# Monotonicity of Two-Band Spectral Vegetation Index in General Form Under a Two-Endmember Linear Mixture Model

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**Abstract – Spectral vegetation indices using red and NIR bands (two-band VIs) have been used as information for vegetation monitoring. However, spatially averaged values of two-band VI have biases (known as scaling effect) that are induced by surface heterogeneity and nonlinearity of VI's model equations. This study tries to understand the mechanism of the scaling effect for a general form of two-band VI assuming a two-endmember linear mixture model. It is proved that spatially averaged VI value changes monotonically for a certain sequence of resolution cases. The findings (monotonicity of two-band VI) lead us to identify an error bound of two-band VI caused by the scaling effect. The derived results would also provide useful information for cross-calibration among VI products from various sensors/platforms.**

**Keywords:** scaling effect, two-band VI, linear mixture model, monotonicity, spatial resolution, NDVI, SAVI, EVI2

## 1. INTRODUCTION

Spectral vegetation index (VI) involving band algebraic manipulation plays an important role in monitoring vegetation status, which also used as the proxy of biophysical parameters such as leaf area index, fraction of vegetation cover, and photosynthetically active radiation (Goward, 1991; Jiang, 2008). The VIs using red and near infrared (NIR) bands, called two-band VI has been frequently employed for various purposes owing to its simplicity. For example, Normalized difference vegetation index (NDVI) (Tucker, 1979), soil adjusted vegetation index (SAVI) (Huete, 1988), and recently introduced enhanced vegetation index 2 (EVI2) (Jiang, 2007) fall into this category.

Monitoring results by the two-band VIs would suffer from uncertainties caused by scaling effect. It has been known that the sources of uncertainties are surface heterogeneity and nonlinearity of VI model equation regarding spatial parameters. The scaling effect would expect to be observed regardless level of spatial resolution as long as heterogeneity exists at subpixel level. For example, let's consider observation of vegetation canopy by submeter resolution. Even for this case, heterogeneity would still exist due to texture differences in leafs, shade, and stems.

This study is to investigate the mechanism of scaling effect in two-band VIs. To facilitate our analytical approach, it is assumed that the surface is composed of two endmemers, namely, vegetation and non-vegetation. We first model variation of area-averaged two-band VIs along with the number of pixels (representing spatial resolution) within a fixed area. Our focus is its monotonicity as a function of spatial resolution for two reasons. If two-band VI changes monotonically, 1) error bound of scaling effect in two-band VI can be identified theoretically, and 2) scaling effect can be inferred based on the reflectance spectrum at any resolution (sub-meter to 1 km). Information about the monotonic behavior would help to cross-calibrate VI products among sensors of different spatial resolution. The objective of this study is to analytically prove the monotonic behavior of two-band VI (expressed in a general form) along with spatial resolution.

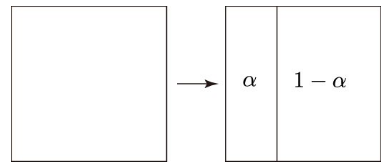


Figure 1. Illustration of resolution transfer based on our framework. Resolution level shifts from resolution level 1 into 2.

## 2. BACKGROUND

The monotonic behavior of NDVI as a function of spatial resolution had been investigated under the two-endmember assumption (Yoshioka, 2008). Yoshioka et al. concluded that NDVI changes monotonically along with resolutions, if the sequence of resolution cases satisfies a certain condition. The condition imposed on the resolution sequence is expressed as a rule regarding pixel partitioning process to model resolution transfer illustrated in Fig. 1. For any sequence of resolution cases produced by the partitioning rule, the spatially averaged NDVI changes monotonically from coarser to finer resolutions. As a result, the maximum or minimum occurs at the coarsest or highest resolution cases. We define a series of resolution as a 'resolution class'. And each resolution level is represented by a number of pixel within a fixed area. Moreover, by comparing spatially

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Table A. Coefficients in Eq. (2) for various two-band VIs. The constant $L$ is often assumed to be 0.5.

