Analytical study for the identification of the corresponding couples of points on the frames of a plane trajectory.

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Abstract. We present a method for the numerical computation of the coordinates of corresponding couples on the photographic images of a plane trajectory. The method is based on the analytic projectivity, between the planes of the photographic frames, generated by few known points of the trajectory which may be obtained experimentally.

In the survey of space lines , as for instance in the photogrammetric survey of trajectories of bullets or rockets, in order to obtain the trajectory with numerical method, is of great importance the identification, on the two plates, of the couples of the corresponding points in the images of the trajectory.

In terrestrial photogrammetry, in general, is defined normal the position of the plates when are absent convergence, drift of the photograph**s,** oblique photography and, moreover, the base is horizontal. In the restitution this positioning allows to determine immediate the couples of the corresponding points in the two images of the trajectory.

When the plates are non in the same plane the direct identification is not possible since the ordinates are not the same. The identification of the couples of corresponding points in the two plates is difficult if one wants to proceed with a graphical method since is required the use of the theorem of the nodal points and, consequently, of their location which, in general**,** are not in a useful position to obtain rapidly the graphical identification of the couples of corresponding points**.**

In a study Piazzolla Beloch (1954) notes that translating the photographic images in the symmetric position, with respect to the optic centre, relative to that occupied at the time of the taking (obtaining thus a perspective image inversely equal to the original one), one obtains that the nodal points are near the principal points which gives a useful setting for the identification of the couples of corresponding points.

In the same study of is also suggested an apparatus for the solution of the problem in a more general case. I found therefore of interest to solve the problem of the identification of the corresponding points also analytically.

We first note that the graphical methods cannot give an analytical solution except when we know the equations of the image curves in the plates in given coordinate systems, which is difficult to obtain. The scope of this research is precisely to overcome this obstacle which is present when one wants to give an analytical treatment to the graphical procedures of restitution.

The method suggested here consists in finding the projectivity relation between the planes of the images through the plane of the trajectory avoiding thus the need of the equations of the images of the trajectory in the plates.

In order to obtain the equations of the projective correspondence in a simple manner it is useful to discuss the following theoretical simpler but more general problem..

1. We begin considering the problem between two straight lines. Let *r* and *r'* be non coplanar straight lines as shown in figure 1, and let *t* and *t'* linear coordinates in *r* and *r'* with

$t' = (\alpha t+\beta)/(\gamma t+\delta)$

the equation of a projectivity Σ relating *r* and *r'* . Let also *P(X,Y,Z)* and *P'(X',Y',Z')* two generic points in space where a Cartesian system O(x,y,z) is introduced. Let us now project *r* and *r'* from *P* and *P'* respectively associating the straight lines which project points of *r* and *r'* corresponding in Σ; we obtain two sets of projective straight lines.

If *s* is the straight line intersection of the planes *Pr* and *P'r'* and $\bar{\Sigma}$ the projectivity generated by the sets of lines with centres *P* and *P'* respectively; the problem of determining the region of the couples *P* and *P'* for which $\bar{\Sigma}$ is the identity, is identical to that of determining the couples P and P' for which the sets of straight lines with centres in P and P' are perspective. The problem is also identical to that in which if Q and Q' are two corresponding point in r and r' respectively the points *P, P', Q, Q'* are coplanar which suggests a very simple method for the solution.

In fact assuming the straight line r as z axis in the Cartesian system, if

$x = at' + a', \qquad y = bt' + b', \qquad z = ct' + c'$

are the equations of the straight line r', the coplanarity of *P, P', Q, Q'* implies the following identity

$$\begin{vmatrix} X & Y & Z & 1 \\ X' & Y' & Z' & 1 \\ a(\alpha t+\beta)/(\gamma t+\delta)+a' & b(\alpha t+\beta)/(\gamma t+\delta)+b' & c(\alpha t+\beta)/(\gamma t+\delta)+c' & 1 \\ 0 & 0 & t & 1 \end{vmatrix} \equiv 0 \quad (1)$$

