CONSTRUCTION OF IMAGE - DEFORMATION TENSOR FOR SPACE - PLAVE PROJECTIVITY AND UNIQUE SOLUTION TO PROBLEM OF RELATIVE ORIENTATION

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1. Introduction

If the causes of gross errors, that is the crowded residuals at a point after adjustment, not necessarily due to poor observations, has been ever known in aerial triangulation, the course of development of analytical photogrammetry might be different from what we find today. An editorial discussion on the gross errors, by the late Prof B Hallert (1969) with Prof A J van der Weele (1969), had inspired the Author to begin this theoretical research. As the aerial triangulation alone can not be blamed for this, the paper will consider modifications according to

(i) the geometrical model of the physical transformation induced by photography; and

(ii) the mathematical model of the optical deformation induced by stereo-model in the empirical orientation method.

Consequently new condition for relative orientation is derived not from the usual geometrical condition but from the material condition of model.

Study of relative orientation problem is a study of parallax equations, particularly y-parallax equation. Stability of the relative orientation is governed by uniqueness of y-parallax equation. With these facts in mind, the paper discusses and compares the well-known classical principle of analytical relative orientation propounded by Prof S. Finsterwalder in 1899, with the new principle based on the new mathematical and geometrical models.

Construction of the image-deformation tensor, the heart of the new principle of relative orientation, is based on the first principle inorder to portray the physical nature of photogrammetry. Alternative construction which is not intended to discuss in the paper, could be derived from a certain deformation tensor; however, with no easily visible forms.

2. Basic Theory: Deformation of Solid Analytical Model

2.1 The Mathematical Model

Assume that a mathematical model of a stereo-pair is a solid continuum,[*1] deformable like an elastic body and has identical elastic properties at all points (homogenity) and in all directions (isotrophy). Suppose that under action of orientation elements (B_x, B_y, B_z, ω, ϕ and κ) the model would be deformed and displaced similar to motion of a deformable body in space (Hunter, 1976): a body (model) moves in space with respect to a reference coordinate system (a rigid motion) and is subject to distortion (model deformation) which makes the body occupy different regions of space at different times.

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*1 It is expressed as material solid with a definite shape, and displays the same properties for volumes of all sizes, whether large or small.

If a general cartesian tensor of the second rank T_{ij} represents transformation from one region B_o to another B (Fig. 1a), then this tensor decomposes into symmetric and antisymmetric tensors respectively. Physically, symmetric tensor characterising strain[#2] tensor executes a deformation of model (relative orientation), whereas anti-symmetric tensor characterising a spin tensor associates with rotation of rigid body (part of absolute orientation).

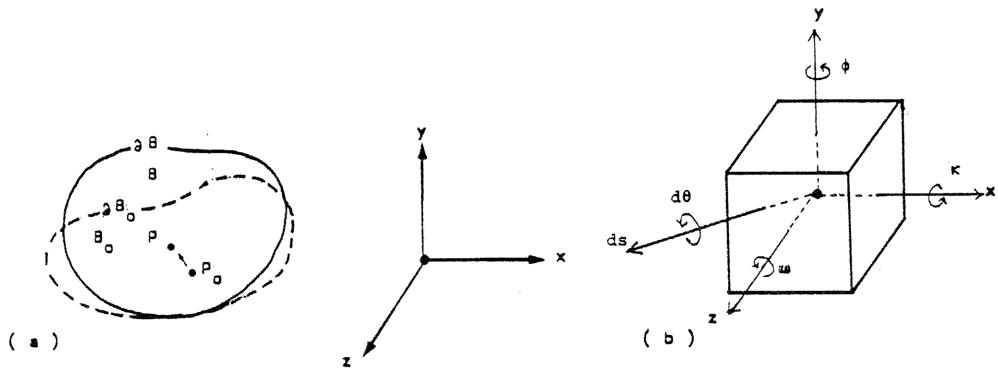

FIG. 1 DEFORMATION OF AN ELASTIC BODY AND POINT P .

