A NEW TREATMENT FOR THE ADJUSTMENT OF TRILATERATION NETWORKS

By

Fouad Zaki, PH.D, Technical Advisor.
Mona El Kady, PH.D, Director.
Manal Ahmed, Eng.
Survey Research Institute,
Water Research Centre, Ministry of Public Works, Egypt.

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Abstract:

In the least-squares adjustment of a trilateration network composed of interlacing braced quadrilaterals and centered polygons, the current procedure is to compute approximate values of the angles forming the geometrical figures of the net. These values are then used to get the correction for the measured sides either as a least-squares triangulation adjustment with condition equations or by applying the least-squares method of variation of parameters.

In either case, the question arises why then take the difficulty of measuring lengths, if at the end we are computing approximate values of the angles which could have been measured more easily and effectively with a theodolite.

In this paper, a proposed least-squares adjustment of the trilateration problem is introduced which does not rely altogether on the computation of any angle. Besides, it reduces the number of conditions to only one for each geometrical figure (a braced quadrilateral or a central polygon) instead of four or more conditions.

A computer program using the "Basic language" has been devised for such treatment with two different applications for the two common figures, together with a comparison with the current procedure applied, until now, even in the most advanced and newest treatises on the subject.

KEY WORDS: Adjustment, Trilateration, Software.

INTRODUCTION

Conventionally, objects in the field are located in the horizontal position by angles, distances, or a combination of both. A point can be located in relation to two other points by measuring two quantities: either a direction from each of the two points (triangulation), or a direction and a distance from one of the two points (traverse), or a distance from each of the two points (trilatcration) [1]. Thus in Fig.1, if A and B are too fixed points with known plane rectangular coordinates, and C and D are new points whose coordinates are to be determined, then four measurements are therefore necessary and sufficient for the determination of the unknown points. These measurements could be:

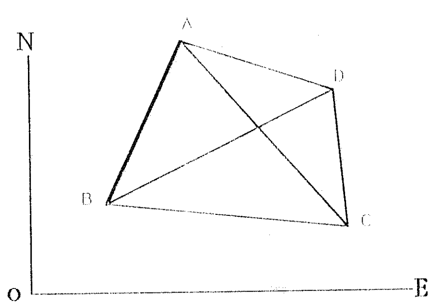

Fig1(a)

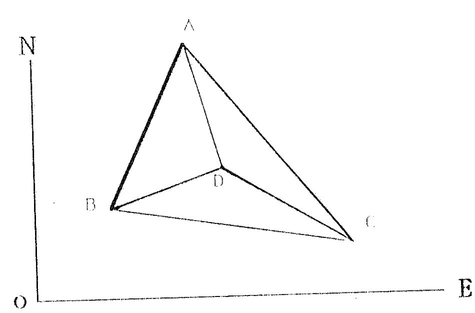

Fig1(b)

(a) angles D A B, D B A, C A D and C D A
(b) distances A D, B D, A C, or
(c) angles A B D, C A D and distances B D, A C, or any other suitable combination of four measurements.

However, if only four measurements are made, there will be no independent check, and there will be the possibility of an undetected mistake or blunder in the measurements.
If additional measurements are taken as checks, they can also be used to obtain better estimates of the unknowns.

In the Framework of Fig.1 a, there are eight interior angles and five distances (AB is assumed fixed) giving thirteen possible measurements to determine four unknowns, i.e., nine "redundant measurements" while in Fig. 1b there are nine interior angles and five distances giving fourteen possible measurements to determine four unknowns, i.e.,ten redundant measurements. Whether or not all thirteen (fourteen) measurements are made is a matter to be decided taking into consideratiom the time taken to make the additional measurements and also the time necessary for computation (although this is not now of primary importance due to the introduction of large electronic computers, except for saving of storage).

For such "overdetermined" problems, the most probable values (MPV) of the unknowns can be found from the measurements by using either the so-called direct method (condition equations), or the indirect method (observation equations or variation of parameters). Both methods make use of the Least Squares principle (LS), and both can be applied to the same measurements to obtain the same unknows.

TRILATERATION

The usual method of fixing points C and D (Fig.1) has long been the triangulation method, where the eight (nine) interior angles are measured by a theodolite. In this case, we have four (five) redundant measurements ending with four (five) condition equations.
The introduction of EDM instruments has made it possible to replace this usual ground triangulation by trilateration, in which case five distances are measured, namely, AC, AD, BD and CD (AB is assumed fixed) for the fixation of points C and D. Thus we have one redundant measurement ending with one condition equation, which has to be satisfied before computing the horizontal coordinates of points C and D.

