# AN INVESTIGATION OF THE SCOPE OF HIGH ORDER POLYNOMIAL DIGITAL ELEVATION MODELS

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ABSTRACT: Computer-assisted application of functional digital elevation models have had widespread use. Most of these applications are based on local low order mathematical functions. This paper presents some results of work done on relatively large cells, 1 x 1 km square, with high order double Chebyshev polynomial functions.

INTRODUCTION: A functional digital elevation model is taken here to mean a mathematical expression that expresses terrain elevation, z, as an "approximate" function of plan coordinates x and y.
That is,

$$z = f(x,y) \tag{1}$$

This is an extension of a classical definition of digital elevation model (DEM) (Doyle, 1978) to include both the numerical data representing the terrain and the means with which additional data may be interpolated.

Topographically speaking, terrain elevation is not necessarily a function of plan location. But given a set of terrain data $(x_i, y_i, z_i)$, $i = 1,2, \ldots, n$, a mathematical expression can be written to approximate $z_i$ as a function of x and y, abbreviated as (1). The expanded form of (1) will depend on the type of mathematical function employed.

Functional digital elevation models have been used in a variety of forms in terrain surface modelling and/or interpolation. The commonest forms are for example: linear (plane) surface function (Allam, 1978); bilinear surface function (Leberl, 1973; de Masson d'Autume, 1979) or hyperbolic paraboloid surface function (Grist, 1972); multiquadric surfaces (Hardy, 1971); bicubic polynomial surface function (Schut, 1968); and double Fourier series surface function (Maxwell and Turpin, 1968). Most of these forms have one thing in common, namely, they are relatively low order mathematical functions. They are therefore suitable for defining terrain surfaces of small patches (cells). This paper presents some results of work done on relatively large cells, 1 x 1 km square, with high order polynomial surface functions.

A power series polynomial function of z in x and y is a common form of functional digital elevation models. That is,

$$z = a_0+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2 + \ldots \tag{2}$$

Depending on the order of the polynomial, the number of the unknown parameters, $a_i$, is given by m
where

$$m = \frac{1}{2}(k+1)(k+2) \tag{3}$$

and k is the order of the polynomial. The values of $a_i$ in (2) can be determined by least squares surface fitting method using redundant terrain data, or by direct solution of n simultaneous equations. That is, from n observed terrain points, one can form n observation equations of the form (2) which in matrix form is

$$AR = S \tag{4}$$

where A is n x m matrix of surface coefficients,
R is n x 1 vector of surface parameters (unknowns),
S is n x 1 vector of observations.
Assuming a weight matrix of W, the normal equations can be formed as:

$$A^TWAR = A^TWS \tag{5}$$

from which,

$$R = (A^TWA)^{-1}A^TWS \tag{6}$$

But the solution of (6) is very unstable (Clenshaw, 1960) for high orders of polynomial and/or for large values of x and y. This is because it involves inverting a matrix, $A^TWA$, with large elements. For this reason, this method has only been used for low order polynomials. Such polynomials would invariably be suitable only for small patches of terrain.

In projects where relatively large size cells and hence high order polynomials are envisaged, an alternative to power series polynomials are the Chebyshev polynomials. The latter are particularly useful in projects where a fixed size cell is a unit of terrain definition area (Segu, 1984). Because of the large size of the terrain definition cells so adopted, the fitting surface polynomials tend to be of high orders except, of course, for flat terrain. This article will show that there is some scope for this type of surface functions.

DESCRIPTION OF THE PROJECT:

The object of this project is to investigate how the current forms of DEM's (Schut, 1976) can be modified to take into account specific requirements of, say, highway route location by digital computers.

The use of digital terrain models for highway design dates back to mid-1950's (Doyle, 1978). This article is still addressing itself to that same old problem, i.e. use of DEM in highway design, but with a difference.

Location of a highway route corridor through a planning region needs a rational and cost-effective approach. One such approach is to define the relief of the region by mathematical functions on patchwise (cell-wise) basis. The advantages of this methodology is that the cell can, to some extent, be of any size comparable to highway design standards. Furthermore, the digital elevation data can be acquired in any mode compatible with the mathematical function(s) to be adopted in the exercise, but independent of the location of the final alignment of the route. This latter aspect (flexibility in data acquisition pattern) is very advantageous to any route (highway, railway, power line etc.) alignment project because the final alignment is never known at data acquisition stage. Finally, these terrain definition cells can also be adopted as units of definition of a route corridor, hence as route location cells.

