GEOMETRIC REASONING IN 3D BUILDING MODELS USING MULTIVARIATE POLYNOMIALS AND CHARACTERISTIC SETS

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ABSTRACT:

In order to generate complex virtual cities, models of buildings have to be defined in advance. A common approach describes individual components of these buildings, which are in turn restricted by geometric constraints such as orthogonality, parallelity or symmetry of building parts. A major challenge is to ensure consistency and avoid redundancy. Tools are needed which support geometric reasoning and thus the modelling of buildings. This leads to the theory of automatic theorem proving. By using this theory it can be shown that a constraint is deducible from a set of axioms that can be realized by using multivariate polynomials. We draw upon Wu's method and claim that 3D reasoning for building models is feasible.

1. INTRODUCTION

3D building modelling has become increasingly important due to a high demand in the context of navigation systems or virtual city tours. The use of city models for noise mapping, disaster management or the calculation of escape routes requires even exact knowledge about the structure of buildings. Building construction and reconstruction and its automatization is therefore an essential task. Hence, it is necessary to give detailed descriptions of models that represent houses, for instance.

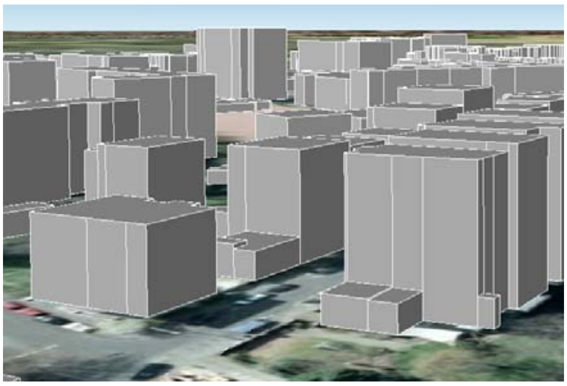

Figure 1. Parallelity and orthogonality are main organizing principles in streets and buildings (aerial image taken from GoogleEarth)

Definitions of models can be built up of primitives, e.g. points or planes that represent walls or roof halves, and therefore their position and their relations have to be restricted by constraints. Since most man-made objects have a regular structure, geometric constraints in 3D mainly include parallelity, orthogonality and symmetry. We focus on analyzing existing city models - the mapping of measured data onto 3D buildings and their constraints is beyond the scope of this paper (the interested reader is referred e.g. to Schmittwilken (2009)). As these constraints have fundamental correlations, geometric reasoning, i.e. deducing properties, can be used to determine which constraints are subsumed by others and therefore can be declared equivalent or rather be eliminated. But how can we show that different constraint sets express the same or how can a consistent and non-redundant model be developed? Consequently, it is of great interest to have feasible methods for interactive systems which meet the requirements of an efficient implementation.

Geometric constraints can be expressed by algebraic equations. As in many cases these constraints contain several parameters, the polynomials are often not linear or even quadratic. While 2D models are easy to cope with, in the transition to the three dimensional space a substantial increase in complexity can be observed, which has to be overcome by the modeller. While there are efficient methods to solve non-linear equation systems numerically, we have to cope with the general validity, in other words, the interest does not lie in finding specific values but in proving theorems on a symbolic level. Against this background, various approaches for automatic theorem proving have been developed in the last three decades which among others are based on multivariate polynomials. A way forward to solve this problem is the construction of the Groebner Bases that leads back to the work of Buchberger (1988).

So far, these approaches have hardly been noticed in the context of computer vision and building modelling. A notable extension is the work of Brenner and Sester (2005) who, however, restrict to the 2D space and emphasize the complexity of the problem. In this paper we use a related approach, namely Wu's method, which is based on characteristic sets. Our main contribution is a method that discovers redundancy and consistency. Thus, we claim that geometric reasoning for buildings and building parts in 3D space is feasible.

This paper is structured as follows: An overview of related work in the area of geometric representations and theorem proving is given in the next section. Section 3 introduces the mathematical foundations, in particular two of the algebraic approaches, and illustrates the constraints that have been used in the context of this paper. Section 4 presents our results and shows that the method is feasible. The paper ends with our conclusions in section 5.

2. RELATED WORK

Automatic theorem proving became popular in the late 70's by the work of Wu (1978,1986), who was able to proof numerous theorems automatically. Various approaches were developed in the last three decades that were applied to perspective viewing (Kapur, 1988) or formula derivation (Chou, 1989).

Constraints play an important role in the representation of man-made objects. Brenner (2004) introduces weak primitives that allow for a relaxation of constraints between geometric primitives. Brenner (2005) also uses multivariate polynomials to recognize redundancy of constraints. In contrast to our approach, Brenner's approach is based on the Groebner Bases (Buchberger, 1988), which will be shortly described later. Brenner's approach is restricted to the 2D case and may not be feasible for interactive systems due to efficiency problems.

Constraint graphs for geometric objects represent the geometric and topological relations between different primitives, such as parallelity between planes. Kolbe (2000) deals with these spatial relations between primitives. He describes roofs by geometric constraints and compares them to observations from aerial images to reconstruct buildings.

Schmittwilken (2007) proposes an ontology and grammar based approach for semantic building modelling. Ontologies defined by UML (Unified Model Language) diagrams and OCL (Object Constraint Language) diagrams are mapped to attribute grammars in order to express semantic constraints.

