

A LEAST-SQUARES APPROACH TO MATCHING LINES WITH FOURIER DESCRIPTORS

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ABSTRACT

A common problem in computer vision, digital photogrammetry and cartography is to find the best match between a given line and a set of candidate lines, based on characteristics of shape. Fourier descriptors have been used successfully to match lines. In this paper we show how the best geometric fit of two matched lines is determined. The translation, scaling and rotation parameters are found by matching the Fourier descriptors with a least-squares adjustment. A mean-square error can be calculated after matching. This offers the advantage of a quantitative measure of goodness of fit. Experimental results using synthetic data demonstrate the feasibility of the proposed algorithm.

KEY WORDS: Pattern Recognition, Machine Vision, Image Matching, Algorithm.

1. INTRODUCTION

To classify a set of patterns or to match two sets of features are common problems in computer vision, digital photogrammetry and cartography. These tasks are broadly known as pattern recognition. The fundamental approach to these problems is to find the best match in shape between a given feature and a set of candidate features. Each candidate feature should be fitted against the given feature one by one, based on their characteristics of shape. This matching process is often conducted to come out with some quantities of intrinsic measure for checking the degree of similarity. The best match is then determined according to the intrinsic measure.

An ambiguity may emerge on the determination of the best match, if there are more than one candidate features having a similar shape to that of the given feature. Because features may be distorted in practice, it is not surprising that the best match in shape is not guaranteed to be a correct match. Under the circumstance, additional information is needed to make a better decision. An extrinsic measure, such as a measure of the relative location, orientation and dilation between features, is considered to be key information to resolve this ambiguity. We therefore suggest that a matching process should generate both of intrinsic and extrinsic measures.

Geometric information of features is often described by lines of the vector form. Image features, such as object boundaries, skeletons, edges or textures can be represented by lines. Although other geometric information such as area, perimeter, number of holes and moments is believed to be useful, the information content of lines is thought to be the most compact, accurate and useful. This paper, therefore, focuses on the matching of lines.

In the last two decades, many techniques, such as polygonal approximation [Pavlidis and Ali, 1975; Greenfeld and Schenk, 1989], $\psi - s$ curves [Ballard and Brown, 1982; Schenk, Li and Toth, 1991], and invariants of Fourier

descriptors [Granlund, 1972; Lin and Hwang, 1987], have been proposed to tackle the problem of matching linear features. These techniques emphasize the use of shape invariants, a kind of intrinsic measure, to discriminate linear features. Although these techniques were reported efficient in some cases, two disadvantages were recognized. First, measuring similarity by comparing the shape invariants between features does not provide a clear statistical sense. Second, they cannot provide any extrinsic measure.

In this paper, a new matching process is proposed. It is designed to generate both of intrinsic and extrinsic measures. Our strategy is to transform each candidate line to be optimally matched, in the condition of the least-squares fit, with the given line. The transformation parameters are solved by means of least-squares adjustment. A statistical quantity, the mean-square error, can be calculated after the adjustment. This quantity presents an ideal intrinsic measure. And the estimated transformation parameters offer an extrinsic measure.

A conventional transformation is usually performed about the origin of the coordinate system. The spatial relationships between the original and the transformed features cannot be explicitly described by using the parameters of a conventional transformation. We, therefore, developed a *centroid-based transformation* which transforms a feature about its centroid - the mean position.

The quantities of intrinsic and extrinsic measures are needed to be referred to a spatial coordinate system. It seems necessary to perform the matching process in the spatial domain. However, it is required to pre-define corresponding points between the lines. Difficulties in finding the corresponding points are expected because of the differences of sampling density, scale and starting point. In order to remedy this problem, an algorithm to perform the matching process in the frequency domain is developed, where the Fourier descriptors of lines are matched. The results of matching in the frequency domain are also interpreted with respect to the quantities desired in the spatial domain.

This paper is composed of 6 sections including introduction and conclusion. Section 2 outlines the Fourier descriptors of closed and open lines. Section 3 describes the centroid-based transformation in the spatial and frequency domains. An algorithm of least-squares matching in frequency domain and interpretation of the results from the matching algorithm are illustrated in section 4. Section 5 presents some experimental results using synthetic data.