## ACQUISITION OF DEM:

Acquisition of DEM data requires a slightly different approach to that of graphic data. The DEM data output from photogrammetric methods, topographic maps, or ground survey methods are numerical. The modes for acquiring these data can generally be classified as: (1) random, (2) regular, (3) semi-regular. From the point of view of automation, the regular and semi-regular modes are well-suited for the purpose.

Regular Mode: This is the regular grid pattern of data acquisition. When this mode is employed in a photogrammetric technique, for example Leberl (1973), the operation can be fully or partially automated (Allam, 1978).

Semi-Regular Mode: This mode operates very much like the regular mode save for the data spacing along scan lines. Scan lines or profiles are set out at regular intervals whereas the data points along them need not be equally spaced. This provision adds extra flexibility to the method in that the operator can add extra points in areas where he deems it necessary. He can adopt a measuring technique which is not rigorous in data spacing. For example, recording contour heights along a profile, plan locations are not equidistant. If an operator chooses to record elevations at regular intervals along a profile or scan line, then the method reverts to "regular mode". So regular mode can be said to be a special case of semi-regular mode.

Random Mode: This pattern is said to be the most appropriate of the three as far as representing the terrain is concerned (Mark, 1979). The method can pick out the salient terrain points such as breaklines, crests, droughts etc. These data can then be used to derive grid data by interpolation if required for example in contour plotting etc. Despite this merit, the operation suffers from one serious drawback in speedy production, namely, it cannot be automated. The method is also rather subjective. Therefore the author sees it to have a restricted application.

Because of the flexibility of the semi-regular mode of DEM data acquisition, this project has adopted this approach although the original data used in this report were available to the author in regular pattern form. These data were acquired by photogrammetric technique on regular pattern at 50 m grid at ground scale. The test data cover a ground area of 2 km by 5 km.

## DATA PROCESSING:

The observed elevation data are modelled to be in a form compatible with the intended use. In this work, the use has been defined as a mathematical expression (1) from which additional elevation data can be obtained by interpolation. So part of data processing operation is to formulate and solve a selected mathematical model. But owing to the limitation in the use of power series polynomial form (2), Chebyshev series polynomial form is adopted for this project.

Chebyshev polynomials are based on trigonometric functions (Fox and Parker, 1968)

$$T_n(x_c) = \cos\ n\Phi \qquad (7)$$

in which

$$\cos \Phi = x_c, \ -1 \leq x_c \leq 1 \tag{8}$$

and n is a degree of cos $\Phi$. The arguments, $x_c$, of a Chebyshev polynomial $T_n(x_c)$ lie between -1 and 1 (a standard range). Therefore in data processing, the independent variables, x, are first transformed into the corresponding Chebyshev polynomial arguments. This transformation, sometimes referred to as normalization, takes the form:

$$x_c = \frac{(x-x_{max}) + (x-x_{min})}{x_{max} - x_{min}} \tag{9}$$

$x_{max}$ and $x_{min}$ are respectively the maximum and minimum values of x's, making $x_{min} \leq x \leq x_{max}$.

Equation (1) in Chebyshev series form becomes

$$z_x = \sum_{n=0}^{k} A_n T_n(x_c) \tag{10}$$

In the case of two variables, x and y say, an equivalent expression - a double Chebyshev series form is

$$z_{xy} = \sum_{n=0}^{k} \sum_{i=0}^{j} A_{n,i} T_n(x_c) T_i(y_c) \tag{11}$$

The expanded form of (10) can be obtained by recurrence relation (Fox and Parker, 1968); and its solution is carried out on curve-fitting principle.

The solution of a double Chebyshev polynomial (11) is also carried out on the principle of curve-fitting as follows: having defined the boundaries of a cell to be modelled, least-squares curves of a prescribed degree are fitted along each profile data parallel to say, x-axis. Then the coefficients of these curves are used as data to fit curves of a prescribed degree along a direction normal to x-axis. This way of solving Chebyshev polynomials presupposes that the terrain data are acquired along parallel lines (scan lines or profiles), e.g. in a semi-regular mode.

