MEASURES FOR THE STRUCTURAL INFORMATION CONTENT OF BINARY IMAGES

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Abstract.
One method to find the noise level of digital images, is based on the visual inspection of bitslices. A bitslice is a binary image which can be made for each bitlevel of an digital image. Visual inspection leads to a decision whether a bitslice is considered as showing spatial structure or not. In this paper a theoretical background is developed for this decision, and from the theory two measures are developed for a numerical evaluation of the structuredness of a binary image. One measure is based on statistical concepts the other measure is based on information theoretical concepts and has the form of a entropy measure. A decision rule based on one of these measures can replace the visual inspection of binary images.

1. The problem.

Digital images are supposed to give spatial information, which means that the pixel values depend on the position of the pixel in the image. The values of neighbouring pixels will correlate when they represent image points of a same object. Objects in the object space are mapped on image segments with features which are in some sense homogeneous. In e.g. Landsat T.M. images one clearly recognizes fields and roads and other terrain objects, in thermal infrared images one clearly recognizes areas with constant temperature etc.
One of the tasks of image processing is to realise an image segmentation and to identify the terrain objects which are related to the image segments. Hence the spatial structure of the image is analysed.
In for instance Landsat T.M. images spectral data of terrain objects are obtained by a sensor and stored in 8-bits per pixel per band. But image analysis shows that the data have a noise component so that the actual useful information is less then 8-bit per pixel per band. One of the effects of this noise is that the spatial structure of the image is blurred. This effect is additional to the fact that structural information about the terrain is lost due to the spatial resolution of the sensor.

In this paper we will discuss some methods to define the structural information content or "structuredness" of an image. The approach will be based on assumption that the image consists of segments which do have, within certain limits, homogeneous pixels values for some spectral band.

Hempenius and Haberäcker proposed to investigate the structuredness of images by inspecting their bitslices (see [4] and also [2] (Ch. 5.2.2)). From e.g. an eight -bit image eight bitslices can be derived, one per bitlevel. A bitslice is then a binary image, and a picture can simply the obtained by making the pixels black and white according to the occurrence of 0 and 1 values at that bit level. Hempenius proposed visual inspection to judge the structuredness of a bitslice, which is related to the spatial clustering of black and white pixels in the image. If no clear clustering can be seen, the bitslice is rejected as containing noise only. Elimination of the noisiest bit levels from the original image gives better results for the geometric image analysis e.g. for edge- and line detection.

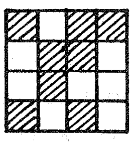
In the sequel some decision rules will be formulated to distinguish between noisy and structured bitslices. These rules will be based on numerical measures for the noise level or alternatively the structuredness of these bitslices, they should replace the visual inspection.

2. Noisy binary images.

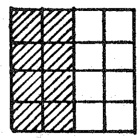
In a noisy binary image, the pixel values 0 and 1 are randomly distributed. Let W_{ij} be the pixel value at position i,j. Then we find for a noisy image

$$P(W_{ij}=1) = P(W_{ij}=0) = \tfrac{1}{2}$$

There is a probability of 50% that the value of an arbitrary pixel i,j is equal to 1, the probability that it is equal to 0 is also 50%.
If 50% of the pixels in an image have a value 1, then this does not necessarily imply, however, that the image is noisy.



8 black $\sim$ 0
8 white $\sim$ 1



8 black $\sim$ 0
8 white $\sim$ 0

In the right hand image 50% of the pixels are black, but the image does have a structure, whereas the left hand images is more noisy.
The right hand image is structured because the pixels are clustered. The clustering of the black and the white pixels should be measured, therefore the values of neighbouring pixels will be compared. In a noisy image we find the following joint probabilities for the values of neighbouring pixels:

$$P(W_{ij}=0,\ W_{i+1,j}=0) = P(W_{ij}=0) \times P(W_{i+1,j}=0) = \tfrac{1}{2}.\tfrac{1}{2} = 1/4$$

And similarly

$$P(W_{ij}=0,\ W_{i+1,j}=1) = 1/4$$
$$P(W_{ij}=1,\ W_{i+1,j}=0) = 1/4$$
$$P(W_{ij}=1,\ W_{i+1,j}=1) = 1/4$$