The equation (1) is an identity with respect to t. From the identity (1) we obtain the following 3 equations

$$\gamma(X'Y-XY')+(ba+b'\gamma)(X-X')+(a\alpha+a'\gamma)(Y'-Y)=0$$

$$(ca+c'\gamma-\delta)(XY'-X'Y)+(b\alpha+b'\gamma)(X'Z-XZ')+(b\beta+b'\delta)(X-X')+ \quad (2)$$

$$+(a\alpha+a'\gamma)(YZ'-Y'Z)+(a\beta+a'\delta)(Y'-Y)=0$$

$$(c\beta+c'\delta)((XY'-X'Y)+(b\beta+b'\delta)(X'Z-XZ')+(c\beta+c'\delta)(YZ'-Y'Z)=0$$

which are the conditions on the coordinated of P and P' for the coplanarity with the generic couple Q and Q', or for $\overline{\Sigma}$ to be an identity or, finally, to obtain that the two sets of straight lines with centres in P and P' respectively be perspective..

We see thus that the equations (2) define a birational cubic transformation of the space in itself such that when one of the centres of projection is given the other is uniquely determined. Moreover it is easily seen that the transformation has the characteristic of an involution and that exist ∞^3 couples of points which satisfy the conditions of the problem.

2. For the future developments it is useful to consider the inverse problem of the previously considered one, that is to determine the projectivity between r and r' when are known the positions of r, r', P, and P'.. In this case the unknown of the problem are $\alpha, \beta, \gamma, \delta$ and the equations (2), which are linear and homogeneous in $\alpha, \beta, \gamma, \delta$ give the solution. In turn $\alpha, \beta, \gamma, \delta$ determine the projectivity between r and r', that is the couple of corresponding points in the two straight lines.

This method may be generalized to the case when r and r' are plane curves projection of a given plane curve, and the following discussion allows to obtain the solution of this problem when the two plane curves are not given by means of equations but graphically which will also allow to obtain the solution with simpler algebra.

3. The problem set in the section 1 for two straight lines may be extended to two planes in the following manner. Let p and p' be two planes (see figure 2) $P(X,Y,Z)$ and $P'(X',Y',Z')$ two generic points off the two planes whose points are corresponding in a projectivity Λ ; we will determine the couple P and P' such that for any couple Q and Q' , in p and p', corresponding in the projectivity Λ, the four points P, P', Q, Q' are coplanar. We will proceed as previously when discussing the analogous problem for the stright lines previously discussed.

Let us then assume that the plane p is the xy plane of the Cartesian system, let

$$X = au'+a'v'+a''= x(u',v')$$

$$Y = bu'+bv'+b''= y(u',v') \quad (3)$$

$Z = cu'+c'v'+c'' = z(u',v')$

be the parametric equations of the plane *p'* and let

$u'=(\alpha u+\alpha'v+\alpha'')/(\gamma u+\gamma'v+\gamma'')$

$v'=(\beta u+\beta'v+\beta'')/(\gamma u+\gamma'v+\gamma'')$ (4)

be the equations of the projective correspondence between the plane *p* and *p'* where *u,v,u'* and *v'* are Cartesian coordinates in the planes *p* and *p'* with *u,v* such that the *x* and *y* axes coincide with the u and v axes respectively.

The conditions that the points *P, P', Q, Q'* be coplanar (that is the sets of straight lines through *P* and *P'*, obtained projecting the corresponding points of *p* and *p'*, be a perspectivity) may be written

$$\begin{vmatrix} X & Y & Z & 1 \\ X' & Y' & Z' & 1 \\ x(u',v') & y(u',v') & z(u',v') & 1 \\ 0 & 0 & t & 1 \end{vmatrix} \equiv 0 \qquad (5)$$

where the coordinates of *Q* and *Q'* are as Q(u,v,0), *Q'(x(u',v'),y(u',v'),z(u',v'))*.

We exclude the case when *p* and *p'* are parallel and assume the reference systems such that the *x* axis and the *u'* axis be parallel to the line of intersection of the planes *p* and *p'* (as in figure 2); then in equations (3) *x = u'* and *y* and *z* depend only on *v'* as follows

$x = v'$

$y= b'v'+b''$ (3')

$x=c'v'+c''$

which imply that the identity (5) may be written as

$$\begin{vmatrix} X & Y & Z \\ X' & Y' & Z' \\ (\alpha u+\alpha'v+\alpha'')/(\gamma u+\gamma'v+\gamma'') & b'(\beta u+\beta'v+\beta'')/(\gamma u+\gamma'v+\gamma'')+b'' & c'(\beta u+\beta'v+\beta'')/(\gamma u+\gamma'v+\gamma'')+c \\ u & v & 0 \end{vmatrix}$$

$\equiv 0$ (5')

where now the identity is meant for the variables u and v.