In this experiment, the test data area was first blocked into regular cells of 1 kilometre by 1 kilometre. The data were then fitted to a double Chebyshev polynomial of degree 1 in x and y. The root-mean-square error (RMSE) at the control points and at the check points (where applicable) were computed and each compared with the pre-defined accuracy of fit (i.e. RMSE = 3 m in this case). In the event of a computed RMSE being higher than 3 m, the degree of fit is incremented by unity and the fitting routine repeated. The degree at which RMSE $\leq$ 3 m is adopted as the degree of best-fit for the cell in question.

Five sets of tests were performed on 10 test cells (FIG. 1). The terrain data in each cell were originally provided on a 50 m regular grid. But in the course of these tests, different data densities were simulated by varying the configuration of the control data available for surface fitting as follows:

Set 1: all 21 x 21 observed data points (per cell) at 50 m regular grid were used to compute Chebyshev polynomial surfaces. There were no checks. The resulting RMSE and the corresponding degrees of fit are shown in Table 1.

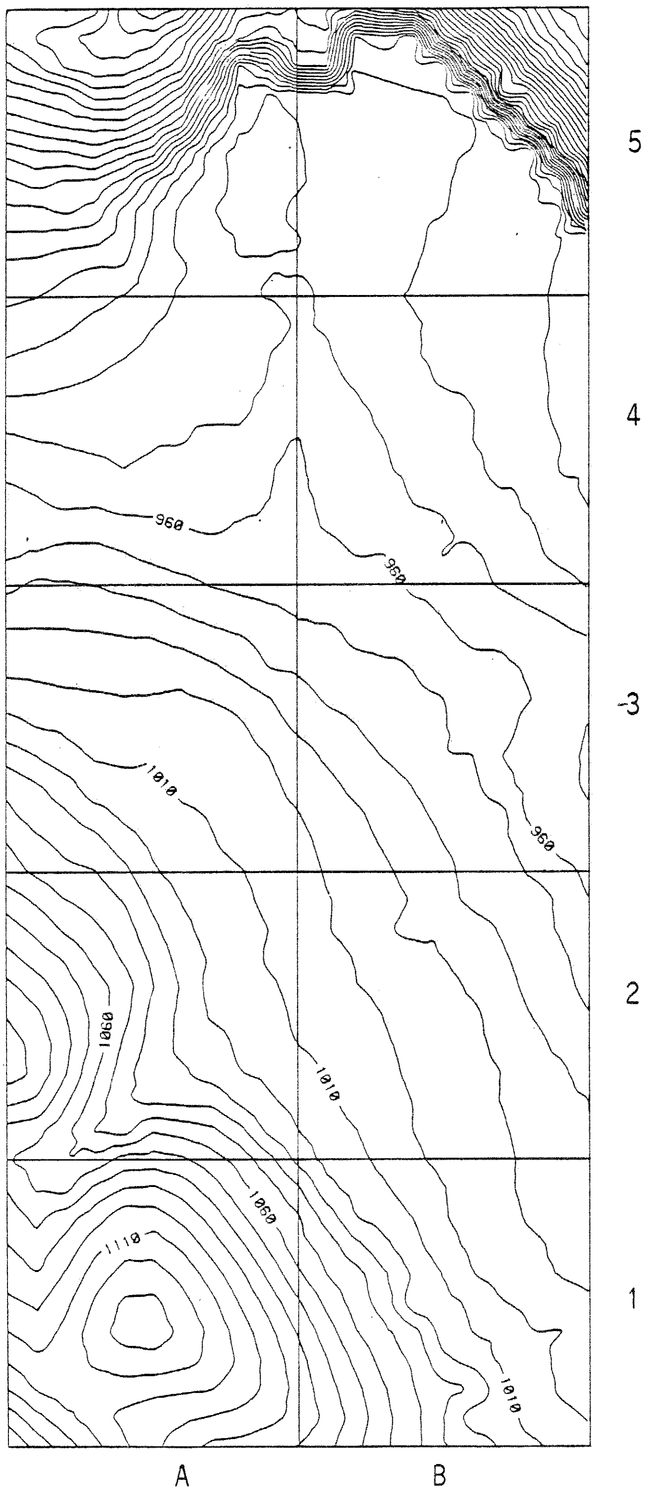


FIG. 1. 2 x 5 TEST CELLS

TABLE 1: ACCURACY OF FIT OF SET 1

| CELL No. | CONTROL POINTS | | RANGE IN ELEVATION (m) |
|---|---|---|---|
| | DEGREE | RMSE (m) | |
| A1 | 4 | 2.34 | 133.00 |
| B1 | 2 | 2.73 | 106.40 |
| A2 | 4 | 2.87 | 102.30 |
| B2 | 1 | 1.85 | 70.30 |
| A3 | 2 | 2.39 | 89.40 |
| B3 | 1 | 2.28 | 65.40 |
| A4 | 1 | 2.55 | 63.90 |
| B4 | 1 | 1.88 | 42.80 |
| A5 | 13 | 2.95 | 208.50 |
| B5 | 19 | 2.35 | 288.20 |