Laurila, - 1983, seems to be the only geodesist who realized this fact in the adjustment of a trilaterated braced quadrilateral (Fig.1 a). However, his treatment for this one condition equation turned to be a geometrical relation between the four corner angles BAD, ADC, DCB, CBA, which he computed from the measured sides.

All other geodesists adjusted the trilaterated quadrilateral either as a triangulated quadrilateral with four geometrical conditions relating the computed angles of the quadrilateral (Moffitt,1975.) or by the method of variation of parameters (Mikhail, 1981)[A]

The New Treatment For Trilateration Adjustment

The "one geometrical condition" connecting the lengths of the sides and diagonals of a plane braced quadrilateral ABCD (Fig.1b) or the length of the sides of a triangular central polygon (Fig.1b) is obtained by multiplying by rows the two matrices.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1^2+y_1^2 & -2x_1 & -2y_1 & 1 \\ x_2^2+y_2^2 & -2x_2 & -2y_2 & 1 \\ x_3^2+y_3^2 & -2x_3 & -2y_3 & 1 \\ x_4^2+y_4^2 & -2x_4 & -2y_4 & 1 \end{bmatrix} \text{ AND } \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \\ 1 & x_4 & y_4 & x_4^2+y_4^2 \end{bmatrix}$$

where (x_r, y_r), $r = 1,2,3,4$ are the cartesian coordinates of the points A,B,C,D respectively.

Since the number of columns in each matrix is less than the number of rows, the determinant resulting from multiplication must vanish identically (Ferrar,1941).

Hence the required geometrical condition is:

$$\Delta = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (AB)^2 & (AC)^2 & (AD)^2 \\ 1 & (BA)^2 & 0 & (BC)^2 & (BD)^2 \\ 1 & (CA)^2 & (CB)^2 & 0 & (CD)^2 \\ 1 & (DA)^2 & (BD)^2 & (DC)^2 & 0 \end{vmatrix} = 0, \qquad (1)$$

where (AB)=(BA),etc. are the lengths of the mutual distances of the four points A, B, C, D. Similarly, for the condition connecting the mutual distances of five points.

Because model (1) is implicit in the measurements s_r ($r = 1,2,3,4,5,6$), it has to be linearized by applying Mclaurin's Theorem of Expansion. Noting that the corrections (δs) applied to the measures (s) of the mutual distances of the four points to made their lengths satisfy such condition are small so that the second and higher order terms in the expansion of (1) can be neglected, the conditional equation(1) may be written :

$$\Delta_0 + \sum \frac{\partial\Delta}{\partial s_r}\, \delta s_r = 0,$$

Where Δ_0 is the value of as estimated by the measured lengths; the summation covering all sides. Or,

$$\begin{bmatrix} \frac{\partial\Delta}{\partial s_1} & \frac{\partial\Delta}{\partial s_2} & \frac{\partial\Delta}{\partial s_3} & \frac{\partial\Delta}{\partial s_4} & \frac{\partial\Delta}{\partial s_5} & \frac{\partial\Delta}{\partial s_6} \end{bmatrix} \begin{bmatrix} \partial s_1 \\ \partial s_2 \\ \partial s_3 \\ \partial s_4 \\ \partial s_5 \\ \partial s_6 \end{bmatrix} = -\Delta_0$$

Which is the form $B\underline{c} = \underline{k}$, (4)

Where B stands for the vector $\left[\frac{\partial\Delta}{\partial s_1}, \frac{\partial\Delta}{\partial s_2}, \frac{\partial\Delta}{\partial s_3}, \frac{\partial\Delta}{\partial s_4}, \frac{\partial\Delta}{\partial s_5}, \frac{\partial\Delta}{\partial s_6} \right]$

C stands for the vector $[\delta s_1, \delta s_2, \ldots, \delta s_6]$

and K stands for $(-\Delta_0)$.

Assuming that the variance- covariance matrix of $\underline{S}$ is given by V i.e., $v = \mathrm{var}(\underline{s})$, then according to the least squares criterion, the best estimates $\hat{c}_j$ of the unknown corrections are obtained by minimizing the quadratic form $\underline{c}^T\, w\, \underline{c}$, where W is the" weight matrix" of the measurements defined by $W = \sigma_0^2 V^{-1}$ (5)

σ_0^2 being the "a periori" variance of a measurement of unit weight. Differentiating, as usual, the quadratic form partially with respect to each c_j , equating each result to zero, and rearranging the results, we get

$$\hat{\underline{c}} = W^{-1} B^T (B W^{-1} B^T)^{-1} \underline{k} \qquad (6)$$

If the measurements are of equal weights, then

$$\hat{\underline{c}} = B^T (B B^T)^{-1} \underline{k} \qquad (7)$$

or,

$$\delta s_r = -\frac{\Delta_0}{\sum \left(\frac{\partial\Delta}{\partial s_r}\right)^2} \frac{\partial\Delta}{\partial s_r}, \quad r = 1,2,\ldots,6 \qquad (8)$$