Let dx, dy, dz are small linear changes of the same point from Po to P, along coordinate axes. However, $\hat{\omega}$, $\hat{\phi}$, $\hat{\kappa}$ are displacement vectors on yz-, zx- and xy- planes, due to small rotations[#3] ω, ϕ, κ ; hence $d\omega_y$, $d\omega_z$; $d\phi_x$, $d\phi_z$; and $d\kappa_x$, $d\kappa_y$; are their components into coordinate axes. Then the linear changes and the displacement components become normal and lateral strains if they are the results based on a unit space-vector acting at P which assumes a shape of parallelpiped of infinitesimal sides (Fig. 1b). Location of the nine quantities of displacement with respect to the three planes is shown in Fig. 2a. Mathematically, the first order form of a Carterian tensor of the second rank

$$T_{ij} = \left\| \begin{matrix} dx & d\kappa_x & d\phi_x \\ d\kappa_y & dy & d\omega_y \\ d\phi_z & d\omega_z & dz \end{matrix} \right\| \qquad (1)$$

depicts the displacement of P due to the six orientation elements. Consequently T_{ij} can be decomposed into a symmetric D_{ij} and anti-symmetric R_{ij} tensors. Further discussion on rotation tensor R_{ij} will not occur in the paper as it is related to absolute orientation. The symmetric tensor D_{ij}, which we will refer to as deformation tensor, with its six independent component can be expressed as:

$$D_{ij} = \left\| \begin{matrix} dx & d\bar{\kappa} & d\bar{\phi} \\ d\bar{\kappa} & dy & d\bar{\omega} \\ d\bar{\phi} & d\bar{\omega} & dz \end{matrix} \right\| \quad \text{where} \quad \begin{matrix} 2d\bar{\omega} = d\omega_y + d\omega_z \\ 2d\bar{\phi} = d\phi_z + d\phi_x \\ 2d\bar{\kappa} = d\kappa_x + d\kappa_y \end{matrix} \qquad (2)$$

Graphically the deformation at the point P is shown in Fig. 2b; inorder to achieve the static equilibrium on three faces of the parallelpiped not shown opposite to the three frontal faces, identical components are acting.

#2 For this paper, strain = lim -> 0 (change in length/original length).
#3 For small rotations, a curved path is assumed as a straight line.

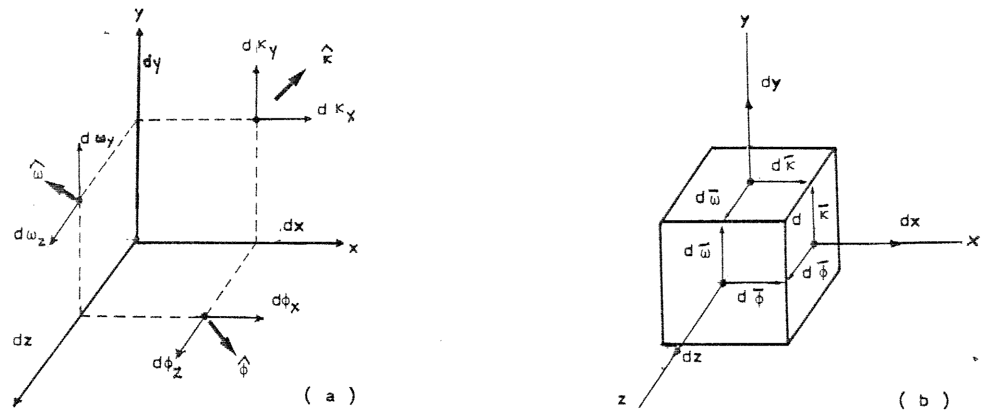

FIG. 2 THE MATHEMATICAL MODEL OF DEFORMATION AT POINT P .

The strain acted upon an elatic body produces stress which is a term used, in the theory of elasticity, to express transmission of force inside the body. With this analogy of elastic theory, the deformation at P due to the incorrect setting of orientational elements during a model reconstruction (relative orientation) produces parallax at this point on stereo-model. The general form of Hooke's Law is expressed by the statement that each of the six components of stress at any point of a body is a linear functions of the six components of strain at the point. By this law, we can loosely described a symmetric stress or parallax tensor, P_{ij} with its components P(m), where m = dx, dy, ... $d\bar{\kappa}$, which is a function of displacement of orientation element.

