ON THE THEORETICAL ACCURACY OF BLOCK ADJUSTMENT BY THE METHOD OF POLYNOMIALS

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Abstract

In this paper the author analyses the theoretical accuracy of block adjustment by the method of polynomials. Accuracy of a newly determined point is represented here by the ratio of the variances of its coordinates and that of unit weight.

In the formula derivation some typical block figures with different distribution of control points are adopted. The results are finally compared with that of the block adjustments by independent models and by bundles.

Introduction

Till now, the block adjustment by the method of polynomials is the most commonly used method in the photogrammetric densification of control points in China. But the problems of accuracy estimation and consequently the necessary control point distribution for a block adjustment with polynomials have left much to be solved. In this analysis some typical block figures with different distribution of control points are adopted, with a view to find the variances of the thus determined coordinates of the new points. The photographs are arranged with a standard overlap of 60% and a side lap of 20%. The polynomials used in the adjustment are the second degree polynomials, the third degree polynomials as well as the conformal ones.

1. Formulae used for the variance estimations

In an adjustment with indirect observations, the observation equation in matrix form is

$$\underline{\underline{V}} = \underline{\underline{A}}\ \underline{\underline{X}} - \underline{\underline{L}} \tag{1}$$

and the normal equation is:

$$\underline{\underline{N}}\ \underline{\underline{X}} = \underline{\underline{C}}$$

with $\underline{\underline{N}} = \underline{\underline{A}}^{T}\ \underline{\underline{P}} \cdot \underline{\underline{A}}$, $\underline{\underline{C}} = \underline{\underline{A}}^{T}\ \underline{\underline{P}}\ \underline{\underline{L}}$ (2)

the covariance matrix of the parameter $\underline{\underline{X}}$ is:

$$\underline{\underline{\sigma}}^{2}_{x} = \sigma_0^2\ \underline{\underline{N}}^{-1} \tag{3}$$

where

σ_0^2 : variance of unit weight;

$\underline{\underline{N}}$: coefficient matrix of the parameter $\underline{\underline{X}}$ in the normal equation;

$\underline{\underline{P}}$: weight matrix of the observation vector $\underline{\underline{L}}$.

$$\underline{\underline{\sigma}}^{2}_{x} = \begin{bmatrix} \sigma_{x_1 x_1} & \sigma_{x_1 x_2} & \cdots & \sigma_{x_1 x_t} \\ \sigma_{x_2 x_1} & \sigma_{x_2 x_2} & \cdots & \sigma_{x_2 x_t} \\ \vdots & & & \\ \sigma_{x_t x_1} & \sigma_{x_t x_2} & \cdots & \sigma_{x_t x_t} \end{bmatrix} \tag{4}$$

As for the corrections of non - linear deformation of each strip in this experiment, the following four set formulae are used for planimetry, and equation (5) and (6) are used for height:

second degree polynomial

$$\delta_x = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy \tag{5}$$

third degree polynomial

$$\delta_x = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 x^3 + a_6 x^2 y \tag{6}$$

appro. second degree conformal

$$\left.\begin{aligned} \delta x &= a_1 + o + a_3 x - a_4 y + a_5 x^2 - 2a_6 xy \\ \delta y &= o + a_2 + a_3 y + a_4 x + 2a_5 xy + a_5 x^2 \end{aligned}\right\} \qquad (7)$$

appro. third degree conformal

$$\left.\begin{aligned} \delta x &= a_1 + o + a_3 x - a_4 y + a_5 x^2 - 2a_6 xy + a_7 x^3 - 3a_8 x^2 y \\ \delta y &= o + a_2 + a_3 y + a_4 x + 2a_5 xy + a_6 x^2 + 3a_7 x^2 y + a_8 x^3 \end{aligned}\right\} \qquad (8)$$

For the j - th point in the i - th strip, the error equation in the case of equation (5) is as follows:

$$- v_{x_{ij}} = a_{0i} + a_{1i}\, x_{ij} + a_{2i}\, y_{ij} + a_{3i}\, x_{ij}^2 + a_{4i}\, x_{ij}\, y_{ij} - dx_{ij} - 1_{x\,ij} \qquad (9)$$

The error equations of the whole block are expressed in matrix form by:

$$\underline{\underline{V}} = \begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \end{bmatrix} \begin{bmatrix} \underline{\underline{a}} \\ \underline{\underline{dx}} \end{bmatrix} - \underline{\underline{1}} \qquad (10)$$

where

$\underline{\underline{a}}$: matrix of the parameters for the non - linear correction, consisting of a_{0i} , a_{1i} , ... a_{4i} for each strip;

$\underline{\underline{dx}}$: matrix of corrections for the approx. values of x - coordinates.

