

Presented Paper

E. VOZIKIS and J. JANSKA  
Institute of Photogrammetry  
Technical University of Vienna  
Gußhausstraße 27-29  
A-1040 V i e n n a

PHOTOGRAPHIC WAY OF LAYING OUT ELLIPTIC CUPOLAS

SUMMARY

Specialists in monument preservation, picture restorers, and architects often need photographic layouts of cupolas. This task can be solved by applying digitally controlled orthophoto devices (such as the Avioplan OR 1), and the corresponding computer programs.

First, the cupola should be approximated by some mathematically defined surface. As such, a general ellipsoid is appropriate. Rotational ellipsoid and sphere are treated as special of this general solution. The parameters of the ellipsoid are determined via adjustment computations, applying some control points given in the three dimension space.

As the surface of an ellipsoid cannot be flattened, laying it out in a plane cannot be performed without distortions. There occur very small distortions when projecting the ellipsoid onto a polyhedron. In this, the projecting rays are kept perpendicular to the surface of the ellipsoid.

As a result, the following major tasks should be performed by the computer programs preparing control data for the Avioplan OR 1:

- to define a quadratic screen on the sides of the polyhedron,
- to orthogonally project the points of the polyhedron onto the ellipsoid,
- to centrally project the points of the ellipsoid onto the photo plane.

The theory and the extensive computer programs have been tried on black-and-white and color photographs taken in the church Karlskirche in Vienna. The geometrical precision and image quality of the resulting photographic layouts are quite satisfactory.

## PRELIMINARY REMARKS

The Wild Company presented a new instrument for differential rectification of photographs to the XIII-th congress of I.S.P. in Helsinki, the Avioplan OR 1 /6/. The software to it, SORA-OPS, developed by the Institute of Photogrammetry of the Technical University of Vienna, has been presented to the same Congress /5/, and it has been reported on some potential applications, as well /2/. The realization of some of these possible applications has been described in the special literature /3/, /4/, /7/, /8/.

The technology of laying out frescos painted on spherical cupolas with possible small distortions has been reported to the V-th International Congress of CIPA in Sibenik /7/. This technology is very useful in restoring frescos, and in their documentation (see the closing words of Dr. H. Foramitti /7/).

However, a large number of cupolas deviate from spheres, and therefore it seems to be necessary to extend the method mentioned to some surface capable of approximating most of them. As such, a general ellipsoid (with three different axis) is appropriate. But there being no way of exactly laying out the surface of an ellipsoid on a plane, some mediating surface has to be applied capable of a precise lay-out. In this case, a polyhedron suites these purposes.

Laying out the ellipsoid on a plane is performed in two major steps:

- first, the ellipsoid is projected on to the surface of the polyhedron, and
- second, the polyhedron is layed out on a plane.

For the aim of projecting the ellipsoid onto the polyhedron, an orthogonal to the surface of the ellipsoid projection is especially suitable. x)

## DEFINITION OF THE POLYHEDRON

As already mentioned, a polyhedron suites especially well our purposes. Its orientation relative to the ellipsoid can be made clear by intersecting the ellipsoid by two groups of planes ( $H_i$ ) and ( $Z_j$ ) (fig. 1). The planes ( $H_i$ ) are so arranged to cut off arc sections of equal length  $S_2$  on the major meridian (fig. 2). The angle  $d\lambda_j$  between two neighbouring planes ( $Z_j$ ), ( $Z_{j+1}$ ) corresponds to a constant arc length of  $S_1$  on the equatorial ellipse (fig. 2).

Points 1,2,3 and 4 are corner points of a plane of the polyhedron. One such plane is referred to as templet. Each templet is provided with its own coordinate system  $x_t, y_t, z_t$  (fig. 3). Side 1-3 defines the  $y_t$ -axis; the axis  $x_t$  is perpendicular to  $y_t$ , and lies within the plane 1-2-3-4; finally the  $z_t$ -axis is perpendicular to this plane.

x) A projection orthogonal to the planes of the polyhedron, or a central projection from the middle point of the ellipsoid both result in gaps at the edges of the polyhedron.

