Parametrization of Sampled Chirps for SAR Range Compression Matched Filtering

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A b s t r a c t

A least squares fit method has been developed which allows approximation of sampled SAR chirps by analytic functions. From the formulation of the general case a factorization approach for amplitude and phase functions is derived. The SAR chirp - sampled by the in-phase and quadrature demodulation technique - is converted to amplitude and phase representation. The phase ambiguity problem can be solved for realistic frequency behaviour and limited sampling noise. Prototype software was implemented in order to test the method for simulated signals. The method shall be used for stability monitoring of the SAR chirp of ERS-1, the first remote sensing satellite of the European Space Agency.

Keywords: SAR, Range compression, Chirp parameters, Phase ambiguity problem

1. Introduction

For digital range compression of Synthetic Aperture Radar (SAR) data the time-reversed complex conjugate of the radar pulse signal is used as matched filter. Assuming a linear frequency modulated radar pulse (chirp), the digital processing is conventionally done using the ideal analytic waveform for the matched filter. This method becomes sub-optimal if the actual radar pulse shows deviations from the pure constant linear frequency modulation. In order to allow monitoring of the radar chirp and corresponding adjustment of the matched filter for range compression, modern SAR systems (such as the one being implemented for ESA's ERS-1 satellite SAR) provide sampling of the chirp.
The chirp samples are regularly transmitted to the ground and can be used for analysis and matched filter adjustment.
Fourier analysis of chirp samples turns out to be not very useful. More appropriate seems to be a parameterization in terms of analytical functions for amplitude and phase. This approach has been investigated here.

2. Representation of SAR chirps by analytic functions

In SAR applications the signal is conventionally sampled in the so-called in-phase and quadrature demodulation technique (I,Q - sampling). Hence, the sampled chirp can be treated as a sequence of complex numbers:

$$z(t_i) = a(t_i) \exp(j\, b(t_i)), \qquad [2.1]$$

where $a(t_i)$ and $b(t_i)$ are samples of the real magnitude and phase functions at times t_i, (i=0, ... , N).

Ideally, the chirp has a linear frequency modulation with a rectangular envelope:

$$z(t_i) = a \exp(j(b_0 + b_1 t_i + b_2 t_i^2)), \qquad [2.2]$$

where a, b_0, b_1, b_2 are constants.

In reality, there may be deviations from this ideal case of a linear chirp. In order to describe its essential characteristics, the amplitude a(t) and the phase b(t) shall be expressed by appropriate analytic functions of time such that the signal can be represented accurately enough with only a few parameters (for example as a matched filter for pulse compression).

A suitable approach for both amplitude and phase functions is to use polynomials in time, i.e. the sampled chirp would be parameterized as follows:

the envelope function of amplitude a (t) is a polynomial of degree L:

$$a(t) = \sum_{k=0}^{L} a_k t^k, \text{ where } a_k \text{ are constants,} \qquad [2.3]$$

and the phase function b (t) is a polynomial of degree M:

$$b(t) = \sum_{m=0}^{M} b_m t^m, \text{ where } b_m \text{ are constants.} \qquad [2.4]$$

The coefficients a_k and b_m shall be determined by a least - squares - fit procedure applied to the sampled complex signal.

3. Phase ambiguity problem

The (I,Q)-sampled chirp can easily be converted to amplitude and phase representation by means of trigonometric relations. However, from this conversion of the sampled signal the phases are only known to modulo 2 n.
For the purpose of the signal phase description as analytic function this ambiguity has at least partially to be solved when expressing the phases of the signal to be fitted as continuous functions in time.
Here assumptions have to be made on the general behaviour of phase functions. The momentary frequency of a signal is given by the first time derivative of the phase function. For a chirp signal the frequency shall at least be monotonically increasing or decreasing, i.e. there are no fluctuations in frequency; mathematically speaking, the time derivative of frequency should be either always positve or always negative. It is sufficient here to consider the case of positive and monotonically increasing frequencies; the other cases can easily be derived. It is clear that the phase ambiguity problem needs only to be solved modulo a multiple of the constant sampling frequency which is assumed to be sufficient to cover the real chirp bandwidth.
Based on these considerations frequency samples are actually obtained from any two successive chirp samples.
Since the frequency function is fitted to a polynomial the phase function is simply obtained by time integration of this polynomial with an arbitrary phase constant which can be adjusted to the phase of the first sample.