The normal equations

$$\begin{bmatrix} \underline{\underline{A}}^T \\ \underline{\underline{B}}^T \end{bmatrix} \begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \end{bmatrix} \begin{bmatrix} \underline{\underline{a}} \\ \underline{\underline{dx}} \end{bmatrix} - \begin{bmatrix} \underline{\underline{A}}^T & \underline{\underline{1}} \\ \underline{\underline{B}}^T & \underline{\underline{1}} \end{bmatrix} = 0$$

$$\begin{bmatrix} \underline{\underline{A}}^T \underline{\underline{A}} & \underline{\underline{A}}^T \underline{\underline{B}} \\ \underline{\underline{B}}^T \underline{\underline{A}} & \underline{\underline{B}}^T \underline{\underline{B}} \end{bmatrix} \begin{bmatrix} \underline{\underline{a}} \\ \underline{\underline{dx}} \end{bmatrix} - \begin{bmatrix} \underline{\underline{A}}^T & \underline{\underline{1}} \\ \underline{\underline{B}}^T & \underline{\underline{1}} \end{bmatrix} = 0$$

$$\begin{bmatrix} \underline{\underline{N}}_{11} & \underline{\underline{N}}_{12} \\ \underline{\underline{N}}_{12}^{T} & \underline{\underline{N}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{a}} \\ \underline{\underline{dx}} \end{bmatrix} - \begin{bmatrix} \underline{\underline{n}}_1 \\ \underline{\underline{n}}_2 \end{bmatrix} = 0 \qquad (11)$$

After the elimination of the parameters $\underline{\underline{a}}$, we obtain

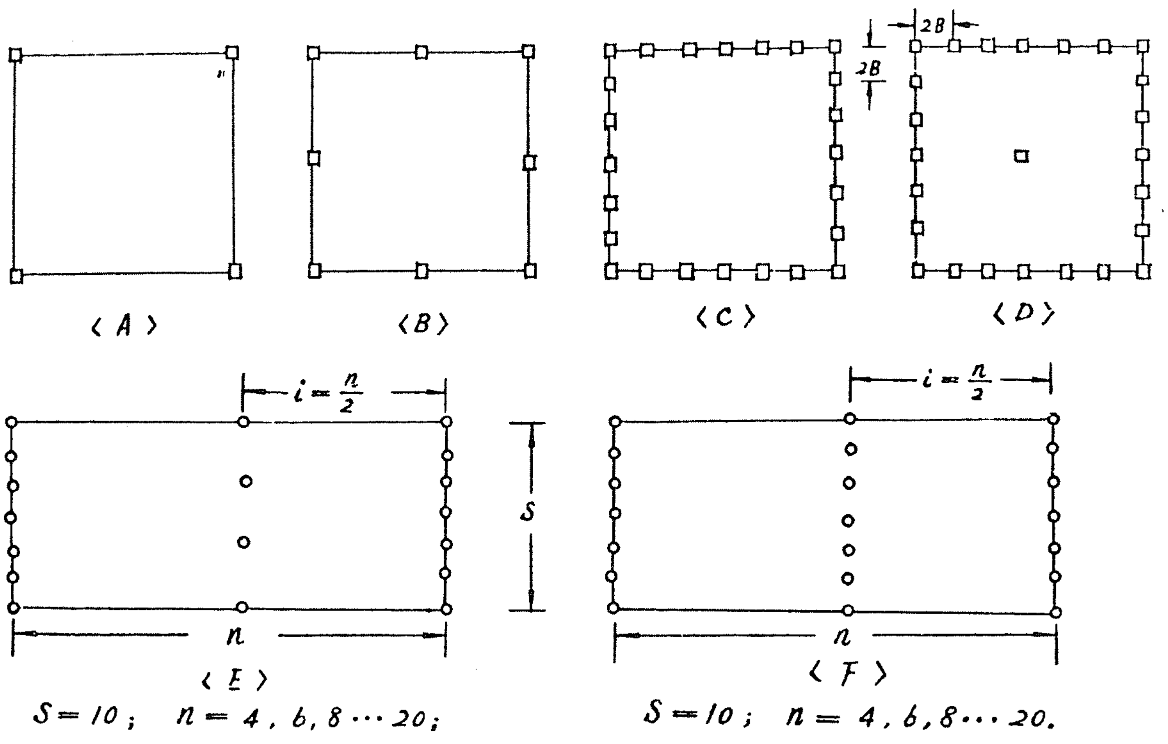
$$(\underline{\underline{N}}_{22} - \underline{\underline{N}}_{12}^{T} \underline{\underline{N}}_{11}^{-1} \underline{\underline{N}}_{12})\ \underline{\underline{dx}} = \underline{\underline{n}}_2 - \underline{\underline{N}}_{12}^{T} \underline{\underline{N}}_{11}^{-1} \underline{\underline{n}}_1$$