2. FOURIER DESCRIPTORS

2.1 Closed lines

A two-dimensional closed line can be described by two periodic functions $x(t)$ and $y(t)$ (Fig. 1). The parameter t is defined as $2\pi l/L$, where L is the perimeter of the closed line and l denotes the arc length along the line from the starting point s to p . According to the theory of elliptic Fourier descriptors [Kuhl and Giardian, 1982; Lin and Hwang, 1987], these two periodic functions can be expressed by Fourier expansions in matrix form as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a_0 \\ c_0 \end{bmatrix} + \sum_{k=1}^{\infty} \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \begin{bmatrix} \cos kt \\ \sin kt \end{bmatrix}, \quad (1)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} x(t) dt; & c_0 &= \frac{1}{2\pi} \int_0^{2\pi} y(t) dt; \\ a_k &= \frac{1}{\pi} \int_0^{2\pi} x(t) \cos kt dt; & b_k &= \frac{1}{\pi} \int_0^{2\pi} x(t) \sin kt dt; \\ c_k &= \frac{1}{\pi} \int_0^{2\pi} y(t) \cos kt dt; & d_k &= \frac{1}{\pi} \int_0^{2\pi} y(t) \sin kt dt. \end{aligned}$$

In Eq.(1), a_0 and c_0 are the mean values of $x(t)$ and $y(t)$ respectively, which indicate the geometric center of the closed line, or so called the *centroid*.

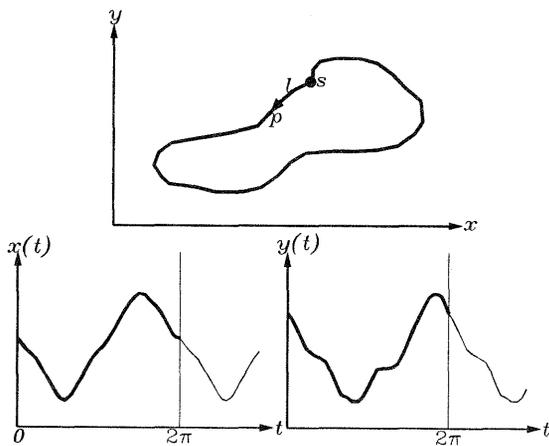


Fig. 1. A 2-D closed line and its periodic functions.

2.2 Open lines

An open line is traced once and then retraced backward so that a closed boundary is obtained (Fig. 2). The Fourier descriptors can then be applied. Let L denote the arc length of an open line and the parameter t is defined as $\pi l/L$. The functions of $x(t)$ and $y(t)$ can be expressed

as periodic functions. A close examination of the periodic functions (Fig. 2) yields two important characteristics. First, they are even functions because $x(-t) = x(t)$ and $y(-t) = y(t)$. This implies that the coefficients of b_k and d_k are all zeros. Second, the integration $\int_{t_1}^{t_2} x(t) \cos kt dt$ is equal to that of $\int_{2\pi-t_2}^{2\pi-t_1} x(t) \cos kt dt$, and it is appropriate to $y(t)$ also. Therefore, an open line can be described with the Fourier expansions as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a_0 \\ c_0 \end{bmatrix} + \sum_{k=1}^{\infty} \begin{bmatrix} a_k \cos kt \\ c_k \cos kt \end{bmatrix}, \quad (2)$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{\pi} x(t) dt; & c_0 &= \frac{1}{\pi} \int_0^{\pi} y(t) dt; \\ a_k &= \frac{2}{\pi} \int_0^{\pi} x(t) \cos kt dt; & c_k &= \frac{2}{\pi} \int_0^{\pi} y(t) \cos kt dt. \end{aligned}$$

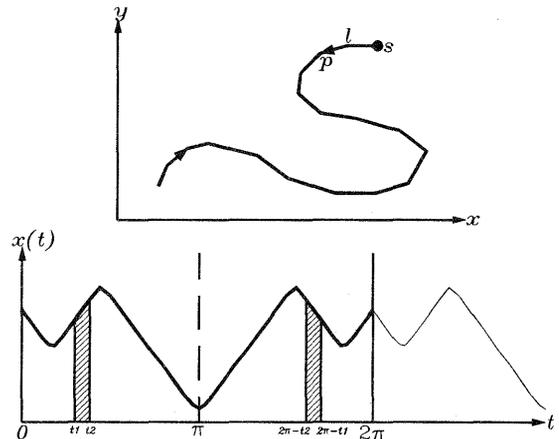


Fig. 2. A 2-D open line and periodic function of $x(t)$.