In a noisy image the value W_{ij} is stochastically independent of $W_{i+1,j}$. This is also true of course for the other neighbours of W_{ij}. From these results follows:

$$P(W_{ij}=W_{i+1,j}) = P(W_{ij}=1,\ W_{i+1,j}=1) + P(W_{ij}=0,\ W_{i+1,j}=0)$$
$$= 1/4+1/4 = 1/2$$

and

$$P(W_{ij}=W_{i+1,j}) = P(W_{ij}=0,\ W_{i+1,j}=1) + P(W_{ij}=1,\ W_{i+1,j}=0)$$
$$= 1/4+1/4 = 1/2$$

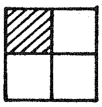
The probability that two neighbours have the same value is 50%, the probability that they do not have the same value is also 50%.

3. Cell Boundaries.

In the sequel pixels will be considered as cells which have four boundaries in common with neighbouring cells. Those at the border of an image have only two or three boundaries in common with their neighbours. Values g will be assigned to the cell boundaries, using the following rules:
for the boundary between cell (i,j) and (p,q) is:

$$g(i,j,p,g) = |W_{ij}-W_{pq}| \qquad (p,q) \in \{(i-1,j),(i+1,j),(i,j-1),(i,j+1)\}$$

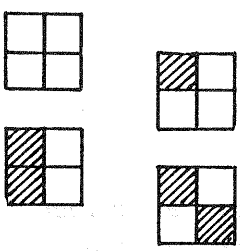
This rule implies that g=1 if $W_{ij} \neq W_{pq}$ and g=0 if $W_{ij}=W_{pq}$, hence if the cell boundary is also a boundary between a black and a white image area then g=1, else =0.



in the figure is $g(1,1,1,2)=1$ $g(2,1,2,2)=0$
$g(1,1,2,1)=1$ $g(1,2,2,2)=0$

By summing the boundary values g over an image we find how many cell boundaries are boundaries between black and white areas. This sum gives information about the clustering of pixels and thus about the image structure. Through this sum there is a relationship of our structure analysis with texture analysis based on grey level cooccurences (see e.g. [3]), which are taken here for neighbouring pixels in an binary image. A criteria for the decision whether an image is structured or not will be based on this sum, but first we will see how many independent cell boundaries there are in an image.

In a 2x2 image segment the following situations may occur:



g1112	g1121	g2122	g1222	Σg
0	0	0	0	0
1	1	0	0	2
1	0	1	0	2
1	1	1	1	4

Other situations can be related to these by rotations or interchanging black and white.
The sum Σg is always even in this image segment, so there is an interdependency between the four cell boundaries e.g.:

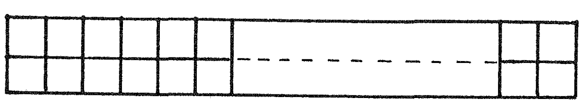
$g(1,2,2,2)$ is dependent on $g(1,1,1,2)$, $g(1,1,2,1)$ and $g(2,1,2,2)$

This relationship can be expressed as

$$g(1,2,2,2) = 2 \text{ modulo } [g(1,1,1,2) + g(1,1,2,1) + g(2,1,2,2)]$$
$$= f[g(1,1,1,2),\ g(1,1,2,1),\ g(2,1,2,2)]$$

If the 2x2 segment is extended to a 2x3 segment, hence two adjacent rows with 3 cells each, a recursive dependency occurs:

$g(1,2,2,2) = f[g(1,1,1,2),\ g(1,1,2,1),\ g(2,1,2,2)]$ and
$g(1,3,2,3) = f[g(1,2,1,3),\ g(1,2,2,2),\ g(2,2,2,3)]$



Substitution of the first expression in the second gives:

$$g(1,3,2,3) = f[g(1,2,1,3),\ f[g(1,1,1,2),\ g(1,1,2,1),\ g(2,1,2,2)],\ g(2,2,2,3)]$$

Similar recursive relationships will be found for all inter row boundaries $g(1,p,2,p)$ with $p \geq 3$.