Customarily, parallax is measured in two dimensions on a projection plane and for a fixed projection distance f, P(dz) = 0. However, the statistic equilibrium of the components of parallax tensor should require that

$$P(dx) + P(dy) + P(dz) = 0 \qquad (3)$$

Without loss of generality, we can show that the six components of P_{ij} could be expressed as function of five displacements of relative orientational elements:

$$P_{ij} = \begin{Vmatrix} P(dx) & P(d\bar{\kappa}) & P(d\bar{\phi}) \\ P(d\bar{\kappa}) & P(dy) & P(d\bar{\omega}) \\ P(d\bar{\omega}) & P(d\bar{\phi}) & -P(dx)-P(dy) \end{Vmatrix} \qquad (4)$$

2.2 The Geometrical Model

The late Prof E.H. Thompson, had shown mathematically (Thompson, 1971) that the central projection of space to plane, the geometrical model of photography is neither unique nor strictly projective transformation. In fact, the central projection is an important particular case of projectivity of a plane into another plane. In general, projectivity can be expressed as a linear mapping M' = L (M) with non-vanishing determinant, transforming points of one plane into points of another plane having one-to-one correspondence; the projectivities of space into space may be similarly expressed. However, this projective relation cannot be achieved uniquely between the object-space (model) and image-plane (photograph). Unique, correspondence between space and plane requires that analytical relative orientation-problem should be solved in metric space of two-dimensional Euclidean smooth manifolds (Effimov, 1980). Unlike the traditional manner of projectivity of the stereo-overlap as a whole, it establishes that to every point of the object-space there exist a neighbourhood[#3] which can be mapped into the image-plane by means of one-to-one correspondence and continuous in both direction.

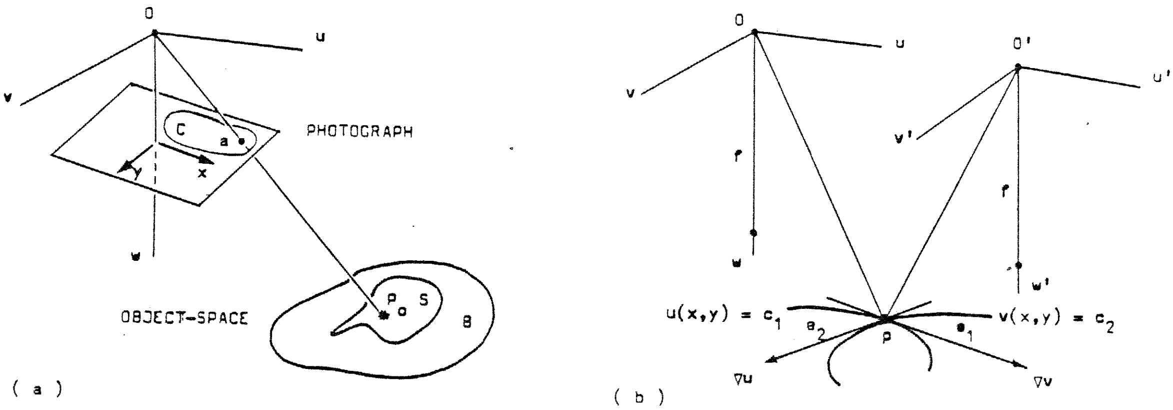

FIG. 3 THE GEOMETRICAL MODELS OF PHOTOGRAPH AND STEREO-MODEL ON PROJECTION PLANE

Consider a right handed three-dimensional coordinate system with origin O and the axes Ou, Ov and Ow (Fig. 3a). A xy-plane having axes x, y and parallel to uv-plane represents, a photograph. In the object-space set B defines a surface and for any of its points P_o, there is a subset $S(P_o)$ which we referred to as neighbourhood. There exists one-to-one correspondence between the points of region C(a) of the photographic plane and points of neighbourhood of P_o by the following equations:

$$\begin{aligned} u &= u\,(x,\ y) \\ v &= v\,(x,\ y) \\ w &= w\,(x,\ y) = f \end{aligned} \qquad (5)$$

where u(x,y), v(x,y) and w(v,y) are functions in the region of the photographic plane. These functions will be referred to as parallax functions. On a projection plane with a fixed projection distance f, the focal length of a camera, the function w = w(x,y) = f.

Thus, the eqs. (5) enables us to obtain model coordinates interms of image coordinates x, y of a point, if functions u and v are single-valued and continuously differentiable, and possess a nonvanishing Jacobian.