3. CENTROID-BASED TRANSFORMATION AND PHASE SHIFT

3.1 Transformation in Spatial Domain

If a linear feature consists of a list of (x, y) coordinate pairs of nodes, a transformation in spatial domain is implemented by transforming all coordinate pairs in the list. Conventionally, such transformation is operated about the origin of the coordinate system. For instance, let the list of (x', y') be the coordinate pairs after transformation. A similarity transformation about the origin is expressed as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = S \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}, \quad (3)$$

where

S	Scale factor;
θ	Rotation angle;
$\Delta x, \Delta y$	Translation.

With this transformation, one can easily discover that the positional change of the transformed feature does not correspond with the translation parameters Δx and Δy , because the centroid of the feature is changed by scaling and

rotation. It is, therefore, appreciated that the parameters of this transformation do not explicitly represent the geometric relationships between the original and transformed features. In order to obtain an explicit form of transformation parameters, the change of the centroid should be isolated from scaling and rotation. This can be accomplished by means of transforming a feature about the centroid, which is called *centroid-based transformation*. For example, a centroid-based similarity transformation is expressed as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = S \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x - x_c \\ y - y_c \end{bmatrix} + \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}, \quad (4)$$

where

x_c and y_c are the coordinates of the centroid.

3.2 Transformation in Frequency Domain

In frequency domain, instead of transforming coordinate pairs, a transformation can directly operate on the Fourier coefficients. This can be seen mathematically, if the coordinate pairs (x, y) and (x', y') in Eq. (4) are substituted by Eq. (1). A notable fact is that it is natural to perform a centroid-based transformation in frequency domain, because coordinates of the centroid are represented by the coefficients of zero harmonic, a_0 and c_0 , and the other coefficients of higher harmonics are independent of the centroid translation. Therefore, a centroid-based transformation in frequency domain can be divided into two parts. The first part is a translation involving just a_0 and c_0 . The second part which deals with a transformation that does not affect the position of the centroid, such as scaling, rotation and shearing, involves the other coefficients. These two parts can be done separately.

For the first part and given that the coefficients a_0 and c_0 are coordinates of the centroid, a translation can be directly added to the coefficients of the zero harmonic. Let a'_0 and c'_0 represent the transformed coefficients, then a translation in frequency domain will be

$$\begin{bmatrix} a'_0 \\ c'_0 \end{bmatrix} = \begin{bmatrix} a_0 \\ c_0 \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}. \quad (5)$$

For the second part, the Fourier coefficients of non-zero harmonics are pre-multiplied by a transformation matrix, which can be a matrix of similarity or affine transformation. The coefficients of each harmonic can be operated separately, because they are orthogonal. For a similarity transformation, the transformation matrix will be a combination of scale factor and rotation matrices. Let the coefficients with a prime be the transformed coefficients, then the transformation in frequency domain is expressed as

$$\begin{bmatrix} a'_k & b'_k \\ c'_k & d'_k \end{bmatrix} = S \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}, \quad (6)$$

where

$$k = 1 \sim \infty.$$

3.3 Phase Shift

If a linear feature is recorded by using a sequential list of (x, y) coordinate pairs along the feature, the first point

to be recorded is defined as the starting point. A change of the starting point does not alter the geometric property of the feature. However, it does change the Fourier descriptors except for the coefficients of the zero harmonic. For a closed line, the starting point can be anywhere along the curve. If a change of the starting point is interpreted as a change of the phase t and denoted as a *phase shift* Δt , then Δt can be an arbitrary value between 0 and 2π . For an open line, the starting point is either one of the two end points. Its phase shift is therefore 0 or π .