Each row has n-1 cell boundaries, for two rows this number is 2x(n-1). Between the rows there is just 1 independent boundary so that the total number is 2x(n-1)+1. For an image with m rows with n cells the total number is:

$$m \times (n-1)+(m-1) = m \times n-1$$

This formula is symmetric for m and n, so that row and columns can be interchanged. That means that one could also count the m-1 cell boundaries in each column and between each pair of adjacent columns one cell boundary. So two different sums can be defined:

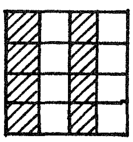
- The first sum is the result of a summation over all the rows and one column (the inter row boundaries are then all taken from the same column), this gives:

$$\sum_{r=1}^{m} \sum_{k=1}^{n} g(k,r,k+1,r) + \sum_{r=1}^{m} g(1,r,1,r+1) = \gamma rk$$

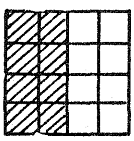
- The second sum is the result of a summation over all the columns and one row (the intercolumns boundaries are then all taken from the same row), this gives:

$$\sum_{k=1}^{n} \sum_{r=1}^{m} g(k,r,k,r+1) + \sum_{k=1}^{} g(k,1,k+1,1) = \gamma kr$$

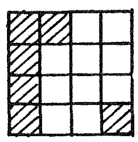
In general γrk and γkr will be different e.g.:



$\gamma rk=12$
$\gamma kr=3$



$\gamma rk=4$
$\gamma kr=1$



$\gamma rk=5$ or 6
$\gamma kr=3$ or 4

The third example shows that γrk and γkr do not necessarily have an unique value. The value depends on the choice of the column to be used for the evaluation of γrk, or the row for γkr.

4. A statistical decision rule to distinguish structured from noisy bitslices

In section 2 we found for noisy binary images:

$$p\{W_{ij}=W_{i+1,j}\} = \tfrac{1}{2}$$

and $p\{W_{ij} \neq W_{i+1,j}\} = \tfrac{1}{2}$

These results imply for the cell boundaries that:

$$p\{g(i,j,i+1,j) = 0\} = \tfrac{1}{2}$$

and $p\{g(i,j,i+1,j) = 1\} = \tfrac{1}{2}$

In an image with m rows and n columns the g-values of $\Gamma = m \times n-1$ independent cell boundaries will be summed. We may expect now that for a noisy image 50% of these boundaries will have a value g=1, that means for the sum:

$$E\{\gamma\} = E\{\Sigma g\} = \tfrac{1}{2}\Gamma$$

For testing whether an image has a spatial structure we start from the null hypothesis Ho that there is no structure, hence the image is noisy and $E\{\gamma|Ho\} = \tfrac{1}{2}\Gamma$

Under this assumption γ is benominally distributed:

$$P(\gamma=G|Ho) = \begin{bmatrix} \Gamma \\ G \end{bmatrix} ½^{\Gamma} = \frac{\Gamma!}{G!(\gamma-G)!} (½)^{\Gamma}$$

and

$$P(\gamma\leq G|Ho) = \sum_{n=0}^{G} \begin{bmatrix} \Gamma \\ n \end{bmatrix} ½^{\Gamma}$$

and of course

$$P(\gamma>G|Ho) = 1-P(\gamma\leq G|Ho)$$

When Γ is large enough (≥ 30) this distribution can be a approximated by a normal distribution with

$$\boxed{\mu_{\gamma} = E\{\gamma\} = ½\Gamma \quad \text{and} \quad \sigma_{\gamma} = ½\sqrt{\Gamma}}$$

This null hypothesis can now be tested against an alternative hypothesis which states that there is structure in the image. Structure means that the black and white pixels are clustered spatially, which implies that there are less cell boundaries on the boundaries between black and white image areas. Therefore we expect under the alternative hypothesis Ha:

$$E\{\gamma|Ha\} \ll ½\Gamma$$

The case that the scene represented by the image has a chess board structure with $E\{\gamma\}\gg ½\Gamma$ will not be taken into consideration here.

A test of Ho against Ha can now be based on the decision rule:

$\gamma \leq Go \rightarrow$ reject Ho accept Ha.

with $p(\gamma\leq Go|Ho) = \alpha$ α is sufficiently small

The interpretation of this rule is that the value of Go has been chosen so that the probability that a noisy image will give a sum γ less than Go is very small. Hence if $\gamma\leq Go$ occurs one may suspect the image is not noisy but that it has some structure.

We learned from the examples at the end of section 3 that the values of γrk and γkr may differ, and that the value of γrk depends on the column used for the evaluation and that γkr similarly depends on the row used for evaluation. A strategy for chosing one of the possible values of γ for testing can be based on the following considerations:

The test aims at finding structured images. The examples of section 3 show that even in structured images one can still find relatively high γ-values (in the examples $½\Gamma=7.5$). If these high values are used in a test, it may fail to detect structured images. Therefore the minimum value of γ should be used in the test.