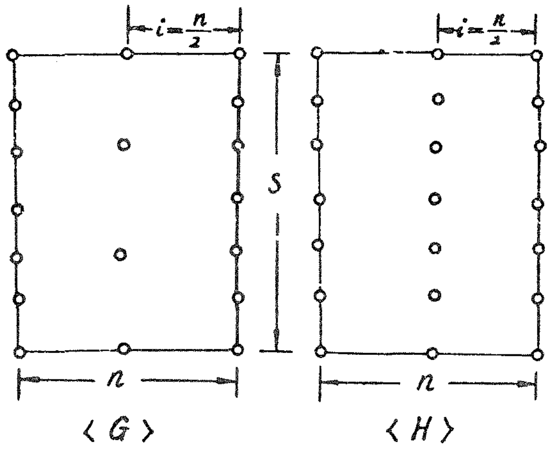
From equations (2), (3), we obtain the variance σ_x^2 :

$$\underline{\underline{\sigma}}_x^2 = \sigma_0^2 (\underline{\underline{N}}_{22} - \underline{\underline{N}}_{12}^{T} \underline{\underline{N}}_{11}^{-1} \underline{\underline{N}}_{12})^{-1} \qquad (12)$$

2. the Experiment

In the typical figure of a block, the model base is equal to B, and the spacing between strips is equal to 2B. (see Fig. 1) To





$n = 10$;

$S = 2, 4, 6, \cdots 20$;