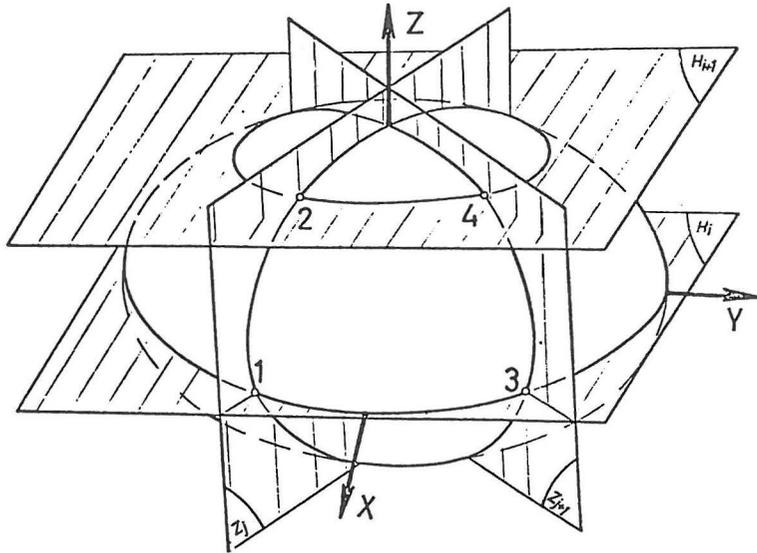


Fig. 1: Definition of the polyhedron

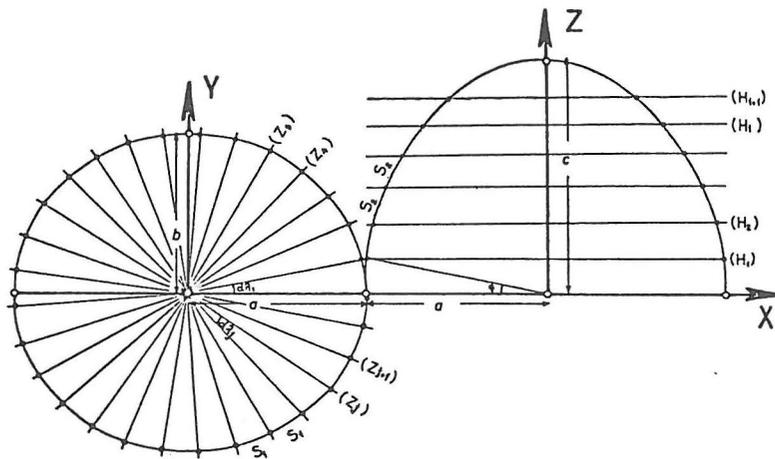


Fig. 2: Equator ellipse and major meridian

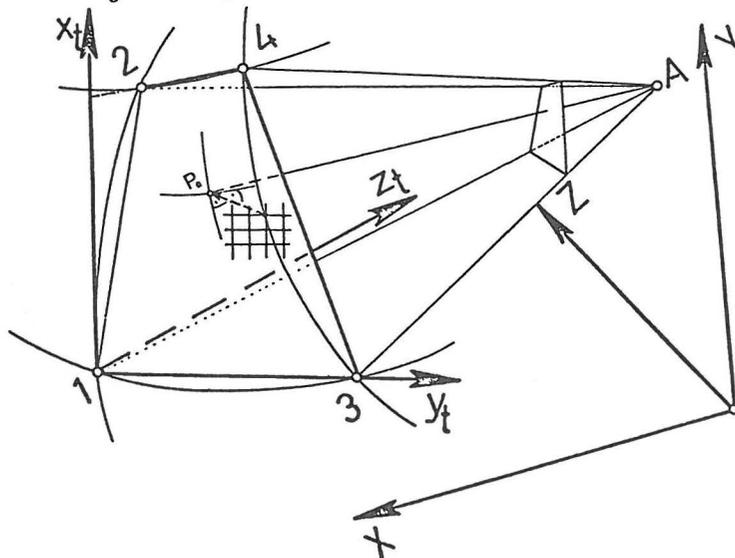


Fig. 3: Differential projection of the ellipsoid onto a plane of the polyhedron

The local systems  $x_t, y_t, z_t$  are connected with the central coordinate system of the ellipsoid  $X, Y, Z$  by a spatial transformation:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_1 \\ Z_1 \end{bmatrix} + R \cdot \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} \quad (1)$$

where  $X, Y, Z$  are the coordinates of some point  $P$  of the templet in the coordinate system of the ellipsoid,  $X_1, Y_1, Z_1$  - coordinates of corner 1 in the same system,  $R$  - the rotational matrix with elements  $r_{ij}$  ( $i=1,3; j=1,3$ ) dependent upon the spatial orientation of the templet,  $x_t, y_t, z_t=0$  - coordinates of point  $P$  in the templet system.

#### PROJECTING THE ELLIPSOID ONTO THE POLYHEDRON VIA DIFFERENTIAL RECTIFICATION

Photographic transformations of very different nature became possible by applying digitally controlled orthophoto projectors. In our case, the differential rectifier Avioplan OR 1 was available. The idea of applying this instrument is the following (fig. 3):

- a quadratic grid is defined in the plane of the resulting image (plane  $x_t, y_t$  of the templets),
- this grid is orthogonally projected onto the object surface (the surface of the ellipsoid), and finally,
- projecting this object surface image of the grid centrally onto the plane of the photograph (exposition center  $A$  applied as projection center).