After converting the signal samples from cartesian to polar representation, phases of the samples are successively compared and factors 2 n are added until the phase from sample n+1 is closest to the phase calculated from the expected frequency at that point in time (assuming for example a linear chirp). This is a reasonable aproach which covers phase sampling errors within plus/minus n. Apart from a multiple of 2 n common to all samples there can not be a sudden jump of 2 n between samples without violating the Nyquist sampling condition which was anticipated to hold. This procedure resolves the ambiguity to the extend needed for the least-squares-fit. It leaves an overall ambiguity of a fixed frequency component which corresponds to a multiple of 2 n which is not significant for the present purpose.

4. General formulation of least-squares fit algorithm

The conventional approach for a least-squares-fit is to choose a linear combination of linear independent basic functions. This leads to the problem of solving a system of linear equations where the rank of the system equals the dimension of the selected basic function space.
Inserting equations [2.3] and [2.4] into [2.1] shows that the resulting chirp function is unfortunately not a simple linear combination of orthogonal base functions, since coefficients appear also in the arguments of the exponentials. Since this leads to a non-linear problem a factorisation approach has to be taken such that the least-squares-fit can be transformed into the task of solving two coupled systems of linear equations.
When the in-phase and quadrature demodulation technique is used (e.g. as for SAR signals) signal samples after A/D conversion can be considered as complex numbers

$$z_i = x_i + j\, y_i = u_i \exp\{ j\, v_i \}, \quad i=1,...,N. \qquad [4.1]$$

taken at times t_i, where x_i and y_i correspond to the in-phase and the quadrature components, respectively.
The polar representation of the complex signal samples can then easily be calculated using standard trigonometric formulae.
The actual signal samples z_i shall be approximated by an analytic function in time t, depending on parameters $a_1,...,a_L$ and $b_1,...,b_M$:

$$z(t) = r(a_1,...,a_L,t) \exp\{j\, p(b_1,...,b_M,t)\} \qquad [4.2]$$

where amplitude r and phase p are real functions with linear independent parametrisation which have to be chosen appropriately.
The least-squares-fit procedure then consists in determining the parameter sets $\{a_n\}$, $\{b_m\}$ such that

$$D = \sum_{i=1}^{N} |z_i - z(t_i)|^2 = \sum_{i=1}^{N} \{(z_i - z(t_i))(z_i - z(t_i))^*\} \qquad [4.3]$$

becomes a minimum.

Equation [4.3] can be transformed by using equ. [4.1] and [4.2]:

$$D = \sum_{i=1}^{N} \{ r^2(a_1,\ldots,a_L,t_i) + u_i^2 - 2\, r(a_1,\ldots,a_L,t_i)\, u_i \cos(v_i - p(b_1,\ldots,b_M,t_i)) \} . \qquad [4.4]$$

A necessary condition for minimum D is that the first partial derivatives with respect to all parameters $\{ a_n \}$ and $\{ b_m \}$ vanish:

$$\frac{d D}{d a_n} = 0 , \; n=1,\ldots,L; \qquad \frac{d D}{d b_m} = 0, \; m=1,\ldots,M \qquad [4.5/6]$$

This leads to the coupled system of transcendental equations:

$$\sum_{i=1}^{N} \{ [r(a_1,\ldots,a_L,t_i) - u_i \cos(v_i - p(b_1,\ldots,b_M,t_i))] \frac{d\, r(a_1,\ldots,a_L,t_i)}{d a_n} \} = 0, \quad n = 1,\ldots,L. \qquad [4.7]$$

$$\sum_{i=1}^{N} \{ [u_i\, r(a_1,\ldots,a_L,t_i) \sin(v_i - p(b_1,\ldots,b_M,t_i))] \frac{d\, p(b_1,\ldots,b_M,t_i)}{d b_m} \} = 0; \quad m = 1,\ldots,M. \qquad [4.8]$$

These expressions represent the general formulation of the least-squares-fit system of equations for complex signal samples given as time functions of amplitude and phase (equ. [4.2]).