□ : planimetric control point

o : height control point

n : number of models in a strip

s : number of strips in the block

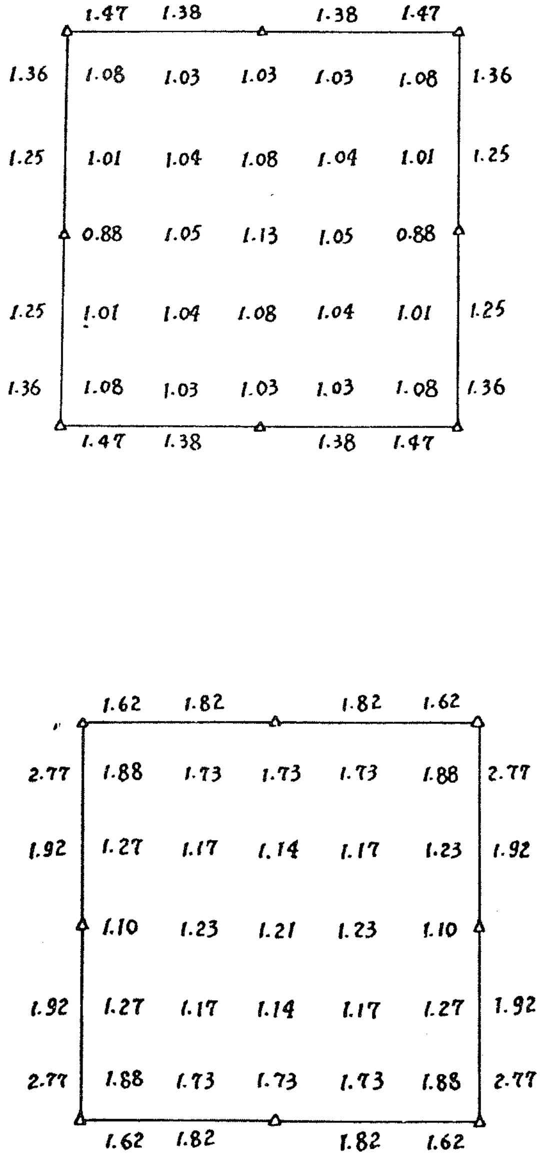
Fig. 1

carry out the computation eight schemes of control point distribution are adopted, among which schemes A, B, C, D are for the planimetric control (square blocks) while schemes E, F, G, H are for the heights.

Due to the limitation of the computer memory in our College, the variance computations are limited to the block sizes of from 4×4 (i.e. four strips of four models each), 6x6, 8x8, until 14×14 for the planimetry and the block sizes of the following two types for the heights.

1) The spacing (i) of height control points varies, while keeping the number of strips S constant, e.g. S = 10, the block sizes from 10X4, 10X6, until 10X20 (equivalent to i = 2, 3, ... 10).
2) The number of strips S in the block varies, while keeping the spacing of height control points constant, e.g. i = 10, the block sizes from 2X10, 4X10, until 20X10.

A part of the computation results is show in Fig. 2 and in Tables 1 → 3. The numerals in Fig. 2 as well as in Tables 1 → 3 are the ratios of the variances of the x - coordinates or the z - coordinates of the determined new points to that of the unit weight.



‹a› conformal second degree

‹b› conformal third degree

Fig.2. variance ratio of x coordinate.

Table 1: For the case of dense peripheral distribution of planimetric control points

block size \ formula / ratio of variances / scheme	C: second deg.	C: second deg. conformal	C: third deg. conformal	D: second deg.	D: second deg. conformal	D: third deg. conformal
	$\sigma^2_{x\,max}/\sigma_0^2$ $\sigma^2_{x\,mean}/\sigma_0^2$	$\sigma^2_{x\,max}/\sigma_0^2$ $\sigma^2_{x\,mean}/\sigma_0^2$	$\sigma^2_{x\,max}/\sigma_0^2$ $\sigma^2_{x\,mean}/\sigma_0^2$	$\sigma^2_{x\,max}/\sigma_0^2$ $\sigma^2_{x\,mean}/\sigma_0^2$	$\sigma^2_{x\,max}/\sigma_0^2$ $\sigma^2_{x\,mean}/\sigma_0^2$	$\sigma^2_{x\,max}/\sigma_0^2$ $\sigma^2_{x\,mean}/\sigma_0^2$
4 x 4	2.643 1.666	1.339 0.928	1.618 1.129	1.642 1.329	1.339 0.919	1.617 1.131
6 x 6	3.311 1.937	1.340 0.839	1.592 0.949	1.714 1.380	1.340 0.820	1.589 1.055
8 x 8	3.815 2.088	1.328 0.795	1.461 1.086	2.154 1.429	1.328 0.775	1.443 1.337
10 x 10	4.187 2.199	1.313 0.768	1.589 1.096	2.316 1.495	1.313 0.768	1.545 1.056
12 x 12	4.299 2.250	1.300 0.749		2.548 1.547	1.300 0.729	
14 x 14	4.561 2.264	1.287 0.721		2.550 1.552	1.287 0.703	
16 x 16	4.584 2.349			2.778 1.641		