The result of these steps is shown in fig. 4.

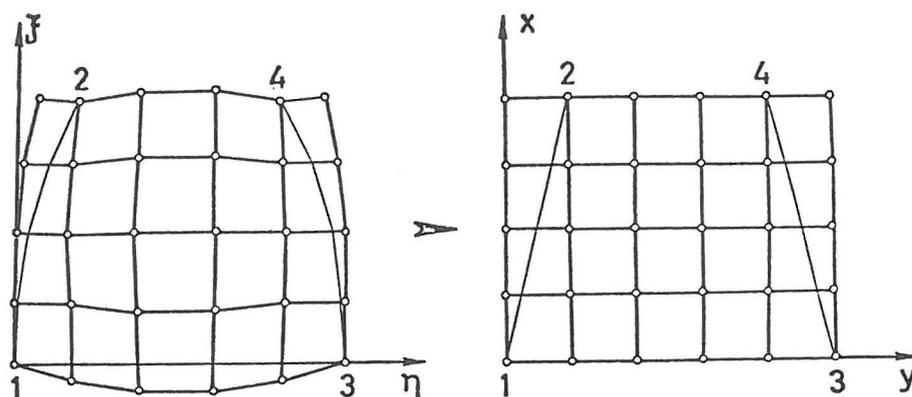


Fig. 4: Transforming the distorted grid (in the photograph) into the quadratic grid in the resulting lay-out (templet).

The quadratic grid defined in the plane of the templet corresponds to a distorted grid in the plane of the photograph. Transforming the individual quadrants of the distorted grid so to correspond to the quadratic grid defined in the templets, is equivalent to projecting the ellipsoid onto the surface of the polyhedron.

To perform this transformation the Avioplan OR 1 needs the image coordinates  $\xi$  and  $\eta$  of the deformed grid written on a magnetic tape. The processor of the Avioplan OR 1 uses these coordinates for controlling the optics of the orthophoto project or in the process of the automatic differential rectification. For this purpose, the processor applies a bilinear transformation (fig. 4):

$$\begin{aligned} x &= e_1 + e_2 \xi + e_3 \eta + e_4 \xi \eta \\ y &= f_1 + f_2 \xi + f_3 \eta + f_4 \xi \eta \end{aligned} \quad (2)$$

To enable the described photographic way of laying out elliptical cupolas, the mathematical algorithms and the corresponding computer program in FORTRAN had to be developed.

The program:

- defines the polyhedron (the templets) within the coordinate system  $X, Y, Z$ ,
- defines the quadratic grid in the  $x_t, y_t$  plane of the templets,
- transforms the grid into the  $X, Y, Z$  system
- projects orthogonally the grid onto the surface of the ellipsoid,
- transforms the available control points and the projected image of the grid from the coordinate system of the ellipsoid  $X, Y, Z$  in the templet system  $x_t, y_t, z_t$ .

Having performed these steps, the conditions are fulfilled necessary for running the second part of the program package SORA-OPS (Software for Off-line Rectification by the Avioplan to gain Orthophotos and of Stereo-orthophotos), /5/. This part of the package defines the exterior orientation, and, for amateur cameras, the interior orientation of photographs, applying control point information. Consequently, the object points are transformed into the photograph, and the image coordinates  $\xi$  and  $\eta$  are written to the magnetic tape serving for the control of the Avioplan OR 1.

To project the object point  $P$  onto the surface of the ellipsoid, the following iterative procedure has been developed (fig. 5):

the direction  $\vec{d}$  should be determined by projecting the point  $P$  of the templet orthogonally onto the surface of the ellipsoid. This unknown direction is first approximated by the direction  $\vec{d}_0$  perpendicular to the surface of the corresponding templet, or, in other words, parallel to the normal vector  $R$  of the templet plane (the templet plane:  $r \cdot e + D = 0$ ;  $e$  = unit vector,  $r$  = a vector pointing from the origin of the coordinate system to some point of the plane,  $D$  = distance between the origin of the coordinate system and the templet plane). The straight line  $g_0: (r - r_p) \times R = 0$ , drawn parallel to this direction through the point  $P$  intersects with the surface of the ellipsoid in point  $P_0$ . The normal vector of the tangential plane in point  $P_0$ :  $r \cdot e_T + D_T = 0$ , serves as the next approximation  $\vec{d}_1$  of the direction sought. The intersection  $P_1$  of a straight line drawn through the point  $P$  and parallel to  $\vec{d}_1: (r - r_p) \times R_T = 0$  with the ellipsoid defines a new tangential plane, and the next approximation  $\vec{d}_2$  can be defined. The iteration is finished when the distance  $\overline{P_1 P_2}$  becomes smaller than an a priori tolerance. The practice of applying this iterative process indicates a very good convergence.