2.3 Image Deformation Tensor

At a point P (Fig. 3b) on the projection plane the coordinates curves $u(x,y) = c_1$ and $v(x,y) = c_2$, where c_1 and c_2 are constants, intersects, orthogonally to each other and they are tangential respectively to the unit vectors e_1 and e_2 in u- and v- directions. With a concept of gradient of function whose direction is normal to the curve defined by the function, the space Cartesian coordinates of the point P are ∇v in u-direction, ∇u in -v direction and f in w-direction respectively. In the right handed coordinate systems Ouvw and O'u'v'w' represent the left and right photographs.

#3 A neighbourhood of a point is a set which contains all points sufficiently near to it. A set is a collection of identifiable objects, its elements. A set can be referred to as a space and elements as a point.

The coordinate axes u, u' are roughly in the flight direction and that Ow and O'w' axes are parallel and normal to the projection plane. The problem of relative orientation on the projection plane is to find the linear relations between the two bundles of vectors from the left and right perspectives centres O and O'. At the point P, it is achieved by a dyadic product[#4], or natural product of two vectors of the form:

$$i.\nabla u \quad + \quad j.\nabla u \quad + \quad k.f \qquad (7).$$

where $\nabla = \cdot_x + \cdot_y$ and also known as the gradient. The result gives a P_{ij} symmetric tensor of the second rank which executes affine orthogonal transformation of image points into a surface of the model. By differential vector calculus, some components of the tensor P_{ij} can be expressed as:

$$\text{(a)} \quad \Delta u = \nabla u \cdot \nabla u \quad \text{and} \quad \text{(b)} \quad \Delta v = \nabla v \cdot \nabla v \qquad (8)$$

where $\Delta = \cdot_{xx} + \cdot_{yy}$ and also known as Laplace's operator. Then P_{ij} is expressed in components form as:

$$P_{ij} = \begin{Vmatrix} \Delta v & \nabla v.\nabla u & f.\nabla v \\ \nabla v.\nabla u & \Delta u & f.\nabla u \\ f.\nabla v & f.\nabla u & f.f \end{Vmatrix} \qquad (9)$$

where, with the same argument given earlier, $f.f = -\Delta v - \Delta u$. As mentioned before, u and v are continuously differentiable functions satisfying the following Cauchy-Riemann conditions:

$$\text{(i)} \quad u_x = v_y \qquad \text{(ii)} \quad u_y = -v_x \qquad (10)$$

Using these conditions (eq. 10), obviously gives us that

$$\nabla v.\nabla u = v_{xy} + u_{yx} \qquad (11)$$

and lastly it can be shown that

$$-\Delta v - \Delta u = u_{yy} - v_{xx} - \nabla (u_x + v_y) \qquad (12)$$

However the last term inside the bracket which is a physical meaning of change in projected area, vanishes since the projection distance $w = f =$ const. Thus the final expression for P_{ij} with its components expressed in partial derivatives of the unknown parallax-functions u,v is:

$$P_{ij} = \begin{Vmatrix} v_{xx} + v_{yy} & v_{xy} + u_{yx} & f.v_x + f.v_y \\ v_{xy} + u_{yx} & u_{xx} + u_{yy} & f.u_x + f.u_y \\ f.v_x + f.v_y & f.u_x + f.u_y & -v_{xx} - u_{yy} \end{Vmatrix} \qquad (13)$$

#4 Components of the product is a scalar coefficient times a pair of unit vectors (direction cosines) which is called a unit dyad.

Further the symmetric tensor P_{ij} can be decomposed into two symmetric tensors[#5]. With its components in all possible ∂x derivatives of functions u, v represents y-parallax tensor, whereas components of x-parallax tensor comprise of all possible ∂y derivatives. The tensor Eq. 13 characterise the general image-deformation at a point. However, for a plotting machine, when Δv, Δu axes are parallel to u-, v- axes respectively, components of the tensor corresponding to components dx, dy of D_{ij} will be:

$$\text{(a)} \quad v_{xx} = 0, \quad v_{yy} = 1; \qquad \text{(b)} \quad u_{xx} = 1, \quad u_{yy} = 0 \tag{14}$$