Table 2 : For the case of sparse peripheral distribution of planimetric control points

block size \ formula / ratio of variances / scheme	A	B	
	$\sigma^2_{x\,max}/\sigma_0^2$ $\sigma^2_{x\,mean}/\sigma_0^2$	$\sigma^2_{x\,max}/\sigma_0^2$ $\sigma^2_{x\,mean}/\sigma_0^2$	
	conformal second deg.	conformal second deg.	conformal third deg.
4 X 4	10.130 6.830	1.558 1.256	1.964 1.455
6 x 6	13.879 8.606	2.609 1.536	17.416 7.019
8 X 8	16.405 9.850	2.583 1.588	33.420 10.388
10 X 10	18.236 10.792	2.978 1.657	40.913 11.685
12 X 12	20.023 11.478	3.078 1.699	
14 X 14		3.213 1.658	

Table 3 : Variance ratio for the heights

block size (formula / ratio of variances / scheme)	second degree $\sigma^2_{z\,max}/\sigma^2_0$ / $\sigma^2_{z\,mean}/\sigma^2_0$		block size (formula / ratio of variances / scheme)	second degree $\sigma^2_{z\,max}/\sigma^2_0$ / $\sigma^2_{z\,mean}/\sigma^2_0$	
	E	F		G	H
			2 X 10	1.744 / 1.521	1.654 / 1.336
10 X 4, i=2	1.642 / 1.192	1.580 / 0.934	4 X 10	1.723 / 1.301	1.654 / 1.121
10 X 6, i=3	1.702 / 1.158	1.595 / 0.941	6 X 10	1.723 / 1.212	1.654 / 1.029
10 X 8, i=4	1.721 / 1.142	1.622 / 1.096	8 X 10	1.723 / 1.162	1.654 / 0.978
10 X 10, i=5	1.723 / 1.131	1.654 / 0.945	10 X 10	1.723 / 1.131	1.654 / 0.945
10 X 12, i=6	1.719 / 1.122	1.681 / 0.945	12 X 10	1.723 / 1.109	1.654 / 0.923
10 X 14, i=7	1.712 / 1.115	1.703 / 0.946	14 X 10	1.723 / 1.093	1.654 / 0.906
10 X 16, i=8	1.724 / 1.109	1.722 / 1.167	16 X 10	1.723 / 1.081	1.654 / 0.893
10 X 18, i=9	1.741 / 1.114	1.737 / 0.944	18 X 10	1.723 / 1.072	1.654 / 0.883
10 X 20, i=10	1.755 / 1.099	1.749 / 0.943	20 X 10	1.723 / 1.063	1.654 / 0.875

From the analysis of Fig. 2 and Table 1—3, the following conclusions can be obtained:

1) It can be seen from Fig. 2 that the weakest point of densification do not lie in the centre of the block, but rather along the perimeters of the block when the non - linear corrections was made by the conformal formulae in the planimetric coordinate adjustment.

2) From Table 1 it is seen that the theoretical accuracy is best for the use of the second degree conformal formulae and is worst for the use of the ordinary polynomials in the planimetric adjustment.

3) The theoretical accuracy of the planimetric coordinates is less affected by the block size in case of dense peripheral control distribution (C, D, in the Fig. 1), while adopting the conformal formulae, the theoretical accuracy shows even gradual increase with the block size. (see from scheme C of Table 1).

On the contrary, the theoretical accuracy of the planimetric coordinate adjustment is decreasing considerably with the block size when the ordinary polynomials are used in the adjustment. (seen from scheme C of Table 1).

4) When one planimetric control point is added in the centre of the block with dense peripheral control, the theoretical accuracy of the planimetric coordinate is obviously increased in the case of the use of the second degree polynomials, while the theoretical accuracy is practically not affected in the case of conformal formulae. (compare scheme C of Table 1 with scheme D).

5) In the adjustment of planimetric coordinates with sparse peripheral control distribution (Fig. 1, A, B), the theoretical accuracy decreases cosiderably with the size of the block.

6) In the height adjustment, the theoretical accuracy of the determined heights is decreased with the enlargement of the spacing i between the height control points. (see scheme E in Table 3).

7) In the height adjustment when the spacing between the height control points is kept constant, the maximum variance ratio of the determined heights is practically constant with the increase of the number of strips in the block. The mean variance ratio is somewhat decreased. (see scheme G and H in Table 3).

8) The difference of the height control configuration between Fig. 1 E and Fig. 1 F affect the accuracy of the determined heights rather slightly.