Various implementations of automatic theorem proving techniques have been developed which partly support graphical sketches but are in turn restricted to the 2D case (Gao, 2004). In order to perform the computational tasks which are part of the algorithms we make use of the software package *Epsilon* that was implemented by Wang (2004) for the mathematical tool Maple. Beside other functions, it also allows for polynomial eliminations or the proving of theorems.

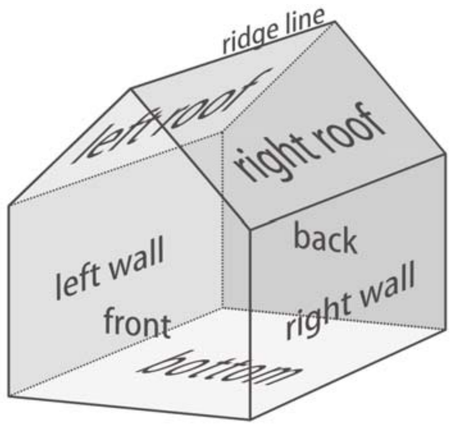

Figure 2. Components of a gable roof house

3. BACKGROUND

This paper describes a method for 3D geometric reasoning with multivariate polynomials. Therefore, geometric constraints have to be defined which are the basis of reasoning methods that use automatic theorem proving. In the following subsections we illustrate the geometric constraints that are necessary for a gable roof house and explain two methods of automatic theorem proving, namely the Groebner Base Method as well as Wu's method of characteristic sets.

3.1 Geometric Constraints

Before we are able to perform geometric reasoning, we have to define a house by its constraints. Figure 2 identifies the components that characterize a gable roof house.

Using a cuboid and a prism, the definition of a house includes basic constraints between planes and points, such as parallelity or orthogonality. The following constraints, which are still regarded independent from their actual representation, are common for man-made objects and describe a gable roof house:

- *2 x parallelity of walls:*
  left wall || right wall, front || back
- *13 x orthogonality of walls/roof:*
  bottom ⊥ right wall, bottom ⊥ left wall, bottom ⊥ front,
  bottom ⊥ back, right wall ⊥ front, right wall ⊥ back,
  left wall ⊥ front, left wall ⊥ back,
  back ⊥ right roof, back ⊥ left roof,
  front ⊥ right roof, front ⊥ left roof

There are further constraints necessary to restrict the position of the roof:

- *1 x incidence:*
  roof meets house block
- *1 x symmetry of roof:*
  symmetrical slope in roof (equal angles of roof areas)
- *1 x oppositional position*
  ridge line is at the top of the house

Altogether, the constraint set contains 17 geometric constraints. However, it is questionable whether all these constraints are needed or if a subset is sufficient to express the same conditions. In fact, in our example in figure 2 we only need 8 of 17 constraints to ensure that the planes forming the house have the intended position. The property, for example, that the right wall is perpendicular to the bottom has already been fulfilled by demanding that the right wall is parallel to the left wall, which is in turn perpendicular to the bottom. Hence, we have several statements of the form A *and* $B \Rightarrow C$. Therefore, we want to deduce the redundant constraints automatically by using multivariate polynomials which leads to symbolic approaches of automatic theorem proving. In the following, we give a short example to illustrate the connection between geometric constraints and multivariate polynomials.

In order to avoid the complexity of 3D reasoning we will present a first example in the 2D space. Figure 3 shows a constellation of three lines where the following theorem should hold true:

If line l_1 is perpendicular to line l_2 and line l_2 is perpendicular to line l_3, then l_1 is parallel to l_3.

To keep track of the constraints we use the following abbreviation: $l_1 \perp l_2$ *and* $l_2 \perp l_3 \Rightarrow l_1 \parallel l_3$. The same theorem can now be expressed with multivariate polynomials. A line l_i can be represented by its normal form $a_i \cdot x + b_i \cdot y + c_i = 0$. Thus, orthogonality as well as parallelity in 2D have their polynomial counterpart as follows:

$$\begin{aligned} & l_i \perp l_j: \; a_i a_j + b_i b_j = 0 \\ & l_i \parallel l_j: \; a_i b_j - b_i a_j = 0 \end{aligned} \tag{1}$$

Accordingly, our theorem which consists of two hypotheses h_1 and h_2 and one conclusion c can now be stated with polynomials:

$$\begin{aligned} & h_1 : l_1 \perp l_2 \wedge h_2 : l_2 \perp l_3 \;\; \Rightarrow \;\; c : l_1 \parallel l_3 \\ & a_1a_2 + b_1b_2 = 0 \wedge a_2a_3 + b_2b_3 = 0 \;\; \Rightarrow \;\; a_1b_3 - b_1a_3 = 0 \end{aligned}$$

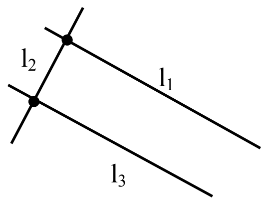

Figure 3. Deduction of parallelity in 2D

With this new representation it is possible to answer the questions of consistency and redundancy by algebraic methods. We consider a model of a complex geometric object that has to satisfy the constraints $h_1,\ldots, h_s$, that is, the model is given by the zeros of the set of polynomials representing the constraints. As a consequence, the key observation is that the satisfaction of geometric relations and therefore the possible positions of the geometric objects can be reduced to finding common zeros of the polynomials. This leads to the theory of a variety which is a set of n-tuples serving as roots of a conjunctive set of different polynomials:

$$V(\{h_1,\ldots h_s\}) := \{(a_1,\ldots a_n) \in k^n : h_i(a_1,\ldots a_n) = 0 \;\; \forall 1 \le i \le s\} \tag{2}$$