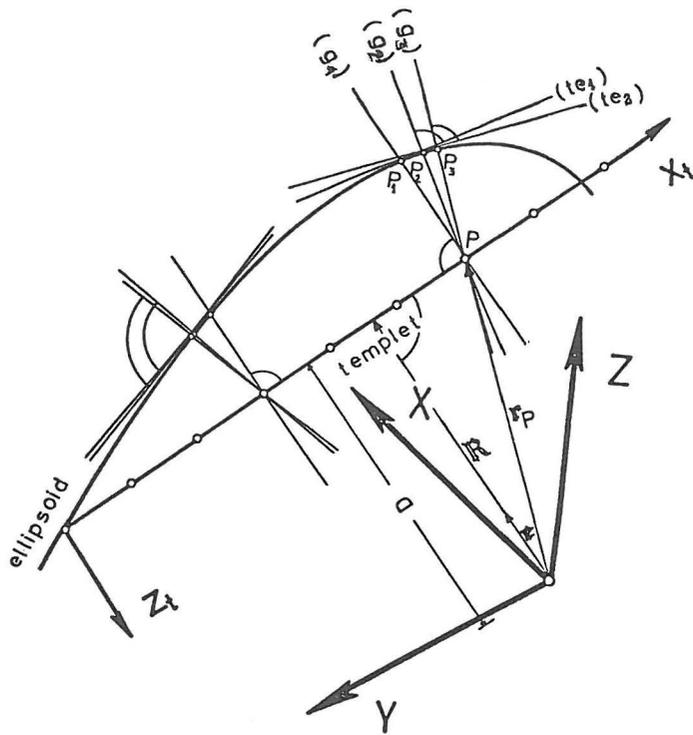


Fig. 5: The iterative orthogonal projection of some grid point of the templet onto the ellipsoid

#### LAYING OUT THE AREA AT THE APEX OF THE CUPOLA

Although the described process could be applied for the entire surface of the ellipsoid, stopping the process at a certain level ( $H_i$ ) suites better the practical purposes (fig. 6). The area above this level - the apex of the cupola - is handled then as one single templet. This templet is bordered by a polygon. The sides of this polygon correspond to the higher edges of the polyhedron. In theoretical aspects, there is no difference between processing the planes of the polyhedron and that of the apex area.

#### TEST EXAMPLE

To test the procedure, the cupola of the church Karlskirche in Vienna has been processed. The photographs have been taken with a Wild P 31 camera at day-light. From each camera position, directions and distances to other camera positions have been measured in two sets with a theodolite and an electronic infrared distancer. The spatial coordinates of the camera positions have then been determined by a simultaneous three dimensional adjustment in a local coordinate system  $X', Y', Z'$  (fig. 7). As control points, well identifiable points of the painted figures of the cupola have been used. These points have been determined by geodetic intersection in the coordinate system  $X', Y', Z'$ .

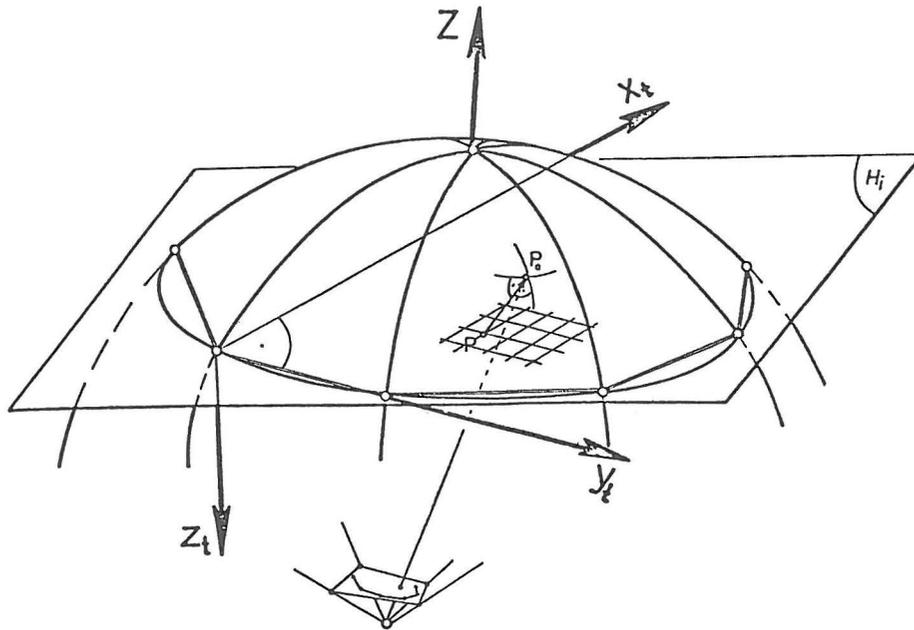


Fig. 6.: Differential transformation of the apex area

A FORTRAN program has been developed to determine the spatial orientation and the parameters of the ellipsoid fitting optimally the cupola. The process is based upon least squares techniques, and performs the following major steps:

- adjustment to define a second grade surface approximating optimally the surface determined by the control points:

$$\alpha_{11}X'^2 + \alpha_{22}Y'^2 + \alpha_{33}Z'^2 + 2\alpha_{12}X'Y' + 2\alpha_{23}Y'Z' + 2\alpha_{13}Z'X' + 2\alpha_{14}X' + 2\alpha_{24}Y' + 2\alpha_{34}Z' + 1 = 0 \quad (3)$$

- the following values will be counted:

$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{34} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & 1 \end{vmatrix} \quad \delta = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{vmatrix} \quad (4)$$

$$T = \alpha_{22}\alpha_{33} + \alpha_{33}\alpha_{11} + \alpha_{11}\alpha_{22} - \alpha_{23}^2 - \alpha_{13}^2 - \alpha_{12}^2$$

$$S = \sum_{i=1}^3 \alpha_{ii}$$

Having these values, the necessity of applying a general ellipsoid with three different axis can be determined /1/,

- transformation parameters are counted, allowing to go over from (3) to the following form:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1 \quad (5)$$

with a,b,c being the half axis of the ellipsoid,

- transformation of the control points from the X',Y',Z' system into the X,Y,Z system:

$$X = X'_0 + R_1 \cdot X' \quad (6)$$

where

$$= [X \ Y \ Z]^T$$

$R_1 =$  rotational matrix

$$X'_0 = [X'_0 \ Y'_0 \ Z'_0]^T$$

$$X' = [X' \ Y' \ Z']^T$$

In the case of the church Karlskirche in Vienna, the program yielded the following results:

$$a = 14,08 \text{ m}, \quad b = 9,45 \text{ m}, \quad c = 16,12 \text{ m}.$$

Fig. 8 shows one of the photographs, and fig. 9 the templets layed out from the equator to  $\phi = 77^{\circ}21'$  and from the major meridian to  $\lambda = 34^{\circ}33'$ . The coordinates of the templet corners 1,2,3,4 had been counted, and with them, the templets were scribed in the film with a CORAGRAPH DC 2.

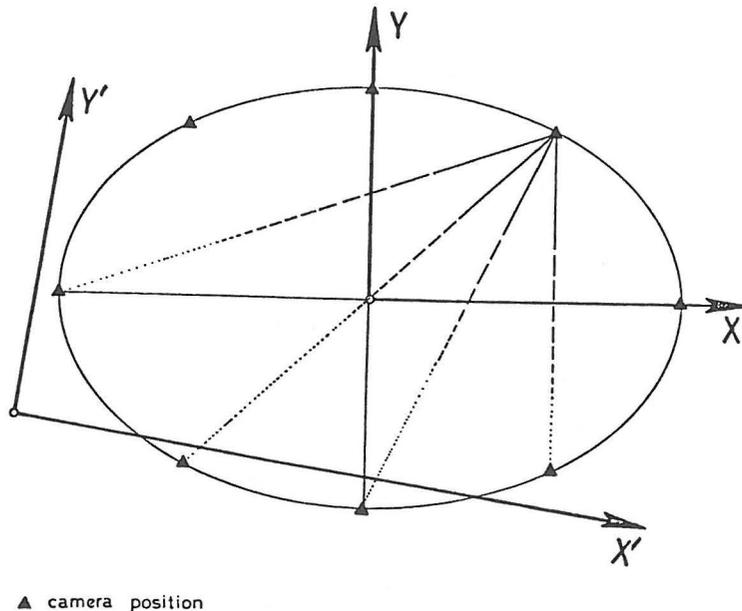


Fig. 7: Scheme of camera stations

To gain an impression about the entire cupola after having layed it out on a plane, all templets have been transferred into a reference coordinate system, and shown in fig. 10 as drawn by a CORAGRAPH DC 2.

### CLOSING REMARKS

In this paper a photographic way of laying out elliptical cupolas is presented. The task is solved by applying digitally controlled differential image transformation. The process is illustrated by applying it to a test example. The essential characteristics of this method are:

- it yields photographic lay-outs of a very good image quality,
- it handles both black-and-white and color photography,
- its precision can be refined and adjusted to any practical purpose by lowering the size of the templates and of the grid step in the computational stages, and by lowering the length of the slit of the orthophoto projector,
- both precision and amateur cameras can be applied,
- as compared with methods of digital image processing it is much more economical, and
- the method is independent of the coordinate system of the control points, as the spatial position and parameters of the ellipsoid are determined by the program itself.