In order to be able to make a "Basic"computer program for applying this method, we have to evaluate the determinant and its derivatives with respect to the sides, as functions of the measured lengths. Putting $AB = S_1$, $AC = S_2$, $AD = S_3$, $BC = S_4$, $BD = S_5$ AND $CD = S_6$,

$$\begin{aligned} \Delta = {} & 2S_1^2 S_6^2 (-S_1^2+S_2^2+S_3^2+S_4^2+S_5^2-S_6^2) \\ & + 2S_3^2 S_4^2 (S_1^2+S_2^2-S_3^2-S_4^2+S_5^2+S_6^2) \\ & + 2S_2^2 S_5^2 (S_1^2-S_2^2+S_3^2+S_4^2-S_5^2+S_6^2) \\ & - 2S_1^2 S_2^2 S_4^2 - 2S_1^2 S_3^2 S_5^2 - 2S_2^2 S_3^2 S_6^2 - 2S_4^2 S_5^2 S_6^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial\Delta}{\partial S_1} = {} & 4S_1 S_6^2 (-S_1^2+S_2^2+S_3^2+S_4^2+S_5^2-S_6^2 \\ & -4S_1^3 S_6^2 + 4S_1 S_3^2 S_4^2 + 4S_1 S_2^2 S_5^2 - 4S_1 S_2^2 S_4^2 - 4S_1 S_3^2 S_5^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial\Delta}{\partial S_2} = {} & 4S_2 S_5^2 (S_1^2-S_2^2+S_3^2+S_4^2-S_5^2+S_6^2) \\ & -4S_2^3 S_5^2 + 4S_2 S_1^2 S_6^2 + 4S_2 S_3^2 S_4^2 - 4S_2 S_1^2 S_4^2 - 4S_2 S_3^2 S_6^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial\Delta}{\partial S_3} = {} & 4S_3 S_4^2 (S_1^2+S_2^2-S_3^2-S_4^2+S_5^2+S_6^2) \\ & -4S_3^3 S_4^2 + 4S_3 S_1^2 S_6^2 + 4S_3 S_2^2 S_5^2 - 4S_3 S_1^2 S_5^2 - 4S_3 S_2^2 S_5 \end{aligned}$$

$$\begin{aligned} \frac{\partial\Delta}{\partial S_4} = {} & 4S_4 S_3^2 (S_1^2+S_2^2-S_3^2-S_4^2+S_5^2+S_6^2) \\ & -4S_4^3 S_3^2 + 4S_4 S_1^2 S_6^2 + 4S_4 S_2^2 S_5^2 - 4S_4 S_1^2 S_2^2 - 4S_4 S_5^2 S_6^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial\Delta}{\partial S_5} = {} & 4S_5 S_2^2 (S_1^2-S_2^2+S_3^2+S_4^2-S_5^2+S_6^2) \\ & -4S_5^3 S_2^2 + 4S_5 S_1^2 S_6^2 + 4S_5 S_3^2 S_4^2 - 4S_5 S_1^2 S_3^2 - 4S_5 S_4^2 S_6^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial\Delta}{\partial S_6} = {} & 4S_6 S_1^2 (-S_1^2+S_2^2+S_3^2+S_4^2+S_5^2-S_6^2) \\ & -4S_6^3 S_1^2 + 4S_6 S_3^2 S_4^2 + 4S_6 S_2^2 S_5^2 - 4S_6 S_2^2 S_3^2 - 4S_6 S_4^2 S_5^2 \end{aligned}$$

If, as assumed, AB fixed, then $\partial\Delta/\partial S_1 = 0$.

The full computer program, written in Basic, is shown in the next section. When applied to different problems, the iterations were few showing a rapid convergence

APPLICATIONS

The proposed method was applied to the adjustment of three braced quadrilaterals given in Laurila [2], Moffitt [3],and Mikhail [4] respectively. The input data and the correctiond obtained, are shown in the following sheets.

REFERENCES

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Mikhail, E.M. 1981. Surveying Theory and Practice. Mcgraw Hill Book Company, New York.