Subtracting the last row from the first three and expanding with the elements of the last column, the matrix (5') is reduced to the following one

$$\begin{vmatrix} X-u & X'-u & \alpha u+\alpha'v+\alpha''-u(\gamma u+\gamma'v+\gamma'') \\ Y-v & Y'-v & b'(\beta u+\beta'v+\beta'')+(b''-u)(\gamma u+\gamma'v+\gamma'' \\ Z & Z' & c'(\beta u+\beta'v+\beta'')+c''(\gamma u+\gamma'v+\gamma'') \end{vmatrix} \equiv 0 \qquad (5'')$$

In order to satisfy the identity (5'') for all couples u, v, must be nil all coefficients of u, v, uv, u^2, v^2 and the constant term, which give the following equations

$$(c'\beta+c''\gamma)(Y-Y')-(b'\beta+b''\gamma)(Z-Z')+\gamma(ZY'-Z'Y)=0$$

$$(c'\beta'+c''\gamma')(X-X')+\alpha'(Z-Z')+\gamma'(Z'Y-ZY')=0$$

$$(c'\beta+c''\gamma)(X'-X)-(c'\beta'+c''\gamma')(Y-Y')+\alpha'(Z-Z')+\gamma(Z'X-ZX')+$$
$$(Z'-Z)(b\beta'+b''\gamma')+\gamma'(ZY'-Z'Y)=0$$

$$(c'\beta''+c''\gamma'')(Y-Y')-(b'\beta''+b''\gamma')(Z'-Z)+)+(\alpha-\gamma'')(Z'Y-ZY')+$$
$$+(c'\beta+c''\gamma)(XY'-X'Y)+(b'\beta+b''\gamma)(ZX'-Z'X)=0$$

$$(c'\beta''+c''\gamma'')(X'-X)+\alpha''(Z-Z')+\alpha'(Z'Y-ZY')+(c'\beta'+c''\gamma')(XY'-X'Y)+ \quad (6)$$
$$(b'\beta''+b''\gamma'-\gamma'')(ZX'-Z'X)=0$$

$$\alpha''(Z'Y-ZY')+(c'\beta''+c''\gamma'')(XY'-X'Y)+(b'\beta''+b''\gamma'')(ZX'-Z'X)=0$$

We note that the 6 equations of system (6) are not independent and are equivalent to five equations as we show in the following. To this purpose let us consider the inverse problem; that is given p, p', P and P' find the equations of the projectivity Ω between p and p' such that a generic couple of points Q and Q', corresponding in Ω, are coplanar with P and P', Q, Q' for all couples Q,Q'. Projectivities of such type may be obtained projecting over p and p' from P and P' respectively the ∞^3 planes of the space, then the system in which we consider α, α', α'', β, β', β'', γ, γ', γ'' as unknown has at least ∞^3 solutions. We may then conclude that since the system of 6 homogeneous equations (6) in the 9 unknowns α, α', α'', β, β', β'', γ, γ', γ'' has at least ∞^3 solutions, at most 5 equations are independent.

We note that the space of the intersections of the couple of straight lines obtained projecting the corresponding points of p and p' from P and P' respectively is a developable surface where each straight line intersects all the others; this surface is therefore a plane. It is thus clear that the projectivities of the type Ω are only those which one obtains projecting over p and p' from P and P' respectively the ∞^3 planes of the space, which in turn implies that the 6 equations of system (6) are actually 5 independent equations.

We have thus established that the projectivities between the planes p and p' obtained projecting from P and P' respectively the ∞^3 planes of the space have equations (4) whose coefficients satisfy equations (6). Given in an appropriate Cartesian system the planes p and p' and the points P and P', the coefficients of the equations of these projectivities are related by 5 independent homogeneous equations and is possible to express the 9 coefficients by means of 3 of them for instance $\gamma, \gamma', \gamma''$. One obtains thus the equations (4) expressed in such form that the coefficients appearing in them depend linearly from $\gamma, \gamma', \gamma''$.