### ACKNOWLEDGEMENTS

The authors wish to thank Mr. J. Fürst (Institute of Surveying, Technical University of Vienna) for the program of spatial adjustment of networks, Dipl.-Ing. H. Kager and Dr. L. Molnar for stimulating discussions. The CORAGRAPH DC 2 has been made available by the Fund for Promoting Scientific Research (Project No.2518). I am grateful for the financial support of the University-Anniversary Foundation of the Vienna City.

### LITERATURE

- / 1/ BRONSTEIN, I.N. and SEMENDJAJEW, K.A.: Taschenbuch der Mathematik; Verlag Harri Deutsch, Frankfurt, 1966
- / 2/ KRAUS, K.: Applications of a Digitally-Controlled Orthophoto Instrument; presented paper, XIII.ISP-Congress 1976
- / 3/ KRAUS, K. and TSCHANNERL, J.: Zur photographischen Entzerrung abwickelbarer Regelflächen; BuL. 168-170, 1976 or Architektur-Photogrammetrie III, Arbeitsheft 18 des Landeskonservators Rheinland, 117-12, 1977
- / 4/ KRAUS, K. and VOZIKIS, E.: Die Verebnung von Globen mit Hilfe der Differentialumbildung; Der Globusfreund, Nr.25-27, 55-61, Wien 1978
- / 5/ OTEPKA, G. and LOITSCH, J.: A Computer Program for Digitally-Controlled Production of Orthophotos; presented paper, XIII.ISP-Congress, 1976
- / 6/ STEWARDSON, P.: The Wild Avioplan OR 1 Orthophoto System; presented paper, XIII.ISP-Congress, 1976
- / 7/ VOZIKIS, E. and KRAUS, K.: Zur photographischen Verebnung von Kuppeln; V.CIPA-Congress, Sibenik, 1978
- / 8/ VOZIKIS, E.: Differential Rectification of Oblique Photographs of Plane Objects; Photogrammetria, 1979
- / 9/ VOZIKIS, E.: Die Differentialumbildung gekrümmter Flächen mit Beispielen von der Architekturbildmessung; Geowiss. Mitt. der Studienrichtung Vermessungswesen der TU Wien, Heft 17, Wien 1979
- /10/ WAGNER, K.: Kartographische Netzentwürfe, Bibliographisches Institut Mannheim, 1962