If γrp is the value of γrk using column nr. p and similarly γkq uses row nr. q, then we first find:

$$\gamma rk \min = \min_{p} (\gamma rp) \qquad P=1,\ldots,n$$

and

$$\gamma kr \min = \min_{q} (\gamma kq) \qquad q=1,\ldots,m$$

For the test we use:

$$\gamma\min = \min (\gamma rk \min, \gamma kr \min)$$

Experience shows that the statistical test proposed here only works in relatively small image segments. When these segments are too large, hence Γ is large, the test will always lead to a rejection of Ho even if the image is noisy, therefore another approach is given in the following section.

5. An information measure for the spatial structure of binary images.

In this section a measure for the amount of structure will be developed for binary images. This measure will be related to the information measure $I=-\ln p$ in information theory, or the entropy measure $H=-\Sigma p \ln p$ in thermodynamics (see [2] ch 3.1 and [5] ch 5.9.1 and [1] ch 3.3).
We start from the assumptions for a noisy image as formulated in section 2. The measure will be based on the sum of boundary values γ of section 3.

Let an image have Γ independent cell boundaries and let γ be the sum of their g-values, then γ out of Γ cell-boundaries are also boundaries between black and white areas.
If we write $\bar{\gamma}=\Gamma-\gamma$ then there are:

$$R\gamma = \frac{\Gamma!}{\gamma!\bar{\gamma}!}$$

different images possible which give this result. The sum γ makes use of all rows and one column or of all columns and one row. Therefore the value R gives the number of possible realisations for one particular combination of rows and columns. An image with n columns and m rows has nxm of these combinations, therefore the total number of realisations is:

$$\bar{R}\gamma = n.m.R\gamma$$

The largest number of realisations will be found for $\gamma=\frac{1}{2}\Gamma$, that is for a noisy image as defined in section 2. Hence for an image with no structure we find:

$$Ro = R_{\frac{1}{2}\Gamma} = \frac{\Gamma!}{(\frac{1}{2}\Gamma)!(\frac{1}{2}\Gamma)!} \qquad (\Gamma \text{ should be even})$$

$$\bar{R}o = nm\ Ro$$

The more γ deviates from $\frac{1}{2}\Gamma$ the smaller $R\gamma$ will be, i.e. the less realisations are possible, the more structure the image has, hence:

$$\bar{R}\gamma < \bar{R}o$$

or

$$\frac{\bar{R}o}{\bar{R}\gamma} = \frac{Ro}{R\gamma} > 1$$

This measure can be used to indicate the amount of spatial structure in a binary image. This amount can be expressed as an information measure in bits by taking the logarithm with base 2:

$$Istruct = \log 2\ \frac{Ro}{R\gamma} = k \ln \frac{Ro}{R\gamma} \quad \text{with } k = \frac{1}{\ln 2}$$

hence

$$Istruct = k[\ln Ro - \ln R\ ]$$

with

$$\ln Ro = \ln \Gamma! - 2\ln(\tfrac{1}{2}\Gamma)!$$

and

$$\ln R\gamma = \ln \Gamma! - \ln\gamma! - \ln \bar{\gamma}!$$

according to Sterlings formula is for γ large enough:

$$\ln \gamma! \approx \gamma \ln(\gamma-1)$$

Further elaboration gives:

$$\ln Ro = \Gamma\ln(\Gamma-1) - 2.\tfrac{1}{2}\Gamma\ln(\tfrac{1}{2}\Gamma-1) \approx -\Gamma\ln\tfrac{1}{2}$$

$$\ln R\gamma = \Gamma\ln(\Gamma-1) - \gamma\ln(\gamma-1) - \bar{\gamma}\ln(\bar{\gamma}-1) \approx -\gamma\ln f - \bar{\gamma} \ln \bar{f}$$

$$\text{with } f = \frac{\gamma-1}{\Gamma} \text{ and } \bar{f} = \frac{\bar{\gamma}-1}{\Gamma}$$