4. Before proceeding to the computation which will lead to the determination of the correspondence between the planes *p* and *p'* few considerations are useful. It is clear that three couples of points on *p* and *p'* respectively which are projection from *P* and *P'* of three points located on a generic plane s, but not on a straight line, of the space determine uniquely a correspondence of the type Ω.

Moreover if we want to proceed to the numerical computation of the coefficients of such projectivity using these elements, it is sufficient to substitute in equations (4) (here all the coefficients are expressed as functions of $\gamma, \gamma', \gamma''$), the coordinates *u, v, u', v'* of these three couples of corresponding points; we find thus 6 linear equations in the 3 unknowns $\gamma, \gamma', \gamma''$. It is clear that only 3 of these 6 equations are independent; in fact, with reference with figure 3 we see that the couple *Q* and *Q'* cannot be given arbitrarily since, when we chose the couple *Q, Q'* we must find the straight line $Q_0 C_2$ (due to the nodal points theorem); thus fixing *Q* the point *Q'* depends on only one parameter. That is each point of a corresponding couple determines only one of the parameters $\gamma, \gamma', \gamma''$ and not 2 as it may seem when substituting the coordinates *u, v, u', v'* in the equations (4) (where the coefficients are all expressed as functions of $\gamma, \gamma', \gamma''$) finding two equations relating $\gamma, \gamma', \gamma''$.

It is thus clear that 3 couples of corresponding points on *p* and *p'* respectively, which are not on the same straight line) determine one and only one projectivity.

5. Going to the application assume that the planes *p* and *p'* are photographic plates and P and P' the objectives of the cameras where are set the plates, It is clear that if we take a picture of a plane trajectory and we have in the plates 3 couples of points resulting from 3 points on the trajectory we may assume to have a picture of the plane of the trajectory and with the 3 couples of the corresponding points single out the projectivity between the two plates. The by means of these 3 couples of corresponding points we may obtain all the couples of corresponding points which result from the same points of the trajectory considering them as corresponding point in the projectivity established between the plates.

As an application of the results previously obtained let us consider the case when the cameras have the objective at the same elevation and, indicating with p and p' the plates and O_1 and O_2 the centres of projections, that is the centres of the objectives, assume the reference systems $O(x, y, z)$; u, v, u', v' such that O_1 coincides with the first principal point of the first plate C_1, $x = u$, $y = v$ and the origin of the system u', v' coinciding with the principal point of the second plate C_2 as shown in the figure 4.

Moreover let us indicate with $d_1 = O_1 C_1 = O_2 C_2$ the principal distances, with ω the angle $p'y$, with g the angle $p'z$ and b_1, b_2 the angles, reduced to the horizon of the planes p and p'. It is seen that

$$g = 90° - \omega = 90° - 180° + (b_1 + b_2) = -90° + (b_1 + b_2) \qquad (7)$$

and the coefficients of the equations (3') are

$$b' = -\cos(b_1 + b_2) \qquad b'' = y_2 + d\sin(b_1 + b_2)$$

$$c' = \sin(b_1 + b_2) \qquad c'' = z_2 - d\cos(b_1 + b_2) \qquad (8)$$

Indicating with $0, y_2, z_2$ the coordinates of O_1 and with $0, 0, d$ those of O_2 the equation (5') becomes

$$\begin{vmatrix} 0 & 0 & d & 1 \\ 0 & y_2 & z_2 & 1 \\ \dfrac{(\alpha u+\alpha' v+\alpha'')}{(\gamma u+\gamma' v+\gamma'')} & \dfrac{b'(\beta u+\beta' v+\beta'')}{(\gamma u+\gamma' v+\gamma'')}+b'' & \dfrac{c'(\beta u+\beta' v+\beta'')}{(\gamma u+\gamma' v+\gamma'')}+c'' & 1 \\ u & v & 0 & 1 \end{vmatrix} \equiv 0 \qquad (9)$$