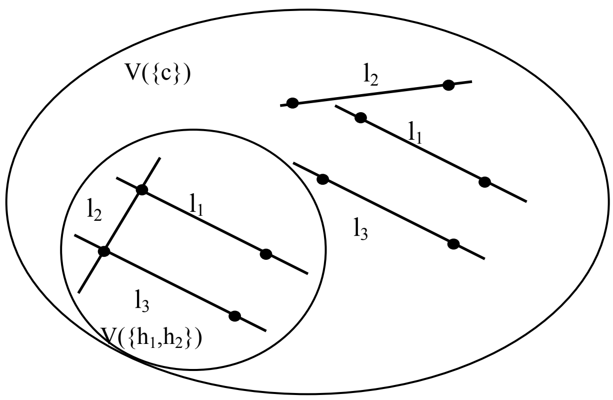

Figure 4. Relation between varieties

Redundancy can simply be identified by a relation between two varieties. Assuming we have a constraint set $\{h_1,\ldots,h_s\}$ and a possibly redundant constraint c, the aim is to show that the zeros of $\{h_1,\ldots,h_s\}$ are a subset of the zeros of c: $V(\{h_1,\ldots, h_s\}) \subseteq V(c)$. Figure 4 illustrates this fact by the given 2D example of three lines. The set of zeros $V(\{h_1,h_2\})=V(\{a_1a_2+b_1b_2, a_2a_3+b_2b_3\})$, which is determined by the two orthogonality constraints, will not be restricted further if we add the constraint of parallelity because of its containment relation $(V(\{a_1a_2+b_1b_2, a_2a_3+b_2b_3\}) \subseteq V(\{a_1b_3-b_1a_3\}))$. The parallelity constraint does not add any new information.

In order to compute with these varieties the concept of ideals is needed (Cox, 2007). A subset *I* of a polynomial ring $k[x_1,\ldots,x_n]$ is called an *ideal* if

$$\begin{aligned} & (1)\; 0 \in I \\ & (2)\; g,h \in I \Rightarrow g + h \in I \\ & (3)\; h \in I \wedge p \in k[x_1,\ldots x_n] \Rightarrow p \cdot h \in I \end{aligned} \tag{3}$$

Indeed, a variety is not defined by its equations but by the ideal generated by these. The *ideal generated by polynomials $h_1,\ldots,h_s$* is defined as

$$\langle h_1,\ldots,h_s \rangle := \left\{ \sum_{i=1}^{s} p_i h_i : p_i \in k[x_1,\ldots x_n] \right\} \tag{4}$$

That is, all polynomial consequences of the h_i's are elements of this ideal and define together the same variety. Referring to the 2D example the ideal is defined by

$$\langle a_1a_2 + b_1b_2, a_2a_3 + b_2b_3 \rangle = \left\{ \begin{array}{l} p_1 \cdot (a_1a_2 + b_1b_2) + \\ p_2 \cdot (a_2a_3 + b_2b_3) \end{array} \right\}.$$

Thus, also the conclusion c mentioned above can be obtained by some multiplication and an addition of the two hypotheses h_1 and h_2.

$$\begin{aligned} a_1b_3 - b_1a_3 &= b_3a_2^{-1} \cdot (a_1a_2 + b_1b_2) - b_1a_2^{-1} \cdot (a_2a_3 + b_2b_3) \\ &\in \langle a_1a_2 + b_1b_2, a_2a_3 + b_2b_3 \rangle \end{aligned} \tag{5}$$

The constraint c is therefore part of the ideal and if added to the constraint set $\{h_1, h_2\}$ does not change the variety. If the conclusion has more than one polynomial, they can be handled separately.

Before the two methods can be described in the next sections some more definitions are needed. A *multivariate polynomial* is a polynomial in more than one variable. The unknowns in the multivariate polynomials can be divided into *independent* and *dependent* variables. That is, on the one hand, we have parameters that can be chosen arbitrarily, and on the other hand, we are interested in the indeterminates that are dependent from other values for constraint satisfaction. Given an ordering on the variables $x_1 < \ldots < x_c < \ldots < x_n$, the class c of a polynomial (class(h)=c) is the smallest index so that the polynomial h is element of the ring $k[x_1,\ldots,x_c]$. This index c also defines the *leading variable LV(h)* = x_c and consequently its *leading coefficient LC(h)*, the so-called *initial* of h. Every polynomial can be expressed with respect to its leading variable x_c where the a_i's are themselves polynomials not containing x_c:

$$h = a_m \cdot x_c^{\;m} + a_{m-1} \cdot x_c^{\;m-1} + \ldots + a_1 \cdot x_c^{\;1} + a_0 \in k(x_1,\ldots,x_c)$$

In this case LC(h) = a_m. The leading degree of a polynomial h is defined by the degree of h in x_c (deg(h, x_c) = m). There is an important relation concerning the occurrence of variables that exists between two polynomials: h is reduced with respect to f if the degree of h in its leading variable x_c is smaller than the degree of f in x_c (deg(h, x_c) < deg(f, x_c)), where the index c is the class of the polynomial f (c = class(f) > 0).