Thus, y-parallax tensor

$$\overset{(y)}{P_{ij}} = \begin{Vmatrix} 0 & v_{xy} & f.v_x \\ v_{xy} & 1 & f.u_x \\ f.v_x & f.u_x & -v_{xx} \end{Vmatrix} \tag{15}$$

and x- parallax tensor

$$\overset{(x)}{P_{ij}} = \begin{Vmatrix} 1 & u_{yx} & f.v_y \\ u_{yx} & 0 & f.u_y \\ f.v_y & f.u_y & -u_{yy} \end{Vmatrix} \tag{16}$$

are valid for our special theory of orientation and always symmetric, independent of the coordinate system. Also they are invariant under rotations, because of the tensor character of its components and by virtue of its physical meaning of being a deformation tensor.

3. Condition for relative orientation: Strain Energy Principle

Since energy developed in an elastic body due to deformation cannot be dissipated inside the body, work done with the result of this deformation is stored inside the body as strain energy. The energy of deformation of the body, for our problem, is equal to half the work done by external forces acting through the displacement due to orientation elements. The work done, from the unstressed state of taking photograph to the state of model deformation of orientation phase, is always positive. Consequently, it is a sufficient condition that the strain-energy-function exists and hence, it is unique (Gurtin, 1972).

Consider the case where on the analytical model there are no body forces and only surface displacements u, v at a point are given. The work done is expressed as the volume integral of the strain-energy-function (W) taken through the volume of the body (B); thus strain energy (U) is expressed as (Love, 1944).

$$U = 1/2 \int_B W \, dV \tag{17}$$

where dV is a small change in the volume of the body.

Moreover, the strain-energy-function is the homogeneous quadratic function of the strain components. Mathematically it can be shown that inner-product of strain and stress tensors would provide the said requirement for

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#5 The sum of two tensors of same order is a tensor of that order; $C_{ij} = A_{ij} + B_{ij}$

strain-energy-function. Thus eq. (17) in a more specific manner is expressed as

$$U = 1/2 \int_B E_{ij} \cdot S_{ij} \, dV \qquad (18)$$

where E_{ij} and S_{ij} are strain and stress tensors.

Since our interest is a deformation at a point, the integral sign can be disregarded for later equations. Moreover, the deformed analytical model, in the usual manner, produces y-parllax and x-parallax as a form of strain energy. By virtue of eq. (18) we can derive the important results of y-components of strain energy U(y) (or y-parallax) and x-component of strain energy U(x) (or x-parallax) as

$$\text{(a)} \quad U(y) = 1/2 \; D_{ij} \cdot P_{ij}^{(y)}$$
$$\text{(b)} \quad U(x) = 1/2 \; D_{ij} \cdot P_{ij}^{(x)} \qquad (19)$$

Here, the right hand side of the eq. (19), the inner product of the second rank tensors is a second order tensor. After substituting the components of tensors (eqs. 2, 15, and 16) in Eq. 19, the final form of parallax equations in partial differentials of displacement functions u, v are

$$\text{(a)} \quad U(y) = 1/2 \, dy - 1/2 \, v_{xx} \, dz + v_{xy} \cdot d\bar{\kappa} + f.v_x d\bar{\phi} + f.u_x.d\bar{\omega}; \text{ and}$$
$$\text{(b)} \quad U(x) = 1/2 \, dx - 1/2 \, u_{yy} \, dz + u_{yx} \cdot d\bar{\kappa} + f.v_y d\bar{\phi} + f.u_y .d\bar{\omega} \qquad (20)$$

For our present study, the aim of relative orientation is, to eliminate y-parallax, that is the strain energy U(y) must be a true minimum; this is expressed mathematically as

$$D_{ij} \cdot P_{ij}^{(y)} = 0 \qquad (21)$$

This condition for analytical relative orientation is only based on changes of its configuration of the body on the model surface. On the contrary, the classical relative orientation using collinearity or coplanarity conditions has to use perspective centre(s) which is not on the surface of the model.