Subtracting the first row form the others and expanding with the elements of the last column we find

$$\begin{vmatrix} 0 & \alpha u+\alpha' v+\alpha'' & u \\ y_2 & b'(\beta u+\beta' v+\beta'')+b''(\gamma u+\gamma' v+\gamma'') & v \\ z_2-d & c'(\beta u+\beta' v+\beta'')+(c''-d)(\gamma u+\gamma' v+\gamma'') & -d \end{vmatrix} \equiv 0 \qquad (9')$$

where the identity is with reference to the variables u and v. Expanding the determinant into an equation and setting to zero the coefficients of all coefficients of u, v, uv, u^2, v^2 and the constant term, we obtain the following system

$$\begin{aligned}
&(d-z_2)(b'\beta+b''\gamma) + y_2(c'\beta+c''\gamma) - y_2 d\gamma = 0 \\
&(d-z_2)\alpha' = 0 \\
&(d-z_2)(b'\beta'+b''\gamma-\alpha) + y_2(c'\beta'+c''\gamma') - y_2 d\gamma' = 0 \\
&y_2 d\alpha + (d-z_2)(b'\beta''+b''\gamma'') + y_2(c'\beta''+c''\gamma'') - y_2 d\gamma'' = 0 \qquad (10) \\
&(d-z_2)\alpha'' - y_2 d\alpha' = 0 \\
&y_2 d\alpha'' = 0
\end{aligned}$$

To simplify the form of the solution we set

$y_2 d = B$

$d - z_2 = A$ (11)

$(d\text{-}z_2)b' \text{-} y_2\, c' = M$

$(d\text{-}z_2)b'' + y_2\,(c'' - d) = N$

and find

$B\alpha + M\beta'' + +N\,\gamma'' = 0$

$-A\alpha + M\beta' + N\gamma' = M$ (10')

$M\beta + N\gamma = 0$

$\alpha' = \alpha'' = 0$

and setting

$\alpha = 1$

we find

$\beta'' = -\,(N\gamma'' + B)/M$

$\beta' = -(N\gamma' + A)/M$ (12)

$\beta = -N\gamma/M$

$\alpha' = \alpha'' = 0$

The result is that the projectivity between the planes p and p' which one obtains with takes of a generic plane is of the type:

$u = u'/(\gamma u' + \gamma' v' + \gamma'')$

$v = 1/M - (N\gamma u' + (A - N\gamma')v' - B - N\gamma)/(\gamma u' + \gamma' v' + \gamma'')$ (13)

In equations (13) the coefficients A, B, M, N, obtained from equation (11) are determined by means of d, y_2, z_2, b_1, b_2 appearing in equation (8) which should be measured before the shooting. In order to determine γ, γ', γ '' one should have 3 couples of corresponding points on p and p' which result from 3 not aligned points of the plane of the trajectory.

If the cameras are opened at the same time and closed at the same time the initial and end points of the trajectory give 2 couples of corresponding points. In order to obtain the third couple one may close the cameras at the same time for a very short interval which gives the third couple of points.

Having obtained the 3 couples of corresponding points, substituting their coordinates

In equations (13) which give the projectivity between p and p', we find 6 equations, in the 3 unknowns $\gamma, \gamma', \gamma''$, of which only 3 are independent. Considering that the measures are affected by errors one may use all six equations with the least square method. Thus the coefficients γ, γ', γ'' and the equations of the projectivity (13) are determined and one may proceed to the determination of the couples of corresponding points in the frames.

References

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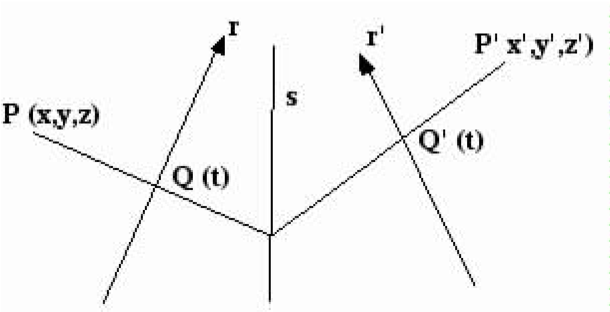

Fig. 1. *Q* and *Q'* are corresponding points in the projectivity relating *r* and *r'*. When *Q* and *Q'* and the projecting points *P* and *P'* are coplanar the projectivity inferred on *s,* the intersection of the planes *Pr* and *P'r*', is a perspectivity.