| | $p_1$ | $p_2$ | $q_1$ | $q_2$ | $r_1$ | $r_2$ | $k$ |
|---|---|---|---|---|---|---|---|
| NDVI | 1 | 1 | -1 | 1 | 0 | 0 | 1 |
| SAVI | 1+$L$ | 1 | -1-$L$ | 1 | 0 | $L$ | 1 |
| EVI2 | 2.5 | 1 | -2.5 | 2.4 | 0 | 1 | 2.4 |

averaged NDVI between two consecutive resolutions, level $j$ and $j+1$, Yoshioka et al. also proved that the trend of averaged NDVI can be determined from comparing one-norm between vegetation and non-vegetation endmembers as written in Eq. (1),

$$V_{i+1}\begin{cases}\geq V_i, & \text{if} \quad \|\boldsymbol{\rho}_V\|_1 < \|\boldsymbol{\rho}_S\|_1, \\ = V_i, & \text{if} \quad \|\boldsymbol{\rho}_V\|_1 = \|\boldsymbol{\rho}_S\|_1, \\ \leq V_i, & \text{if} \quad \|\boldsymbol{\rho}_V\|_1 > \|\boldsymbol{\rho}_S\|_1, \end{cases} \tag{1}$$

where $V_i$ and $V_{i+1}$ are area-averaged NDVI at resolution levels, $i$ and $i+1$, respectively. Variables $\boldsymbol{\rho}_V = (\rho_{V,R}, \rho_{V,N})$ and $\boldsymbol{\rho}_S = (\rho_{S,R}, \rho_{S,N})$ represent vegetation and non-vegetation endmember spectra by red and NIR reflectance. Again, note that Eq. (1) implies that a value of spatially averaged NDVI changes monotonically as spatial resolution becomes higher when the target field is composed of two components regardless of fraction or distribution of endmember components.

Our next step is to prove the monotonicity of two-band VI which is represented by a general form. The proof will be provided by extending the theoretical framework for NDVI in the next section.

## 3. SCALING EFFECT ON TWO-BAND VI

### 3.1 Transformation of General Form

The general representation of two-band VI can be written as a function of reflectance spectrum, $\boldsymbol{\rho} = (\rho_R, \rho_N)$ as follows,

$$v = \frac{p_1\rho_N + q_1\rho_R + r_1}{p_2\rho_N + q_2\rho_R + r_2}, \tag{2}$$

where the coefficients $p_1$, $p_2$, $q_1$, $q_2$, $r_1$, and $r_2$ depend on a choice of two-band VI summarized in Table A. Equation (2) can be transformed into

$$v' = p_3 v = \frac{\rho_N + q_3\rho_R + r_3}{\rho_N + q_4\rho_R + r_4}. \tag{3}$$

where

$$p_3 = p_1 / p_2, \tag{4a}$$

$$q_3 = q_1 / p_1, \tag{4)b}$$

$$q_4 = q_2 / p_2, \tag{4)c}$$

$$r_3 = r_1 / p_1, \tag{4)d}$$

$$r_4 = r_2 / p_2. \tag{4)e}$$

Then reflectance spectrum by red and NIR band, $\boldsymbol{\rho}$ is transformed into $\boldsymbol{\rho}' = (\rho'_R, \rho'_N)$ by multiplying a transformation matrix, $\boldsymbol{T}$

$$\boldsymbol{\rho}' = \boldsymbol{T}\boldsymbol{\rho}, \tag{5}$$

$$\boldsymbol{T} = \begin{bmatrix} -q_3 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6}$$