Set 2: the 11 x 11 terrain data points at 100 m regular grid form the control data for a surface polynomial in each cell. Another set of 10 x 10 data points also at 100 m regular grid are used as checks for the computed surface. The resulting accuracies of fit both at the control points and at the check points and the corresponding degree of fit are shown in Table 2.

TABLE 2: ACCURACY OF FIT OF SET 2

| CELL No. | CONTROL POINTS | | CHECK PTS. RMSE (m) | RANGE IN ELEVATION (m) |
|---|---|---|---|---|
| | DEGREE | RMSE (m) | | |
| A1 | 4 | 2.13 | 2.28 | 133.00 |
| B1 | 2 | 2.88 | 2.70 | 106.40 |
| A2 | 4 | 2.69 | 2.94 | 102.30 |
| B2 | 1 | 1.90 | 1.80 | 70.30 |
| A3 | 2 | 2.45 | 2.35 | 89.40 |
| B3 | 1 | 2.38 | 2.12 | 65.40 |
| A4 | 1 | 2.48 | 2.32 | 63.90 |
| B4 | 1 | 2.00 | 1.83 | 42.80 |
| A5 | 8 | 2.53 | 7.89 | 208.50 |
| B5 | 10* | - | 439.78 | 288.20 |

*Exact-fit degree

Set 3: the 100 check points in Set 2 are now adopted as control points while the 121 control points act as check points. The results are shown in Table 3.

TABLE 3: ACCURACY OF FIT OF SET 3

| CELL No. | CONTROL POINTS | | CHECK PTS. RMSE (m) | RANGE IN ELEVATION (m) |
|---|---|---|---|---|
| | DEGREE | RMSE (m) | | |
| A1 | 4 | 1.90 | 4.56 | 133.00 |
| B1 | 2 | 2.41 | 3.09 | 106.40 |
| A2 | 4 | 2.40 | 4.63 | 102.30 |
| B2 | 1 | 1.72 | 2.11 | 70.30 |
| A3 | 2 | 2.14 | 3.07 | 89.40 |
| B3 | 1 | 2.09 | 2.54 | 65.40 |
| A4 | 1 | 2.28 | 2.87 | 63.90 |
| B4 | 1 | 1.79 | 2.05 | 42.80 |
| A5 | 7 | 2.94 | 18.35 | 208.50 |
| B5 | 9* | - | 602.28 | 288.20 |

*Exact-fit degree

Set 4: the 121 control points used in Set 2 are once again used as control in this step. But now all the remaining 320 observed terrain data points are used as check points. The distribution of the control data is regular, but that of the check points is semi-regular. The corresponding results are given in Table 4.

TABLE 4: ACCURACY OF FIT OF SET 4

| CELL No. | CONTROL POINTS | | CHECK PTS. RMSE (m) | RANGE IN ELEVATION (m) |
|---|---|---|---|---|
| | DEGREE | RMSE (m) | | |
| A1 | 4 | 2.13 | 2.46 | 133.00 |
| B1 | 2 | 2.88 | 2.74 | 106.40 |
| A2 | 4 | 2.69 | 3.03 | 102.30 |
| B2 | 1 | 1.90 | 1.85 | 70.30 |
| A3 | 2 | 2.45 | 2.42 | 89.40 |
| B3 | 1 | 2.38 | 2.24 | 65.40 |
| A4 | 1 | 2.48 | 2.58 | 63.90 |
| B4 | 1 | 2.00 | 1.84 | 42.80 |
| A5 | 8 | 2.53 | 7.16 | 208.50 |
| B5 | 10* | - | 250.99 | 288.20 |

*Exact-fit degree

Set 5: the functions of the two sets of data in Set 4 are reversed. That is, the control data of Set 4 are used as check points and the check points become the control data. So the control points take a semi-regular pattern, whereas the check points take a regular pattern. The results of this set are shown in Table 5.