In order to verify the correctness of the above conclusions, we have made some error computation with simulated photographs carrying with them fictitious random errors of $m_x = m_y = \pm 10\ \mu m$ in their image coordinates. The results are summarized in Tables 4 A and 4 B and have shown cosistency with the above conclusions

Table 4A

scheme / block size / accuracy	A		B		C		D	
	10X20	6X20	10X20	6X20	10X20	6X20	10X20	6X20
m_X (mm)	±0.067	±0.061	±0.048	±0.045	±0.035	±0.038	±0.036	±0.038
m_Y (mm)	±0.087	±0.102	±0.042	±0.040	±0.034	±0.034	±0.034	±0.037

Table 4B

scheme / space / block size / accuracy	E		E	
	$i = 5$		$i = 10$	
	10 X 20	6 X 20	10 X 20	6 X 20
m_Z (mm)	± 0.063	± 0.074	± 0.068	± 0.080

3. Formulae for the accuracy estimation

Based upon the preceding results, formulae for the accuracy estimation are derived for the case of the conformal second degree formulae and of the second degree polynomials in the planimetry and for the case of the second degree polynomials only in the heights.

From the data in Table 1 - 3, according to the least squares principle, fittings with straight lines are made. The results are as follows:

(1) Formulae for the accuracy estimation of the planimetric coordinates:

scheme A (conformal second degree formula)

$$\frac{\sigma^2_{x\,max}}{\sigma_o^2} = 6.077 + 1.207\ S \qquad (13)$$

$$\frac{\sigma^2_{x\,mean}}{\sigma_o^2} = 4.918 + 0.574\ S \qquad (13)\ a$$

scheme B (conformal second degree formula)

$$\frac{\sigma^2_{x\,max}}{\sigma_o^2} = 1.374 + 0.144\ S \qquad (14)$$

$$\frac{\sigma^2_{x\,mean}}{\sigma_o^2} = 1.235 + 0.037\ S \qquad (14)\ a$$

scheme C (the second degree polynomial)

$$\frac{\sigma^2_{x\,max}}{\sigma_o^2} = 2.342 + 0.157\ S \qquad (15)$$

$$\frac{\sigma^2_{x\,mean}}{\sigma_o^2} = 1.596 + 0.051\ S \qquad (15)\ a$$

scheme C (conformal second degree formula)

$$\frac{\sigma^2_{x\,max}}{\sigma_o^2} = 1.369 - 0.006\ S \qquad (16)$$

$$\frac{\sigma^2_{x\,mean}}{\sigma_o^2} = 0.971 - 0.019\ S \qquad (16)\ a$$

scheme D (second degree polynomial)

$$\frac{\sigma_{x\,max}^2}{\sigma_o^2} = 1.261 + 0.098\ S \qquad (17)$$

$$\frac{\sigma_{x\,mean}^2}{\sigma_o^2} = 1.232 + 0.025\ S \qquad (17)\ a$$

scheme D (conformal second degree polynomial)

$$\frac{\sigma_{x\,max}^2}{\sigma_o^2} = 1.369 - 0.006\ S \qquad (18)$$

$$\frac{\sigma_{x\,mean}^2}{\sigma_o^2} = 0.960 - 0.019\ S \qquad (18)\ a$$

(2) Formulae for the accuracy estimation of the heights:

scheme E

$$\frac{\sigma_{z\,max}^2}{\sigma_o^2} = 1.659 + 0.009\ i \qquad (19)$$

$$\frac{\sigma_{z\,mean}^2}{\sigma_o^2} = 1.190 + 0.010\ i \qquad (19)\ a$$

scheme F

$$\frac{\sigma_{z\,max}^2}{\sigma_o^2} = 1.536 + 0.022\ i \qquad (20)$$

$$\frac{\sigma_{z\,mean}^2}{\sigma_o^2} = 0.966 + 0.003\ i \qquad (20)\ a$$

scheme G

$$\frac{\sigma_{z\,max}^2}{\sigma_o^2} = 1.723 \qquad \text{(when } i=5) \qquad (21)$$

$$\frac{\sigma_{z\,mean}^2}{\sigma_o^2} = 1.462 - 0.046\ S \quad \text{(when } i=5) \qquad (21)\ a$$

scheme H

$$\frac{\sigma_{z\,max}^2}{\sigma_0^2} = 1.654 \qquad \text{(when i=5)} \qquad (22)$$

$$\frac{\sigma_{z\,mean}^2}{\sigma_0^2} = 1.281 - 0.047\ S \qquad \text{(when i=5)} \qquad (22)\ a$$