VI value in Eq. (3) can also be rewritten as a function of $\boldsymbol{\rho}'$, using $q_5 = -q_4 / q_3$, as

$$v' = \frac{\rho'_N - \rho'_R + r_3}{\rho'_N + q_5\rho'_R + r_4}, \tag{7}$$

The spectrum $\boldsymbol{\rho}'$ is then transformed into $\boldsymbol{\rho}'' = (\rho''_R, \rho''_N)$ by adding a translation matrix, $\boldsymbol{D}_1$ as

$$\boldsymbol{\rho}'' = \boldsymbol{\rho}' + \boldsymbol{D}_1, \tag{8}$$

where $\boldsymbol{D}_1$ is

$$\boldsymbol{D}_1 = \begin{bmatrix} \dfrac{r_3}{2} \\ \dfrac{-r_3}{2} \end{bmatrix}. \tag{9}$$

Equation (7) is represented by $\boldsymbol{\rho}''$ as

$$v' = \frac{\rho''_N - \rho''_R}{\rho''_N + q_5 \rho''_R + r_5}, \tag{10}$$

where $r_5$ is defined by

$$r_5 = r_4 + \frac{(q_5 - 1)}{2} r_3. \tag{11}$$

Again, the spectrum $\boldsymbol{\rho}''$ is transformed into $\boldsymbol{\rho}''' = (\rho''_R, \rho''_N)$ by adding a translation matrix, $\boldsymbol{D}_2$ as

$$\boldsymbol{\rho}''' = \boldsymbol{\rho}'' + \boldsymbol{D}_2, \tag{12}$$

where $\boldsymbol{D}_2$ is defined by

$$\boldsymbol{D}_2 = \frac{-r_5}{1+q_5}\begin{bmatrix}1\\1\end{bmatrix}. \tag{13}$$

Final form of the transformed equation for two-band VI becomes

$$v' = \frac{\rho'''_N - \rho'''_R}{\rho'''_N + q_5 \rho'''_R}. \tag{14}$$

The coordinate system of transformed space consist of $\boldsymbol{\rho}''' = (\rho'''_R, \rho'''_N)$ can be restored back into the original coordinate ( $\boldsymbol{\rho} = (\rho_R, \rho_N)$ ) by

$$\boldsymbol{\rho} = \boldsymbol{T}^{-1}(\boldsymbol{\rho}''' - \boldsymbol{D}_1 - \boldsymbol{D}_2). \tag{15}$$

It implies that if the $v'$ is monotonic along with resolution changes, $v$ (the VI value in the original domain) is also monotonic. In the following subsection, we focus on the proof of monotonicity regarding $v'$ in Eq. (14).

**3.2 Monotonicity of $v'$ defined by Eq. (14)**

A reflectance spectrum is modeled by weighted sum of the two endmember spectra. The two-band VI under the two-endmember model is written as a function of modeled spectrum, $\boldsymbol{\rho}_M = (\rho_{M,R}, \rho_{M,N})$ based on Eq. (14) as

$$v = \frac{\rho_{M,N} - \rho_{M,R}}{\rho_{M,N} + k\rho_{M,R}}, \tag{16}$$

where $k = q_5$ is a coefficient depending on a choice of VI (corresponds to $q_5$ in Eq. (14)). The modeled spectrum $\boldsymbol{\rho}_M$ is represented by a weighted sum of vegetation and non-vegetation endmember spectrum, $\boldsymbol{\rho}_V = (\rho_{V,R}, \rho_{V,N})$ and $\boldsymbol{\rho}_S = (\rho_{S,R}, \rho_{S,N})$ using the weight as a fraction of vegetation cover (FVC) $\omega$ as

$$\boldsymbol{\rho}_M = \omega\ \boldsymbol{\rho}_V + (1-\omega)\ \boldsymbol{\rho}_S. \tag{17}$$

Under the LMM framework, two-band VI represented by Eq. (14) then becomes

$$v = \frac{\omega(\rho_{V,N} - \rho_{S,N} - \rho_{V,R} + \rho_{S,R}) + \rho_{S,N} - \rho_{S,R}}{\omega(\rho_{V,N} - \rho_{S,N} + k\rho_{V,R} - k\rho_{S,R}) + \rho_{S,N} + k\rho_{S,R}}. \tag{18}$$