Going back to y-parallax eq. (20,a), the major problem left to be solved is to find the unknown functions u,v. However, a unique solution cannot be promised by just assuming equations to represent these functions without providing extra conditions

4. Unique Solution: Boundary Value Problem

These conditions are generally classified as boundary conditions which may not necessarily be the geometrical shape of a neat model; rather it would be appropriate to level curves on its surface. The boundary conditions have a tremendous influence on the solution. The problem of finding unknown functions with continuous derivatives is known to possess a unique solution if Laplace equation and boundary condition are satisfied (Hunter, 1976). As an equivalent statement to our problem, we have

$$\text{(a)} \quad \Delta u = 0 \; ; \quad \Delta v = 0 \text{ on the domain B, and} \qquad (22)$$

(b) $u_n = g_1; \quad v_n = g_2$ on the boundary ∂B,

where $\cdot_n$ is a derivative into direction normal to boundary curve, and g_1, g_2 are certain assigned values for displacement at the boundary. This type of boundary value problem is usually known as boundary value problem of the second kind.

The partial differential equation of the second degree (eq. 20a) in canonical form will be hyperbolic paraboloid of the form $x^2/a^2 - y^2/b^2 = z$ (Fig. 4a). Assuming a = b, then a family of level curves on this surface would be a family of hyperbolas $xy = \pm c$ or $x^2 - y^2 = \pm c$ depending upon the orientation of coordinate axes. These two types of curve will be used to set up boundary conditions.

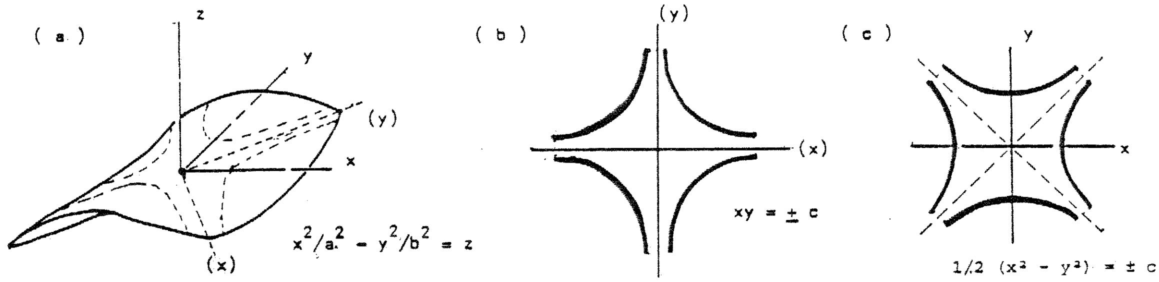

FIG. 4 HYPERBOLIC-PARABOLOID SURFACE OF THE MODEL AND TWO DIFFERENT BOUNDARY CURVES

4.1 (a) First Boundary Value

The unknown parallax functions u, v are solved from the first boundary value (Fig. 4b) giving the boundary conditions:

$$u_y = +xy \quad \text{and} \quad v_x = -xy \qquad (23)$$

With known values of all derivatives of u, v and eq (20) could be expressed in Cartesian coordinates of the photographs as

$$\text{(a)} \quad U(y) = 1/2\, dy + 1/2\, y\, dz - x\, d\bar{\kappa} - f\, xy\, d\bar{\phi} + 1/2\, f y^2\, d\bar{\omega}$$

$$\text{(b)} \quad U(x) = 1/2\, dx - 1/2\, x\, dz + y\, d\bar{\kappa} - 1/2\, f\, x^2\, d\bar{\phi} + f\, xy\, d\bar{\omega} \qquad (24)$$

The eqs. (24) are the well known differential equations which we will refer to as Classical Differential Parallax (CDP) equations.

4.1(b) Second Boundary Value

The unknown parallax functions u, v are solved from the second boundary value (Fig. 4c) giving the boundary conditions

$$u_x = -1/2\, x^2 + 1/2\, y^2 \quad \text{and} \quad v_y = +1/2\, x^2 - 1/2\, y^2 \qquad (25)$$

With known values of all derivatives of u, v the eq. (20) could be expressed in Cartesian coordinates of the photograph as

$$\text{(a)} \quad U(y) = 1/2\, dy - 1/2\, y\, dz + x d\bar{\kappa} + fxy\, d\bar{\phi} - 1/2\, f\, (x^2 - y^2)\, d\bar{\omega}$$
$$\text{(b)} \quad U(x) = 1/2\, dx - 1/2\, x\, dz + y d\bar{\kappa} + 1/2\, f(x^2 - y^2)\, d\bar{\phi} + fxy\, d\bar{\omega} \qquad (26)$$

These are the new parallax equations which we will refer to as Partial Differential Parallax (PDP) equations. The y^2 - term of $d\bar{\phi}$ in x- parallax and x^2-term of $d\bar{\omega}$ in y-parallax are neglected in the usual derivation of

CDP equations when the differential terms of second order are not considered.