3.2 Groebner Bases

One way of looking at redundancy and consistency of constraints is the Groebner Base Method (Buchberger, 1988). The idea is to solve the ideal membership problem by using polynomial long division of multivariate polynomials. A remainder of zero indicates that the polynomial is in the ideal and thus redundant. Applying the algorithm to the basis set the result of the division algorithm is not unique but depends on the order of monomials and the divisibility of the leading terms.

To tackle this problem the polynomial set used during division has to hold a special structure. With regard to the solution of a constraint system, the variety is independent of the actual polynomials which the constraint set is composed of. Instead, it only depends on the ideal generated by these constraints. The property that the original constraint set can be replaced by a set of polynomials generating the same ideal leads to the following statement:

$$V(\{h_1,\ldots,h_s\}) = V(\{\langle h_1,\ldots,h_s\rangle\}) = V(\{\langle g_1,\ldots,g_t\rangle\}) = V(\{g_1,\ldots,g_t\})$$

Thus, in order to have the special property that the polynomial reduction is unique the canonical set used in Buchberger's approach is the Groebner Basis. A *Groebner Basis* is a subset $G = \{g_1, \ldots, g_t\}$ of an ideal I where $\langle LT(h_1),\ldots,LT(h_s)\rangle = \langle LT(I)\rangle$, with LT(I) denoting all the leading terms of polynomials that are part of the ideal I. A Groebner Basis can be computed by Buchberger's algorithm (Buchberger, 1988).

Returning to our problem of redundancy, in order to check whether a geometrical conclusion c is deducible from a set of hypotheses $h_1,\ldots, h_s$ the following steps are necessary:

1. *Definition:* Translate the geometric constraints into polynomial equations: a set of hypotheses $\{h_1,\ldots, h_s\}$ and a conclusion c

2. *Groebner Basis:* Construct a Groebner Basis *G* of the set of hypotheses $h_1,\ldots, h_s$

3. *Proof:* To show that $V(\{h_1,\ldots, h_s\}) \subseteq V(c)$, it is necessary to check c is in the ideal generated by $h_1,\ldots, h_s$. This is realized by dividing the conclusion by the polynomials of the Groebner Basis *G*. If the remainder $rem(c, G) = 0$ the theorem is true, that is, the constraint *c* is redundant.

Brenner (2005) observes that the computation of the Groebner Bases can take substantial time so that the method may not be feasible for interactive systems. Alternatively, it has been shown that Wu's method, which is presented in the next section, can be more efficient in geometric theorems and is also able to solve more complex problems (Cox, 2007). The feasibility still depends on how constraints are represented by polynomials, but together with a suitable representation Wu's method proves to be applicable for user interactivity.

3.3 Wu's Method

Wu's method was first stated in the late 70's. Similar to the Groebner Basis Method, a statement $H \Rightarrow C$ can be proven. In contrast, the method uses the so-called pseudodivision and the output answers the question whether the theorem is generically true, that is, true under some degenerate conditions, the so-called *subsidiary conditions*. In conventional proofs in Euclidean geometry it is often assumed that geometric objects are in a general position without specifying further details. Thus, in many textbooks subsidiary conditions needed for the validity of a theorem remain implicit. The advantage of Wu's method is that these implicit subsidiary conditions are generated and made explicit by the theorem prover. The theorem, for instance, that three points P1, P2 and P3 define a plane unique is false. Only by adding the condition that P1, P2 and P3 are not collinear the theorem will be true. In contrast to conventional proofs, the defective theorem in Wu's method will be true under the subsidiary condition that P1, P2 and P3 are not collinear.

The main idea is to show that the zeros of one set of polynomials which do not vanish on the degenerated cases are included in another set of zeros. To achieve this, the polynomials are divided by each other like in the one-variable case. Therefore, a special form of triangulated structure of the equation system is needed, a so-called *characteristic set.* For a given ordering of the variables $x_1 < \ldots < x_s$ each of its polynomials h_i can be expressed as a polynomial in its highest dependent variable y_i.

$$\begin{aligned} h_1(u_1,\ldots,u_d,x_1) &= a_m \cdot x_1^{\,m} + \ldots + a_1 \cdot x_1^{\,1} + a_0 = 0 \\ &\ddots \\ h_s(u_1,\ldots,u_d,x_1,\ldots,x_s) &= 0 \end{aligned}$$

Here u_i are independent variables, x_i are ordered indeterminates (i.e. dependent variables) and a_i are themselves polynomials which do not include the highest dependent variable (e.g. x_1 in h_1). In addition to an ordinary triangulated equation system, a *characteristic set* is a minimal ascending chain. An *ascending chain* requires that for all indices i < j class(h_i) < class(h_j) and furthermore h_j is reduced with respect to h_i ($\deg(h_j, x_{class(hi)}) < \deg(h_i, x_{class(hi)})$). The constraint set $\{h_1 = x_1^2,\ h_2 = x_1^4 + x_2\}$ with $x_1 < x_2$, for example, is a triangular set but not an ascending chain because the degree of x_1 is lower in h_1 than in h_2. In contrast, $\{h_1 = x_1^2,\ h_2 = x_1 + x_2\}$ has this special property. The characteristic set is computed by pseudodivision that will be described later. An algorithm for this computation can be found in Buchberger (1988).