5. Factorization approach for amplitude and phase

The system [4.7], [4.8] is in general not solvable in closed form.
However, as long as the chosen phase function is sufficiently close to the actual signal samples, the phase angles

$$v_i - p(b_1,\dots,b_M,t_i); \quad i=1,\dots,N. \qquad [5.1]$$

are small. Therefore a linearisation approach is made:

$$\sin(v_i - p(b_1,\dots,b_M,t_i)) =$$

$$v_i - p(b_1,\dots,b_M,t_i); \quad i=1,\dots,N. \qquad [5.2]$$

$$\cos(v_i - p(b_1,\dots,b_M,t_i)) = 1; \quad i=1,\dots,N. \qquad [5.3]$$

With this approximation one gets the following simplified least-squares-fit equations:

$$\sum_{i=1}^{N} \left\{ [\, r(a_1,\dots,a_L,t_i) - u_i \,] \frac{d\, r(a_1,\dots,a_L,t_i)}{d\, a_n} \right\} = 0, \quad \text{for } n = 1,\dots,L. \qquad [5.4]$$

$$\sum_{i=1}^{N} \left\{ [\, u_i\, r(a_1,\dots,a_L,t_i)\, (v_i - p(b_1,\dots,b_M,t_i)) \,] \frac{d\, p(b_1,\dots,b_M,t_i)}{d\, b_m} \right\} = 0; \quad m = 1,\dots,M. \qquad [5.5]$$

These are the general linearised sets of least-squares-fit equations for the functional form [4.2] of the complex signal samples. The subset [5.4] does not now depend on the phase function p which implies that the amplitude fit is performed independently with this approximation.
The equations [5.4] represent the conventional form of least-squares-fit equations. After solving these equations the resulting function r can be inserted into the set [5.5] and these equations can then be solved similarly.

6. Polynomials for Amplitude and Phase

If the amplitude function r and the phase function p are chosen to be polynomials of order M and L, respectively,

$$r(a_1,\ldots,a_L,t) = \sum_{n=1}^{L} a_n t^{n-1} \qquad [6.1]$$

$$p(b_1,\ldots,b_M,t) = \sum_{m=1}^{M} b_m t^{m-1} \qquad [6.2]$$

the sets [5.4] and [5.5] of equations result in:

$$\sum_{i=1}^{N} \sum_{n=1}^{L} \{a_n t_i^{n-1} - u_i\} t_i^{k-1} = 0, \quad k = 1,\ldots, L; \qquad [6.3]$$

$$\sum_{i=1}^{N} u_i r(t_i) \sum_{m=1}^{M} \{b_n t_i^{m-1} - v_i\} t_i^{n-1} = 0, \qquad [6.4]$$

$$n = 1,\ldots,M.$$

Re-ordering the summation brings [6.3] and [6.4] to the conventional form of systems of linear equations:

$$\sum_{n=1}^{L} c_{n,k} a_n = d_k, \quad k = 1,\ldots, L, \qquad [6.5]$$

where

$$c_{n,k} = \sum_{i=1}^{N} t_i^{n+k-2}, \qquad [6.6]$$

and

$$d_k = \sum_{i=1}^{N} u_i t_i^{k-1}. \qquad [6.7]$$

$$\sum_{m=1}^{M} f_{m,n} b_m = g_n, \quad n = 1,\ldots, M, \qquad [6.8]$$

where

$$f_{m,n} = \sum_{i=1}^{N} u_i r(a_1,\ldots,a_L,t_i) t_i^{m+n-2}, \qquad [6.9]$$

and

$$g_n = \sum_{i=1}^{N} u_i v_i r(a_1,\ldots,a_L,t_i) t_i^{n-1}. \qquad [6.10]$$

7. Results with simulated chirps.

For analysis of the method described above prototype software has been developed which also allows test signals to be generated as follows. The total number N of signal samples, the order L of amplitude polynomial, the order M of phase polynomial and the sampling rate can be selected as parameters. Since frequency is the time derivative of phase, the order of frequency polynomial is (M-1).

The synthesized signal has then been formed by predefining (L+1) equidistant amplitude samples and M equidistant frequency samples. These samples have been used to create interpolation polynomials of order L and (M-1) for amplitude and frequency, respectively. The phase polynomial results from the frequency polynomial by integration and setting the phase constant arbitrarily to zero.

Based on these polynomials all individual signal samples (in I,Q format) are then calculated with the sampling rate defined. It is clear that the sampling rate and frequency samples must be consistent with respect to the Nyquist condition.

In order to have more realistic signals, uniformly distributed random noise can be added to both amplitude and phase and limited independently. The advantage of this test approach is that the polynomials resulting from the least-squares-fit can be directly compared with the polynomials used for the creation of the test signals. In the absence of noise, the generation polynomials and the least-squares-fit polynomials shall be identical. This has been used as a consistency check for the software.

The following parameters have been used for the test:

Number of samples:		N = 32,
Order of amplitude polynomial:		L = 4,
Amplitude sample 1:	1.0	
Amplitude sample 2:	2.0	
Amplitude sample 3:	4.0	
Amplitude sample 4:	2.0	
Amplitude sample 5:	1.0	
Order of phase polynomial:		M = 3,
Frequency sample 1:	1.0 Hz	
Frequency sample 2:	2.0 Hz	
Frequency sample 3:	4.0 Hz	
Sampling rate:		32 Hz

The following tables show the test results The coefficients are listed in order of increasing power.

M and P indicate the values limiting the uniform random noise which was added to the signal samples after calculation from generating polynomials;
M is the percentage of maximum amplitude, P is the percentage of phase 2 n.

Coefficients of amplitude polynomial:

Generated signal	Least-squares-fit result with random noise added to signal			
M	0.0	10.0	10.0	0.0
P	0.0	0.0	1.0	10.0
1.0000	1.0000	1.2165	1.2165	1.0000
-15.1398	-15.1398	-16.4205	-16.4205	-15.1398
129.288	129.288	126.841	126.841	129.288
-234.652	-234.652	-222.841	-222.841	-234.652
121.111	121.111	112.342	112.342	121.111

Comparison of the coefficients shows that phase noise has no effect on the amplitude as it should be the case with the factorization aproach.

Coefficients of phase polynomial:

Generated signal	Least-squares-fit result with random noise added to signal			
M	0.0	10.0	0.0	0.0
P	0.0	0.0	1.0	10.0
0.0	0.0	0.0	-0.002613	5.563612
6.283185	6.283185	6.283185	6.229420	15.370678
3.242934	3.242934	3.242934	3.617274	165.38268
4.463394	4.463394	4.463394	4.095795	-132.89817

Analysis of these coefficients shows that for small phase noise the aproach works, whereas noise of 10 percent causes a breakdown since the linearization approximation for small angles does not hold anymore.
As expected, amplitude noise does not have an effect on the phase polynomial

8. Conclusion

Tests with synthesized SAR chirps show that the method can be used for parametrization if the frequency behaviour is reasonably smooth and the signal noise is of the order of a few percent. Because of the factorization approach the method is mainly limited by phase noise and not so much by amplitude noise. It is intended to refine the method with simulated signals for the case of ERS-1 SAR chirp and to apply it later to samples from the real satellite system.

9. References

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