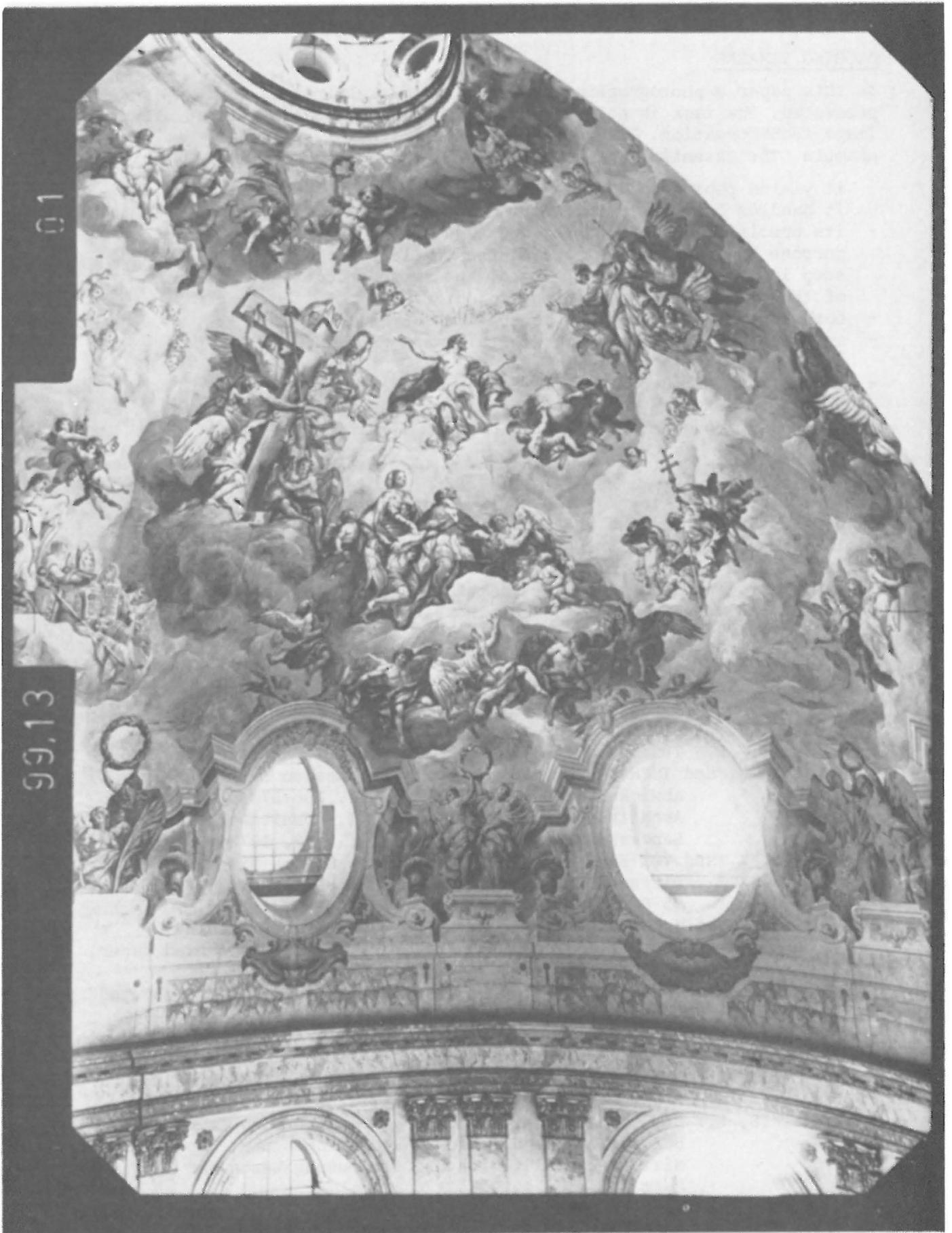


Fig. 8: Karlskirche, Vienna; Photograph taken with a Wild P 31 camera

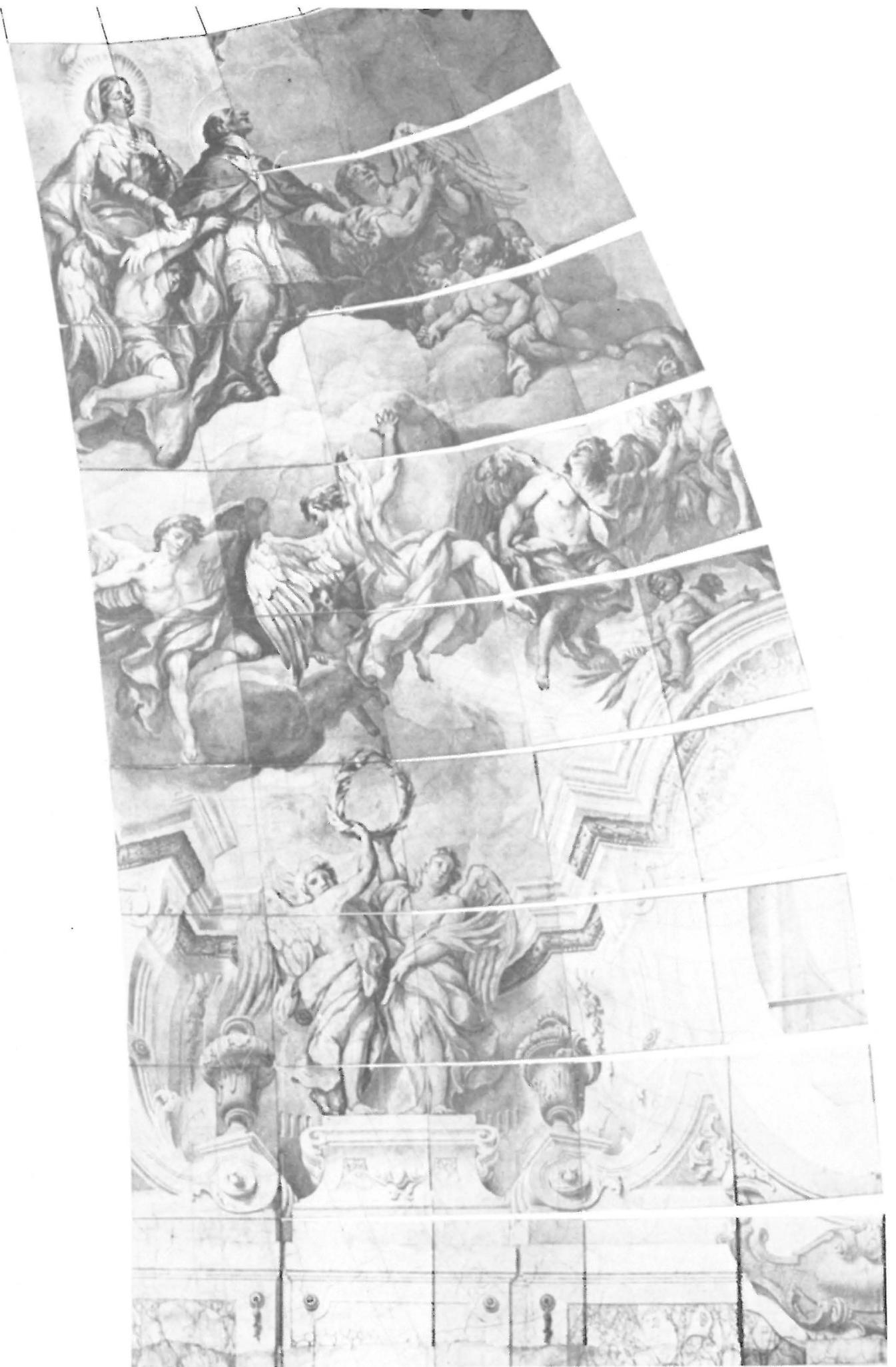


Fig. 9: Templets mounted from  $\varphi=0^{\circ}$  to  $\varphi=77^{\circ}21'$  