Wu's method can be outlined in three steps:

1. *Definition:* Define the theorem $Hyp \Rightarrow Con$ in form of multivariate polynomial equations $h_i = 0$ where *Hyp* is the hypothesis and *Con* the conclusion. Optionally, add subsidiary conditions $d_i \neq 0$ to the hypothesis.

2. *Characteristic Set:* Transform the hypothesis into a triangulated equation system subject to the dependent variables of the geometric constraints. While the conclusion with $c(u_1,\ldots,u_d,x_1,\ldots,x_s) = 0$ remains unchanged, we obtain a new constraint set *Hyp'* with $x_1<\ldots<x_s$ that fulfills the properties of a characteristic set.

3. *Proof:* Prove $Hyp' \Rightarrow Con$, that is realized by showing $V(\{h_1,\ldots,h_s / d_1 \cdot \ldots \cdot d_t\}) \subseteq V(c)$. This proof is also done by pseudodivision. If the final pseudoremainder equals zero, the zeros of *Hyp'* are also zeros of *Con* except from degenerated cases $d_1,\ldots,d_t$, i.e. the theorem is *generically* proven true.

Pseudodivision

The crucial operation in Wu's method is the *pseudodivision*. In some sense Wu's method resembles the triangulation algorithm of linear systems. The main difference is the replacement of division of real numbers by pseudodivision of multivariate polynomials. Pseudodivision of two multivariate polynomials c and h is considered as a division between univariate polynomials, e.g. in the highest variable x of the divisor h. It differs from the polynomial long division in that it is allowed to multiply the dividend c with a factor $d(h)^k$, k > 0:

$$d(h)^k \cdot c = q \cdot h + r \tag{6}$$

d(h) equals the initial of h defined in section 3.1, whereas q denotes the quotient and r the remainder. For our purposes pseudodivision can be extended to more than one dividend:

$$d(h_1)^{k_1} \cdot \ldots \cdot d(h_s)^{k_s} \cdot c = q_1 \cdot h_1 + \ldots + q_s \cdot h_s + r \tag{7}$$

In the context of theorem proving, *c* is identified as conclusion and $[h_1, \ldots, h_s]$ denoted as hypothesis. In particular, we are interested in the remainder r that shows whether our theorem is true. Referring to equation (6) and (7) we can now define the computation of the pseudoremainder recursively:

$$prem(c,h,x) = r$$
$$prem(c, [h_1,\ldots,h_s]) := prem((\ldots prem(c, h_s, x_s),\ldots), h_1, x_1)$$

Therefore, against the background of equation (7), the theorem $[h_1,\ldots, h_s] \Rightarrow c$ is generically true if, first, we obtain a zero remainder, and second, the initials *$d(h_i)$'s* which correspond to the undegenerated conditions do not equal zero. The algorithm of Wu's method and the pseudodivision have been implemented in Maple by Wang (2004).

2D example

We demonstrate the procedure of Wu's method by the introductory example in section 3.1. Although Wu's method originally is a point coordinate based method we stick to the pointless representation that we will also use in 3D. Given three lines in 2D space, that is, l_i: $a_i \cdot x + b_i \cdot y + c_i = 0$, i = 1,2,3, we would like to prove that two orthogonalities imply a parallelity. Therefore, the set of hypotheses is:

$$h_1)\ l_1 \perp l_2 : a_1a_2 + b_1b_2 = 0$$
$$h_2)\ l_2 \perp l_3 : a_2a_3 + b_2b_3 = 0$$

whereas the conclusion to deduce is

$$c)\ l_1 \parallel l_3 : a_1b_3 - b_1a_3 = 0\ .$$

After defining our theorem in polynomial equations, the dependent variables, on whose choice and ordering Wu's method is based, have to be specified. We demonstrate the notion of dependent variables by building up a step-wise geometric construction of the objects of the theorem. In order to receive the constellation of figure 3, for instance, we start with the construction of line 1 so that its parameters *a_1, b_1* and *c_1* are completely independent. While building line 2 as a second step it can easily be seen that one variable has to be dependent in order to fulfil hypothesis h_1. We decide to choose a_2 as dependent variable. Finally, the position of line 3 has to be set. If we choose a_3 dependent, this will lead to the variable ordering $a_2 < a_3$, on which the construction of the characteristic set is based.