4.2 Conditions on the Domain

Functions derived from the first boundary values does not satisfy the given Laplace equations (eqs. 22a) which can be derived from Eq. 10. In other words, the functions which provide the CDP equations are not continuously differentiable; this is quite contrary to the initial assumption. However, the functions derived from the second boundary values satisfy the Eqs. 22a. Therefore, PDP equations are derived from functions which will be referred to as a continuous function whereas the CDP equations are derived from the functions which will be referred to as non-continuous functions. Those functions that satisfy the Laplace equations are also harmonic functions.

4.3 Analysis

For better physical meaning, we will choose the parallax functions u and -v which gives similar Cauchy-Riemann conditions (eq. 10) with only the signs changed. Hence, the results can be arranged in the following forms:

$$\text{(a)}\quad u_x + v_y = 0 \quad \text{and} \quad \text{(b)}\quad u_y - v_x = 0 \qquad (27)$$

The two new relations (Eq. 27), by differential vector calculus, can be shown that (a) Div. F = 0 and (b) Curl. F = 0, where the displacement vector field in the projection plane by changes in the orientation elements is defined as F = i.u + j.v.

The continuous functions of PDP equations possess sourceless (eq. 27a) and irrotational (eq. 27b) vector fields. By sourceless we mean that the displacements of point are purely due to changes in orientation elements and by irrotational that a point undergoes pure deformation without rotation around a z-coordinate axis. However, non-continuous functions of CDP equations (Eq. 24) provide sources and sinks in its vector field, albeit irrotational. By sources or sinks we mean that points move radially from or towards the centre.

The question of stability of CDP equations cannot possibly be answered without taking into account the singular points in the model at which point displacements are being manufactured (sources) or destroyed (sinks). At the same time a unique projective relation and unique solution for analytical relative orientation demand that the displacement vector field in the projection plane is sourceless and irrotational. Moreover, the displacement vector F in the projection plane is smooth on domain B of the analytical model, since it is differentiable at any point on B; and the gradient of the displacement vector field is a tensor field. By the tensor field we mean functions (eq. 5) that assign to each point on B, a symmetric tensor (eq. 15) to characterise the displacements of point with changes in orientation elements.

4.4 Computation Method: Finite Element Analysis

Mathematical results achieved from the unique solution have strongly suggested a "breakthrough" in computing techniques contrary to the existing method of linearisation to approximate equation and iterative solution.

Moreover, as shown earlier, the relative orientation problem is simply to find the unknown displacement functions for the given physical condition of a problem. Bearing the fact in mind, the Finite Element Method (FEM) (Zienkiewicz, 1977) becomes most appropriate technique for finding the unknown functions. It is indeed a powerful method, suitable for this computer age, that provides mathematical as well as physical approximation to a problem. Although an unconventional approach to our orientation problem, it is a well-established discretisation method of the continuum problem.

5. Conclusion

The six quantities of image deformation at a point with movement of orientation elements and the resulting parallax in the model are the symmetric tensors of the second rank. The unique equaiton for the x- or y-parallax is a two-dimensional conformal equation. By and large relative orientation problem should be viewed as conformal mapping between xy- plane and uv- plane. The inner product of two tensors, mathematics of strain-energy principle provides the intrinsic condition for the analytical relative orientation at a point on the model without appealing to the embracing space such as the perspective centres. In metric space in Euclidean geometry, this condition is developed, not line (collination) or plane (coplanarity) as its fundamental element, but "point" which, of course, outweighs the others in importance.

The intrinsic condition of orientation may overcome some limitations imposed by the classical method in digital imaging technology utilizing electro optical recorder rather than film. On the other hand, the proposed finite element method can provide a variation in the degree of polynomials and changes in sizes and shape of elements in agreement with the type of terrain. Thus a photogrammetrist can monitor the accuracy requirements, at an early stage of data processing such as the relative orientation. Because of that, he does not need to rely on the final accuracy based on the result of the gross error detection followed by adjustment, or vice versa.

6. Acknowledgements

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