These results substituted in

$$Istruct = k[-\Gamma\ln\tfrac{1}{2} + \gamma\ln f + \bar{\gamma}\ln \bar{f}]$$

and when γ en $\bar{\gamma}$ are large enough:

$$\boxed{Istruct \approx k\Gamma[-\ln\tfrac{1}{2} + f \ln f + \bar{f} \ln \bar{f}]}$$

This expression gives then the structural information content of an image when γ out of Γ cell boundaries are boundaries between black and white areas.
The information content per cell boundary is:

$$\boxed{i\ struct = \frac{Istruct}{\Gamma} = k[-\ln\tfrac{1}{2} + f \ln f + \bar{f} \ln \bar{f}]}$$

istruct gives the information measure per cell boundary for the structure in one bitslice of a digital image. Let $i^b struct$ be this measure for bitslice nr. b and let the digital image have n+1 bitslices $b \in \{0,\ldots,n\}$. Then the total amount of information per cell boundary for the digital image is:

$$\boxed{i\ total = \sum_{b=0}^{n} i^b struct}$$

For the analysis of the image structure one could make use of the bitslices for which

$$\boxed{d^b = \frac{i^b struct}{i\ total} > C = \text{critical level}}$$

This decision rule means that bitslices with $d^b \leq C$ are considered as noisy, they contain no significant structural information. This decision rule has been based on the assumption that the bitlevels are not correlated.

6. A modification of the decision rules for the case $P(W=0)\neq\tfrac{1}{2}$

The decision rules in section 4 and 5 have been based on the assumption for the pixels values: $P(W=0) = P(W=1) = \tfrac{1}{2}$. Counting the pixels, however, may give relative frequencies po for W=0 and p1 for W=1, with po$\neq$p1, but of course po=1-p1. This fact can be taken into consideration for the analysis of the structuredness of a binary image.
If the distribution of the black and white pixels in the image is random, the image is noisy, then the probability that a cell boundary is a boundary between black and white areas is now:

$$p(g=1) = pop1 + p1po = 2pop1$$

and similarly

$$p(g=0) = pop0 + p1p1$$

of course $p(g=0) + p(g=1) = 1$

When counting Γ cell boundaries, we may expect in a noisy image that there are:

$$p(g=1) \cdot \Gamma = \bar{\Gamma}$$

transitions from black to white and vice versa, and that

$$\bar{\bar{\Gamma}} = \Gamma - \bar{\Gamma}$$

cell boundaries are no transition.

The parameter γ has then a binominal distribution with

$$p(\gamma = G|Ho) = \begin{bmatrix} \Gamma \\ G \end{bmatrix} p(g=1)^{G} \cdot p(g=0)^{\Gamma-G}$$

If Γ is large enough this density function can be approximated by a normal-density function with:

$$\boxed{\mu_{\gamma} = \bar{\Gamma} \text{ and } \sigma_{\gamma} = [P(g=1) \cdot P(g=0) \cdot \Gamma]^{1/2} = [\bar{\Gamma}.\bar{\bar{\Gamma}}]^{1/2}}$$

This density function should replace those of section 4. Similarly we can write for a noisy image:

$$Ro = \frac{\Gamma!}{\bar{\Gamma}!\bar{\bar{\Gamma}}!}$$

With this Ro the structural information measures of section 5 will be:

$$Istruct = k\Gamma[-F \ln F - \bar{F} \ln \bar{F} + f \ln f + \bar{f} \ln \bar{f}]$$

With $F = \frac{\bar{\Gamma}}{\Gamma}$ and $\bar{F} = \frac{\bar{\bar{\Gamma}}}{\Gamma}$, and f and $\bar{f}$ as before the structural information content per cell boundary is now:

$$\boxed{istruct = k[-F \ln F - \bar{F} \ln \bar{F} + f \ln f + \bar{f} \ln \bar{f}]}$$

7. An example.

As an example we compute the proposed decision rules for a 128x128 pixels region of a TM image band 4 of August 1984. The scene shows an agricultural district North-West of Wageningen. In the top left and bottom left corner there are town areas.

The figure shows the eight bitslices and the table gives the parameter values to be used in the decision rules, first for the assumption po=p1=½ then for the case po≠p1 (ch 6).

