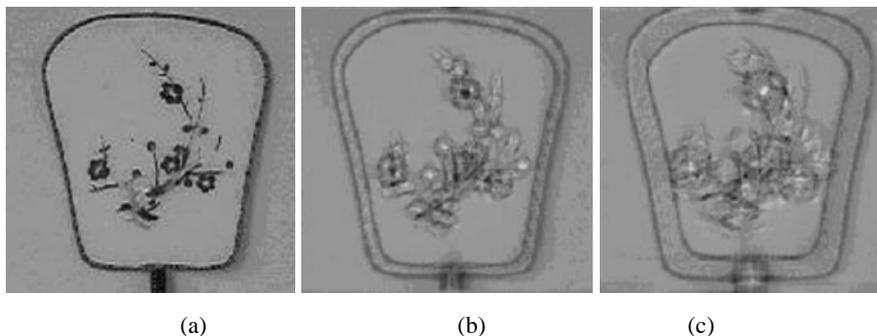
Figure 2. Schrödinger transformation of image
(a) The original image, (b), (c) are Schrödinger transformations of image. The constant *at* is 0.0005, 0.001, respectively.

(a)   (b)   (c)
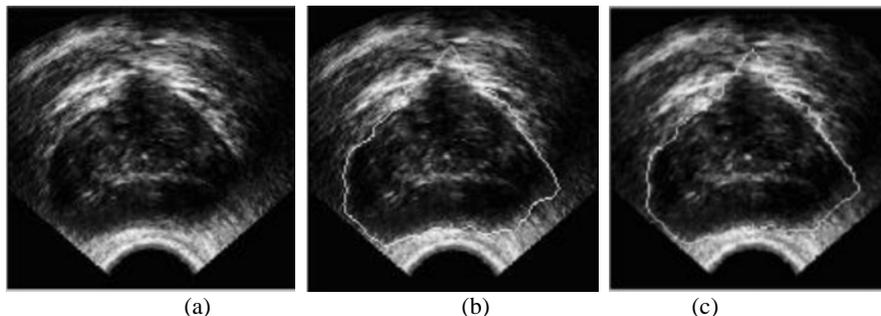
Figure 3. contours extracted by using the quantum contour based approach.
(a) The original image, (b) quantum contour without using Schrödinger transformations(Liantang Lou and Mingyue Ding, 2007), (c) quantum contour based on Schrödinger transformations with the constant *at* is 0.000001.

6. CONCLUSIONS AND ACKNOWLEDGEMENT

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