The accuracy of the straight line fitting above are listed as follows:

Table 5: Accuracy of the straight line fitting

formula \ accuracy	m_{fit}	formula \ accuracy	m_{fit}
(13)	0.74	(17)	0.11
$(13)_a$	0.39	$(17)_a$	0.02
(14)	0.30	(18)	0.01
$(14)_a$	0.10	$(18)_a$	0.03
(15)	0.25	(19)	0.02
$(15)_a$	0.09	$(19)_a$	0.01
(16)	0.01	(20)	0.01
$(16)_a$	0.02	$(20)_a$	0.09

From the formulae (14) a, (16) a, (18) a it can be seen that with the sparse peripheral control distribution, the mean variance of the planimetric coordinate is 1.6 times the variance of the unit weight (s=10), while with the dense peripheral control distribution, the mean variance of the planimetric coordinate is 0.8 times the variance of unit weight (s=10).

From the formulae (19) a, (20) a, it can be seen that with the spacing of the height control points i=10, the mean variance of heights is 1.3 times (scheme E) or one time (scheme F) the variance of unit weight.

4. Theoretical accuracy of the block adjustment with polynomials as compared with the independent model method and the bundle method

The formulae for accuracy estimation in the block adjustment with the method of independent models and of bundles are taken from foreign literatures and are listed below together with those obtained in this paper.

Theoretical accuracy of the planimetric coordinates

method \ scheme	A	B	C
polynomials	$\frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 4.92 + 0.575$	$\frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 1.24 + 0.0375$	$\frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.97 - 0.0195$
independent	$\frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.47 + 0.25\,n_s$	$\frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.83 + 0.05\,n_s$	$\frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.70 + 0.29\,lg\,n_s$
bundle	$\frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.53\,n_s$	$\frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.28 + 0.15\,n_s$	$\frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.87$

Theoretical accuracy of the heights

method \ scheme	F
polynomials	$\frac{\sigma_{z\,mean}^2}{\sigma_0^2} = 0.97 + 0.003\,i$
independent	$\frac{\sigma_{z\,mean}^2}{\sigma_0^2} = 0.34 + 0.22\,i$
bundle	$\frac{\sigma_{z\,mean}^2}{\sigma_0^2} = 0.93 + 0.19\,i$

For the scheme C, the planimetric accuracy of the three methods are abtained as follows (block size: 10 strips, 10 models / strip):

$$\left.\begin{aligned} &\text{polynomial} && \frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.78 \\ &\text{independent} && \frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.99 \\ &\text{bundle} && \frac{\sigma_{x\,mean}^2}{\sigma_0^2} = 0.87 \end{aligned}\right\} \qquad (23)$$

For the height when the spacing between the height control points is 10, the height accuracy of the three methods are obtained as follows:

$$\left.\begin{array}{lll} \text{polynomial} & \dfrac{\sigma_{z\,mean}^2}{\sigma_0^2} = 1.00 \\ \text{independent} & \dfrac{\sigma_{z\,mean}^2}{\sigma_0^2} = 2.54 \\ \text{bundle} & \dfrac{\sigma_{z\,mean}^2}{\sigma_0^2} = 2.83 \end{array}\right\} \qquad (24)$$

From the comparisons of the formulae (23), (24), it seens that the accuracy of the block adjustments with polynomials are not so much worse than that with the independent model methods. On the contrary, in the case of dense peripherial control distribution the theoretical accuracy of the adjustment with polynomials is even better than that of the independent model method. It is to be noticed that the σ_0^2 in the three methos of block adjustment are not equal. Further research works are needed to make the assertions more affirmative.

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