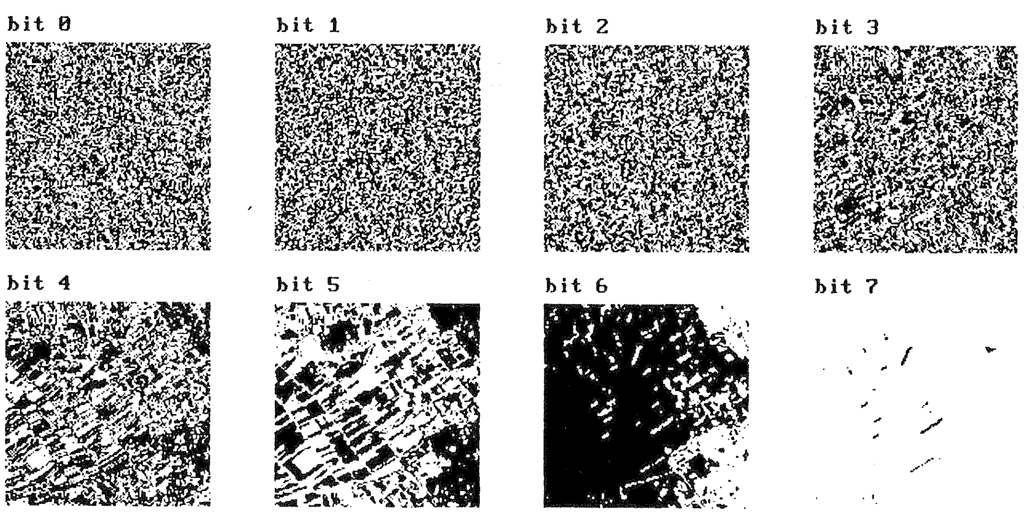


Table:

		$\sigma\gamma = \frac{1}{2}\sqrt{\Gamma} = 64$			$\sigma\gamma = [p(q=1) \cdot p(q=0)]^{\frac{1}{2}} \cdot \Gamma^{\frac{1}{2}}$					
bitslices	γmin	stat	i	i/total	po	$\sigma\gamma$	$\bar{\Gamma}$	stat	i	i/total
0	7927	8041	0.0007	0.0004	0.49	64	8188	8039	0.0007	0.0008
1	7828	8041	0.0013	0.0008	0.47	64	8162	8013	0.0014	0.0016
2	7827	8041	0.0014	0.0008	0.49	64	8188	8039	0.0014	0.0016
3	7313	8041	0.0083	0.0046	0.49	64	8188	8039	0.0083	0.0097
4	5788	8041	0.0629	0.0350	0.47	64	8162	8013	0.0629	0.2925
5	3560	8041	0.2448	0.1360	0.49	64	8188	8039	0.2448	0.2925
6	1527	8041	0.5530	0.3072	0.21	60	5436	5296	0.4730	0.5653
7	144	8041	0.9276	0.5153	0.01	18	324	282	0.0444	0.0530

Column "stat" gives critical value C for statistical decision rule so that $p(\gamma<C|Ho) = 1\% \rightarrow C=\frac{1}{2}\Gamma - 2.33x\sigma\gamma$, $\Gamma=16383$.

Comparing γmin with the critical value for the statistical decision rules, we see that in no case Ho is accepted. That means that the test never leads to a conclusion that a bitslice is noisy. This test is no good discriminator between noisy and structural binary images.
Under the assumption that po=p1 istruct and istruct/total shows that about $998^{\circ}/_{oo}$ of the structural information is contained in the bitslices 3 and higher, and $993^{\circ}/_{oo}$ of the structural information is contained in the bitslices 4 and higher.
For the case that po≠p1 we see that $996^{\circ}/_{oo}$ of the structural information is contained in bitslices 3 and higher and $986^{\circ}/_{oo}$ is contained in bitslices 4 and higher.

These results show that, whereas the statistical test does not discriminate between noisy and structural images, the information measure i struct clearly indicates a jump in information content between bitslices 3 and 4. This latter result confirms the outcome of visual inspection, where structure is firstly observed in bitslice 4. Some minor clustering occurs in bitslice 3 already, but this is not considered as being significant.
The image in this example is 128x128 pixels and the decision rules consider the whole image. If one is interested in smaller parts of the image then the same decision rules can be applied to them. It may be as in bitslice 3 that some parts are structured even when the whole image as such is considered as noisy. Most likely the statistical decision rule will work better in smaller images.

Further investigation is required on the effect of the assumption that bitlevels are not correlated. If correlation does occur the decision rule should be adjusted to take care of it.

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November 1987.

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