Moffitt,F.M.1975. A Surveying. Harper and Row Publishers, New York.

```
10  READ X1,X2,X3,X4,X5,X6
20 DATA  1341.785,2775.364,2167.437,1937.887,2173.715,1511.014
21  Y1=(2*X1*X1*X6*X6)*((-X1*X1)+(X2*X2)+(X3*X3)+(X4*X4)+(X5*X5)-(X6*X6))
22  Y2=(2*X3*X3*X4*X4)*((X1*X1)+(X2*X2)-(X3*X3)-(X4*X4)+(X5*X5)+(X6*X6))
23  Y3=(2*X2*X2*X5*X5)*((X1*X1)-(X2*X2)+(X3*X3)+(X4*X4)-(X5*X5)+(X6*X6))
24  Y4=-(2*X1*X1*X2*X2*X4*X4)-(2*X1*X1*X3*X3*X5*X5)-(2*X2*X2*X3*X3*X6*X6)-(2*X4*
X4*X5*X5*X6*X6)
25  LPRINT "y1="; Y1
26  LPRINT "y2="; Y2
27  LPRINT "y3="; Y3
28  LPRINT "y4="; Y4
29  X =Y1+Y2+Y3+Y4
30  LPRINT "x=";X  :PRINT
31 DATA  1341.785,2775.364,2167.437,1937.887,2173.715,1511.014
35  H1   = 0
40  H2 = 0
50  B(1,1)= H1+H2
60  LPRINT " b(1,1)="; B(1,1)
70  A1 =(4*X2*X5^2*(X1^2-X2^2+X3^2+X4^2-X5^2+X6^2))-(4*X2^3*X5^2)
80  A2 =(4*X2*X1^2*X6^2)+(4*X2*X3^2*X4^2)-(4*X2*X1^2*X4^2)-(4*X2*X3^2*X6^2)
90  B(1,2)= A1+A2
100 LPRINT " b(1,2)="; B(1,2)
110 C1 =(4*X3*X4^2*(X1^2+X2^2-X3^2-X4^2+X5^2+X6^2))-(4*X3^3*X4^2)
120 C2 =(4*X3*X1^2*X6^2)+(4*X3*X2^2*X5^2)-(4*X3*X1^2*X5^2)-(4*X3*X2^2*X6^2)
130 B(1,3)= C1+C2
140 LPRINT " b(1,3)="; B(1,3)
150 D1 =(4*X4*X3^2*(X1^2+X2^2-X3^2-X4^2+X5^2+X6^2))-(4*X4^3*X3^2)
160 D2 =(4*X4*X1^2*X6^2)+(4*X4*X2^2*X5^2)-(4*X4*X1^2*X2^2)-(4*X4*X5^2*X6^2)
170 B(1,4)= D1+D2
180 LPRINT " b(1,4)="; B(1,4)
190 E1 =(4*X5*X2^2*(X1^2-X2^2+X3^2+X4^2-X5^2+X6^2))-(4*X5^3*X2^2)
200 E2 =(4*X5*X1^2*X6^2)+(4*X5*X3^2*X4^2)-(4*X5*X1^2*X3^2)-(4*X5*X4^2*X6^2)
210 B(1,5)= E1+E2
220 LPRINT " b(1,5)="; B(1,5)
230 F1 =(4*X6*X1^2*(-X1^2+X2^2+X3^2+X4^2+X5^2-X6^2))-(4*X6^3*X1^2)
240 F2 =(4*X6*X3^2*X4^2)+(4*X6*X2^2*X5^2)-(4*X6*X2^2*X3^2)-(4*X6*X4^2*X5^2)
250 B(1,6)= F1+F2
260 LPRINT " b(1,6)="; B(1,6)
270 FOR I = 1  TO  6
280 LPRINT B(1,I)
290 NEXT I
300 Q =(B(1,1)^2+B(1,2)^2+B(1,3)^2+B(1,4)^2+B(1,5)^2+B(1,6)^2)
310 LPRINT "q=";Q
320 FOR  I=1  TO  6
330 AX(1,I) = -X/Q *B(1,I)
340 LPRINT AX (1,I)
350 NEXT I
353 READ X(1),X(2),X(3),X(4),X(5),X(6)
355 DATA  1341.785,2775.364,2167.437,1937.887,2173.715,1511.014
360 FOR I = 1 TO 6
370 FR (1,I)= X(I) +AX(1,I)
380 LPRINT FR(1,I)
390 NEXT I
400 END
```

y1= 1.380934E+20
y2= 2.843213E+20
y3= 7.938321E+18
~~y4=-4.303462E+20~~
x= 6.860953E+15
b(1,1)= 0
b(1,2)=-3.509649E+17
b(1,3)= 2.343582E+17
b(1,4)= 2.795437E+17
b(1,5)=-2.681028E+17
b(1,6)= 1.58588E+17
0
-3.509649E+17
2.343582E+17
2.795437E+17
-2.681028E+17
1.58588E+17
q= 3.532741E+35
0
6.816108E-03
-4.551481E-03
-5.429031E-03
5.206838E-03
-3.079946E-03
1341.785
2775.371
2167.432
1937.882
2173.72
1511.011