In order to compute the characteristic set, it is necessary to achieve that the constraint set satisfies the properties of reduction and ascending classes. Because a_2 is part of h_1 and h_2, h_1 has to be pseudodivided by h_2 with respect to the variable a_2 and h_2 is replaced with the remainder of this pseudodivision. As a result we obtain the following triangulated equation system:

$$H' = \{a_1a_2 + b_1b_2, prem(h_2, h_1, a_2)\} = \{a_1a_2 + b_1b_2, a_1b_2b_3 - b_1b_2a_3\}$$

Notice that depending on the way of constructing the lines, the characteristic set and thus the outcome of the proof can be influenced. Assuming that we construct line 2 first, we could choose line 1 and thus constraint h_1 subsequently. We decide to select a_1 as dependent variable while a_2, b_2, b_1 are parameters that can be chosen arbitrarily. Finally, we construct line 3 and declare a_3 as dependent. The computation of the characteristic set is thus based on the variable ordering $a_1 < a_3$. In this case, there is no need to build a characteristic set H', because the hypothesis H has already got the required form: H=H' is a minimal ascending chain with $\deg(h_1, a_1) > \deg(h_2, a_1)$:

$$H' = \{a_1a_2 + b_1b_2, a_2a_3 + b_2b_3\}$$

Notice that h_2 is reduced with respect to h_1 if $prem(h_2, h_1, a_1) = h_2$. In order to prove the theorem, once again recursive pseudodivision is performed.

$$\begin{aligned} R_1 &= prem(c, h_2, a_3) = a_2a_1b_3 + b_3b_1b_2 \\ R_0 &= prem(R_1, h_1, a_1) = 0 \end{aligned} \tag{8}$$

Since the pseudoremainder is zero, the theorem is true under the subsidiary condition that $a_2 \neq 0$. Having *$d(h_1)=a_2$* and *$d(h_2)= a_2$* as initials of the two polynomials, Wu's method outputs the degenerated condition *$a_2 <> 0$.*

If we look at this successive computation of pseudoremainders with $a_1 < a_3$ in detail, we see how the results are connected to pseudodivision and thus to the concept of polynomial consequences and ideals. With regard to (8) the first step is to pseudodivide the conclusion by the second hypothesis. Beside the pseudoremainder we obtain the quotient q_1 and the multiplier m_1 that also is the leading coefficient of h_2, i.e. its initial (cf. equ. (6)):

$$\begin{aligned} m_1(a_1b_3 - b_1a_3) &= q_1(a_2a_3 + b_2b_3) + \\ &\quad prem(a_1b_3 - b_1a_3, a_2a_3 + b_2b_3, a_3) \Leftrightarrow \\ a_2(a_1b_3 - b_1a_3) &= -b_1(a_2a_3 + b_2b_3) + (a_1a_2b_3 + b_1b_2b_3) \end{aligned}$$

In the second step the obtained remainder is pseudodivided by h_1. Once more the multiplier is a_2:

$$\begin{aligned} m_2(a_1a_2b_3 + b_1b_2b_3) &= q_2(a_1a_2 + b_1b_2) + \\ &\quad prem(a_1a_2b_3 + b_1b_2b_3, a_1a_2 + b_1b_2, a_1) \Leftrightarrow \\ a_2(a_1a_2b_3 + b_1b_2b_3) &= a_2b_3(a_1a_2 + b_1b_2) + 0 \end{aligned}$$

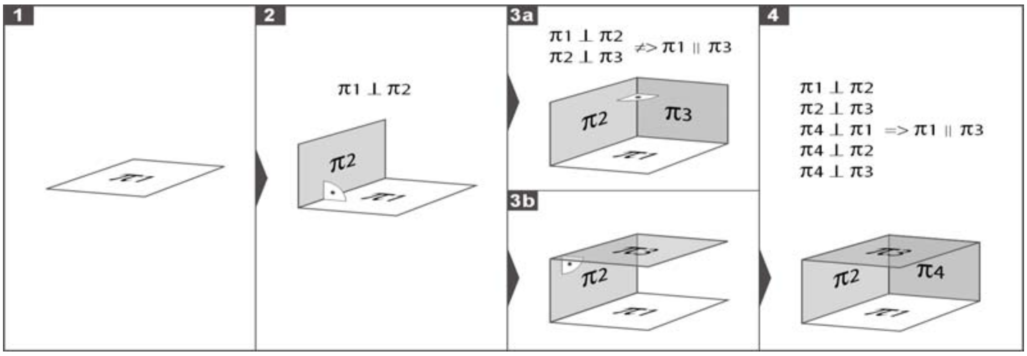

Figure 5. Increasing complexity: Deduction of parallelity in 3D

Finally, step one and two can be combined by a simple substitution of polynomials to express the conclusion in dependency on the hypotheses:

$$a_2(a_1b_3 - b_1a_3) = b_3(a_1a_2 + b_1b_2) - b_1(a_2a_3 + b_2b_3) + 0$$

Notice that with a pseudoremainder of zero this equation leads to the same equation as the ideal based description in (5), and the initial a_2 must not be zero. It can be seen that this second version of variable ordering (a_1<a_3) should be preferred because the initials are simpler and even the same.

Finally, the subsidiary condition can be analysed. They often allow for a geometric interpretation or at least have got an algebraic meaning. Supposed that a_2 equals zero, the dependent variables do no longer appear in the orthogonality constraints and consequently the independent variables can no longer be considered degrees of freedom of the geometric objects. In addition, a_2 appears as denominator in the solution process and thus, it does not have to be zero. Degenerated cases can therefore be excluded by considering subsidiary conditions in order to declare the theorem generically true.