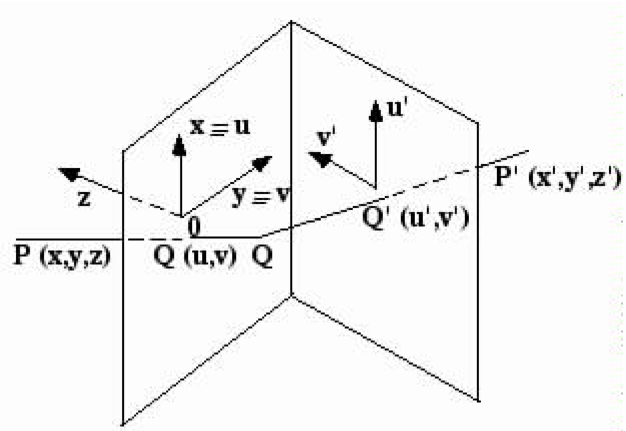

Fig. 2. *Q(u,v)* and *Q'(u',v')* are corresponding points in the projectivity relating *p* and *p'*.

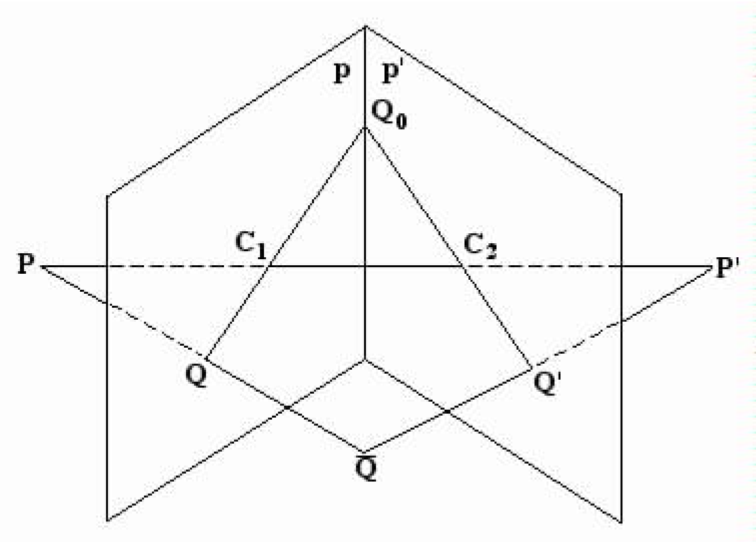

Fig. 3. *P* and *P'* are the projecting points in the frames *p* and *p'*. Because of the theorem of the nodal points the couple of corresponding points in *p* and *p'* respectively may not be given arbitrarily since the lines QC_1 and $Q'C_2$ must intersect.

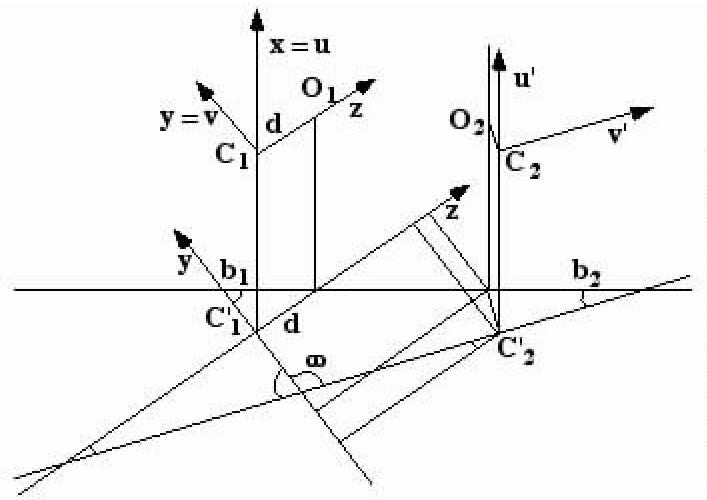

Fig. 4. The frames are in the planes *x,y* and *x',y'* and O_1 and O_2 are the centres of the objectives of the cameras. The origins of the Cartesian systems C_1 and C_2 must be in the nodal points of the respective cameras. The principal distances $O_1 C_1$ and $O_2 C_2$ must be equal. ω is the angle *p'y*, *g* is the angle *p'z* and b_1 and b_2 are the angles of *p* and *p'* respectively reduced to the horizon.