All the five steps were repeated for each of the ten cells.

TABLE 5: ACCURACY OF FIT OF SET 5

| CELL No. | CONTROL POINTS | | CHECK PTS. RMSE (m) | RANGE IN ELEVATION (m) |
|---|---|---|---|---|
| | DEGREE | RMSE (m) | | |
| A1 | 4 | 2.35 | 2.43 | 133.00 |
| B1 | 2 | 2.62 | 3.01 | 106.40 |
| A2 | 4 | 2.86 | 3.04 | 102.30 |
| B2 | 1 | 1.81 | 1.95 | 70.30 |
| A3 | 2 | 2.32 | 2.57 | 89.40 |
| B3 | 1 | 2.23 | 2.40 | 65.40 |
| A4 | 1 | 2.56 | 2.50 | 63.90 |
| B4 | 1 | 1.82 | 2.01 | 42.80 |
| A5 | -** | - | - | 208.50 |
| B5 | -** | - | - | 288.20 |

**Underdefined by the data available.

## RESULTS:

Accuracy of fit of a digital terrain model is often used to assess the performance of any terrain modelling technique. The accuracy of any digital terrain model (DTM) in representing the terrain is dependent on many factors:

(a) accuracy of observed data
(b) pattern of data acquisition
(c) density of control data
(d) mathematical model
(e) interpolation model
(f) type of terrain
(g) distribution and density of check points.

So, to quantify the accuracy of fit of a digital terrain model meaningfully can only be carried out with a corresponding defined set of parameters.

In this project, where the data are processed off-line, the author had pre-selected some parameters: the sizes of the cells (fixed at 1000 m square); the accuracy of mathematical surface modelling (i.e. RMSE < 3 m in this case); the pattern of data acquisition is semi-regular; and terrain modelling is by Chebyshev polynomials. The density of data parameter is variable. A degree of fit is allowed to float to find its own level of best-fit; and the terrain interpolation is carried out with the Chebyshev polynomial coefficients. It remains to test whether the density of the control data affects

(1) degree of fit,
(2) accuracy of the fitted model.
The latter was tested at (a) data points, and (b) check points.

The assessment of the results is based on how accurately the reduced density data can represent the terrain. This is judged on the degree of fit and on the value of the RMSE.

Theoretically, the accuracy of fit at the check points should be as good as that at the control points. From the work done (of which this report is just a part), it was shown conclusively that the accuracy of fit at the check points is even better than that at the control points if the degree of fit is less than half the number of control data points in X or Y (assumed equal).
That is

$$k \leq \frac{1}{2}\sqrt{N}$$

where k is the degree of fit and N is the total number of control data points in a cell. The results in Tables 2, 4 and 5 are a further testimony of this statement.

The results of Table 3 do not quite conform to this theory. An explanation to this non-conformity is that in Set 3, some check points are extrapolated so lowering the accuracy of the results.

In general, if the degree of fit is higher than half the number of data points in X or Y, the accuracy of fit is bound to be poor. Needless to say, exact-fit terrain modelling should be discouraged where possible, as the errors are indeterminate. Values of k (degree of fit) higher than the number of data points in X or Y should never be used in surface modelling as the error values start to oscillate arbitrarily.

Another factor brought out in this project is that the degree of fit, k, is quite stable with change of density of data so long as k remains equal to or less than the number of data points in X or Y. The degrees of fit for cells A1 to B4 in Tables 1 to 5 have remained unchanged while the densities changed considerably.

CONCLUSION:

Terrain approximation by a double Chebyshev polynomial is an ideal way of representing relief in a functional form. The advantages of this technique are apparent in projects where large cell sizes are involved, and also where large numbers of data are acquired along parallel profiles to fit a mathematical surface of high orders. Such a surface would be a general purpose surface from which heights could be derived by interpolation along, say, a defined alignment and at a required density.

The problem of this methodology, however, is in assessing the accuracy of the results so obtained. One method of predicting the accuracy is based on the degree of fit and the number of control data available. The accuracy of fit is also further improved by overlapping data along the common boundaries of the adjacent cells (Segu, 1984). This solves the problem of cracks across the boundaries.

Another problem to this technique, not tackled explicitly in this paper, is that of density of data versus accuracy of fit. Density is a function of type of terrain and of accuracy of the DEM.

When the density factor has been resolved, this technique of approximating terrain elevation by polynomial digital elevation models can meet a lot of demand.

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