4. CONSTRAINTS IN 3D

We now turn to the constraints in 3D space as needed in 3D city models. There are two main problems that arise: First, we observe that the theorem relating to parallelity and orthogonality becomes more complex. The increased complexity evokes a prolonged running time. Whereas in the 2D case the results were obtained after microseconds, the theorem prover failed in the first trials due to the complexity in 3D. The number of constraints that are necessary to deduce parallelity of planes from orthogonalities, for example, increases. In the following, we transfer the 2D example of section 3 in 3D space. Since three planes can be orthogonal in pairs without having two of them parallel (figure 5.3a), we need five instead of two constraints to ensure that parallelity exists (figure 5.4). Thus, we are able to deduce it from orthogonalities. As the number of variables is increased on the one hand and the construction of the characteristic set does not ensure simple initials of the polynomials on the other, the second problem that occurs is that the subsidiary conditions have become more complex. That is why it is important to choose an appropriate representation of geometrical constraints and an advantageous order of independent variables as input of the proof in order to obtain feasible results. These issues will be discussed in the following subsections.

4.1 Representation

Crucial for the efficiency of the procedure and the complexity of subsidiary conditions is the chosen polynomial representation for the constraints that describe a building. We will now discuss the method on the basis of the constellation of figure 2 and 6 respectively. We have to define seven planes – four walls, one bottom and two for the roof – and represent them by using the normal vector:

$$n_1 = (a_1, b_1, c_1)^T$$
$$\prod_1 = a_1x + b_1y + c_1z + d_1 = 0$$

Their positions are restricted by the constraints given in subsection 3.1, which are translated to the following polynomial equations:

- *parallelity*: $n_1 \times n_2 = \mathbf{0}$ (3 equations)
- *orthogonality:* $n_1 \cdot n_2 = 0$ (1 equation)
- *incidence point/plane*: $a_1 \cdot p_1 + b_1 \cdot p_2 + c_1 \cdot p_3 + d_1 = 0$

Although other representations are possible, the advantage of this representation (including the cross product and the scalar product) is that it does not contain any quadratic equations so far, but is bilinear instead. Furthermore, we were careful in choosing unnecessary variables. In order to express the parallelity, for instance, we have chosen the cross product instead of a linear combination of the normal vectors. Additionally, it is a decisive advantage to avoid point coordinates where possible by using e.g. the pointless normal forms of planes. If those steps are taken the complexity is reduced considerably.

Because we have to consider a large set of constraints, it is of great benefit to find a simplification of the polynomials. Obviously, our theorems are invariant to translation and rotation. Therefore, we make the plane representing the bottom face without loss of generality parallel to the x-y-plane by setting its normal vector to *(0,0,1)*. We have recognized that this does not only lead to a reduction of running time but also to interpretable subsidiary conditions. This is mainly due to the substantial reduction of the number of terms that occur in the constraints defined.

Two other issues are worth to be mentioned for the correctness of the house model and thus for the input of the proof. So far we have considered orthogonality and parallelity that have been sufficient to define the house block. However, the constraints of the roof have to be expressed in other terms. Firstly, we have to ensure that the ridge is at the top of the house and not turned downwards. In contrast to the house block, this relation is between a line and a plane and inequations cannot be avoided. Besides, both roof halves should have the same slope which is generally not explicitly available.

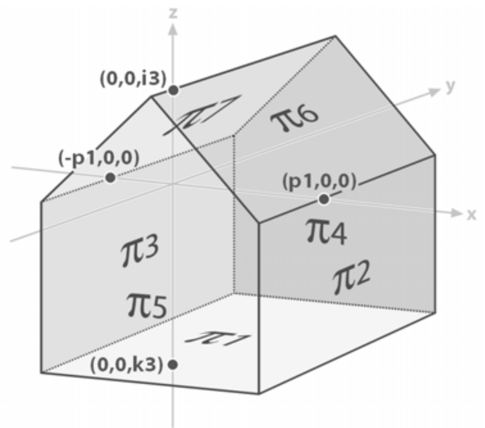

Figure 6. (Graphical) representation of a house by points and planes

As a consequence, we choose the position of the origin of the coordinate system in such a manner that it supports the satisfaction of these two constraints. This means, that the origin lies in the top plane of the house block and contains the axis of symmetry for the roof. The resulting advantage is, that it is possible to express these two constraints using four points with the special property that two of their coordinates equal zero (see figure 6).

The property of symmetry corresponds to the equality of distances from the ridge line – projected on the top of the house block – to each of the edges of the roofs. Therefore, referring to figure 6, symmetry can be translated algebraically by choosing three points *(p1,0,0), (-p1,0,0)* and *(0,0,i3)* which in turn restricts the position of the roof halves.

In order to avoid the roof being oriented downwards, we require that point *(0,0,i3)* lies above the top of the house block and point *(0,0,k3)* underneath (see figure 6). Wu's method does not allow strict inequations. Nevertheless, there are equivalent expressions which only use equations by introducing another variable (Kapur, 1988):

$$x < 0 \Leftrightarrow x \cdot w^2 + 1 = 0 \qquad (9)$$
$$x > 0 \Leftrightarrow x \cdot w^2 - 1 = 0$$

As a result, the constraints can as well be easily expressed in the calculus of Wu's method without increasing the complexity significantly.

4.2 Results

Referring back to section 3.1, we have 17 geometric constraints that are represented by 27 equations. Our aim is to reduce this constraint set automatically in order to filter out the minimal set of constraints. Therefore, we are able to solve the problem of redundancy and inconsistency simultaneously. Consequently, it comes into question which constraints are unnecessary, and which dependent variables should be selected in order to reduce subsidiary conditions.