To prove the monotonicity, we should compare averaged two-band VI between resolution level 1 and 2 (illustrated in Fig. 1). The difference in the values between the two resolutions, $\Delta \boldsymbol{v}$ , is written by

$$\Delta v = \alpha v_{21} + (1-\alpha) v_{22} - v_{11}, \tag{19}$$

where $\alpha$ is the proportional area of one of the two pixels within the fixed area (Fig. 1). The variable $v_{ij}$ represents two-band VI for $j$ -th pixel at resolution level $i$ . The VI value for each pixel in each resolution level can be written as

$$v_{ij} = \frac{\omega_{ij}(\rho_{V,N} - \rho_{S,N} - \rho_{V,R} + \rho_{S,R}) + \rho_{S,N} - \rho_{S,R}}{\omega_{ij}(\rho_{V,N} - \rho_{S,N} + k\rho_{V,R} - k\rho_{S,R}) + \rho_{S,N} + k\rho_{S,R}}, \tag{20}$$

where $\omega_{ij}$ represents FVC for $j$ -th pixel at resolution level $i$. In order to examine behavior of $\Delta \boldsymbol{v}$ , we take a partial derivative of $\Delta v$ with respect to FVC in one of the pixels at resolution level 2, $\omega_{21}$. It becomes

$$\frac{\partial \Delta v}{\partial \omega_{21}} = \alpha (k+1)(\rho_{V,N}\rho_{S,R} - \rho_{V,R}\rho_{S,N})(S_1^{-1} - S_2^{-2}), \qquad (21)$$

where $S_j$ $(j = 1,2)$ is defined by

$$S_j = (\rho_{V,N} - \rho_{S,N} + k\rho_{V,R} - k\rho_{S,R})\omega_{2j} + \rho_{S,N} + k\rho_{S,R} \qquad (22)$$

From Eqs. (21) and (22), the sign of the partial derivative (left-hand-side of Eq. (21)) can be determined based on the variable defined by Eq. (24) below. In summary, we have

$$v_2 \begin{cases} \geq v_1, & \text{if} \quad \eta_k < 1, \\ = v_1, & \text{if} \quad \eta_k = 1, \\ \leq v_1, & \text{if} \quad \eta_k > 1, \end{cases} \qquad (23)$$

where

$$\eta_k = \frac{\rho_{V,N} + k\rho_{V,R}}{\rho_{S,N} + k\rho_{S,R}}. \qquad (24)$$

As mentioned before, multiple use of partitioning rule results in any resolution levels. Therefore, the relationship of spatially averaged two-band VI between resolution level $i$ and $i+1$ can be summarized as

$$v_{i+1} \begin{cases} \geq v_i, & \text{if} \quad \eta_k < 1, \\ = v_i, & \text{if} \quad \eta_k = 1, \\ \leq v_i, & \text{if} \quad \eta_k > 1. \end{cases} \qquad (25)$$

Equation (25) indicates that averaged two-band VI changes monotonically as a function of spatial resolution under the two-endmember assumption regardless of distribution of vegetated surface. Moreover, we found that the trend of VI variation (either increasing or decreasing) simply depends on $\eta_k$. For NDVI and SAVI, the coefficient $k$ becomes unity, while that of EVI2 is not unity (Table A). It implies that NDVI and SAVI will show similar behavior, while the behavior of EVI2 should be somewhat different from that of NDVI and SAVI.

## 3. DISCUSSION

Monotonic behavior of two-band VI as a function of spatial resolution was proved under the two-endmember assumption. The trend of the VI along with spatial resolution depends only on a factor $\eta_k$. The factor $\eta_k$ for the case of NDVI and SAVI is identical, indicating similar behavior for those VIs. On contrary, EVI2 with different value of $\eta_k$ from those two VIs is expected to show differences in its behavior.

In this study, we proved that the two-band VIs with the general form is indeed monotonic within a resolution class as introduced for NDVI previously. Although the number of endmember is limited to two, the fundamental aspect of the scaling effect helps us to further understand the mechanism of the scaling effect. Those limitations and restrictions imposed on the model need to be relaxed for practical application of our findings.

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