and from  $\lambda=0^{\circ}$  to  $\lambda=34^{\circ}33'$ ; original scale 1:50

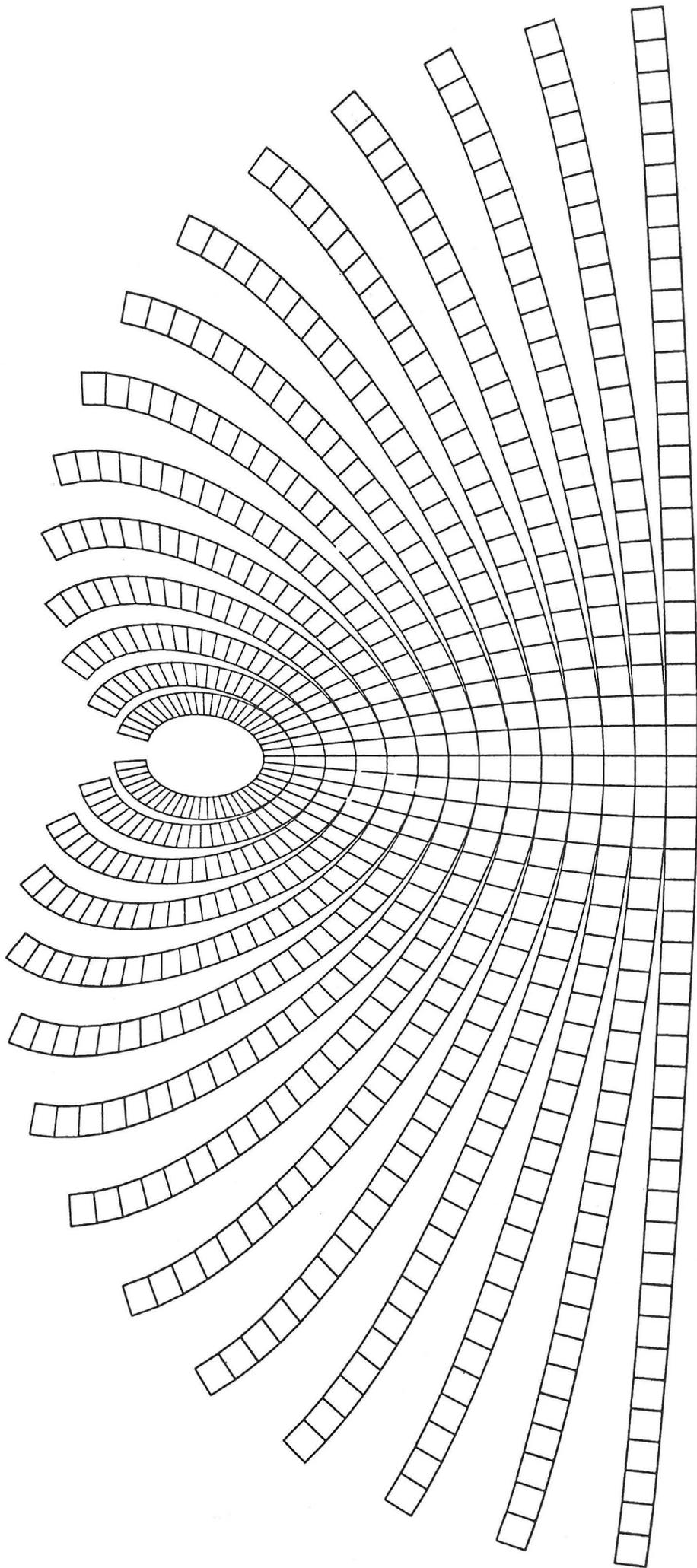


Fig. 10: The polyhedron-surface layed out in a plane  
(scale 1:200; plotted by CORAGRAPH DC 2)