In subsection 4.1, we have already translated our geometric constraints into polynomial equations. Before we are able to check deduction possibilities using Wu's Method, our constraint set C has to be divided into a non-redundant and a redundant constraint set $C_1 \cup C_2$, that leads to a theorem of hypothesis and conclusion: $C_1 \Rightarrow C_2$. Furthermore, we have to select dependent variables with respect to our construction process. We assume that 3 orthogonality constraints, 2 parallelity constraints and 3 constraints expressing the roof position are satisfied by a geometric configuration. These requirements are sufficient to deduce the remaining 7 constraints and thus define a gable roof house. On the whole, it can be proven by Wu's method that 20 equations are necessary whereas the remaining 7 equations can be deduced from these. Indeed, only 11 equations (7 constraints) are required to ensure that the redundant constraints are satisfied automatically. Other divisions into hypothesis and conclusion are possible. If the constraint set were inconsistent the function used would give it out as inconsistent instead of proving the theorem.

As a consequence of an appropriate representation and its normalization, it takes a few milliseconds to get the following result on the Maple's console:

```
The theorem is true under the following
subsidiary conditions:
a3 <> 0, a2 <> 0, b5 <> 0, w1 <> 0, w2 <> 0
QED.
```

As stated before, the list of subsidiary conditions is not unique. The choice of parameters depends on the choice and order of dependent variables which in turn depend on the step-wise construction of a complex object. In order to reduce the number and complexity of subsidiary conditions, their order and choice is very important. The polynomials in independent variables that occur in the denominators of the coefficients as well as the initials in the construction of the characteristic set do not have to equal zero and should therefore be simplified or avoided.

5. CONCLUSION

In this paper, we have shown that symbolic geometric reasoning which uses Wu's method in combination with an appropriate representation can be applied to minimize constraints in building models successfully and thereby proven its feasibility for interactive systems.

Semantic models are defined by constraints. We have shown that constraints of 3D buildings can be represented by multivariate polynomials, and that redundancy of constraints can be recognized by methods of automatic reasoning.

In contrast to Brenner and Sester (2005), who restrict to geometric problems in 2D, we address building models and reasoning in 3D. Whereas their approach is based on Groebner bases, we have used characteristic sets to identify redundancy and inconsistency. However, in the beginning Wu's method has not been sufficient to cope with the complexity that is increased in the 3D space. The key aspect was the choice of an appropriate representation. We reduced the number of variables wherever possible and used polynomials that are rather multilinear than quadratic. We made use of invariance with respect to rotation and translation and assumed that the bottom face is parallel to the horizontal plane.

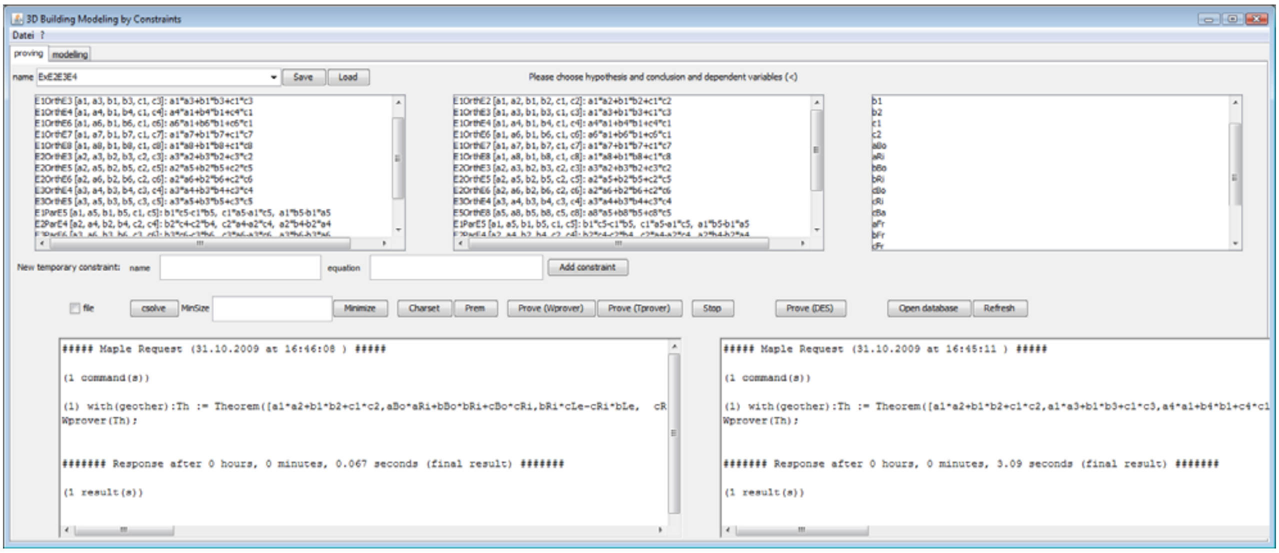

Figure 7. Prototype of our constraint proving module which supports the management and analysis of geometric constraints

The main contribution of this paper is to show that geometric reasoning in the 3D space is feasible for building models. We have implemented a prototype of a constraint proving module that supports users in managing the constraints and analyzing them with respect to redundancy and consistency (cf. figure 7). Our work is part of a larger project dealing with the interactive modeling of 3D building models.

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