According to the theory of Fourier series, a phase shift is accomplished by post-multiplying the coefficients of each harmonic by a phase shifting matrix, which is similar to a rotation matrix. Mathematically, it can be expressed as

$$\begin{bmatrix} a'_k & b'_k \\ c'_k & d'_k \end{bmatrix} = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \begin{bmatrix} \cos k\Delta t & -\sin k\Delta t \\ \sin k\Delta t & \cos k\Delta t \end{bmatrix}. \quad (7)$$

3.4 Combined Effect of Transformation and Phase Shift

The effects of a transformation and a phase shift can be combined in frequency domain. From Eqs. (6) and (7), a combined effect of a similarity transformation and a phase shift will be

$$\begin{bmatrix} a'_k & b'_k \\ c'_k & d'_k \end{bmatrix} = S \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \begin{bmatrix} \cos k\Delta t & -\sin k\Delta t \\ \sin k\Delta t & \cos k\Delta t \end{bmatrix}. \quad (8)$$

Other transformations can be derived in the same fashion as the similarity transformation. Eq. (5) can be used for all kinds of transformation. All what needs to be changed for another type of transformation is the transformation matrix in Eq. (8). For example, an affine transformation in frequency domain with a phase shift can be formulated as

$$\begin{bmatrix} a'_k & b'_k \\ c'_k & d'_k \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \begin{bmatrix} \cos k\Delta t & -\sin k\Delta t \\ \sin k\Delta t & \cos k\Delta t \end{bmatrix}, \quad (9)$$

where

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \text{ is an affine transformation matrix.}$$

4. LEAST-SQUARES MATCHING

4.1 Matching in the Spatial Domain

The matching process for two given lines has been defined in the first section. Let a list of (x, y) coordinate pairs represent a candidate line, which is to be transformed in order to match a given line pattern composed of a list of (x', y') coordinates. In the spatial domain, if corresponding points between the two lines can be defined, each pair of corresponding points can form two observation equations, which can be derived from Eq. (4) as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix} = S \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x - x_c \\ y - y_c \end{bmatrix} + \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}, \quad (10)$$

where

$v_{x'}$ and $v_{y'}$ are residuals.

Having the observation equations, a least-squares adjustment can be conducted to solve the unknowns of S , θ , Δx and Δy , by minimizing the summation of $v_{x'}^2$ and $v_{y'}^2$.

In practice, however, corresponding points between two lines are difficult to define, due to the differences of sampling density, scale and starting point. A possible solution is to model the lines with some mathematical functions and resample the lines at equally spaced points. For instance, one could take the Fourier descriptors to model each line and resample the lines at every $2\pi/n$ interval, in which n is the number of points to be resampled for each line. Total number of $2n$ observation equations can be formed accordingly. Although this method is feasible, it seems not rigorous and efficient. First, the resampling space is difficult to determine. A complicated line requires a small sampling space, but a smaller sampling space increases computation time. Second, the computation is not straightforward. The lines are transformed into the frequency domain when modeling, and are transformed back to the spatial domain when resampling. Therefore, an idea of matching lines in frequency domain, matching Fourier descriptors, emerges to remedy these problems [Zhan and Roskies, 1972].

4.2 Matching in the Frequency Domain

Instead of matching spatial coordinates, Fourier descriptors of each harmonic are matched in the frequency domain. The problem of finding corresponding points no longer exists. The transformation parameters can be directly solved in the frequency domain and they naturally correspond to the idea of the centroid-based transformation, so that the computation becomes efficient and useful.

Since the Fourier descriptors of a closed line and an open line have different properties, the algorithms of closed-line matching and open-line matching are different. In general, open-line matching is a simplified case of the closed-line matching. Because a phase shift does not change the centroid, the translation parameters can be directly calculated for the both cases by using the following formula derived from Eq. (5):

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} a_0 \\ c_0 \end{bmatrix} - \begin{bmatrix} a'_0 \\ c'_0 \end{bmatrix}. \quad (11)$$

For the case of matching closed lines, each harmonic (except the zero one) has 4 coefficients, so that 4 observation equations can be constructed for each harmonic. Let a_k , b_k , c_k and d_k be the Fourier coefficients of a candidate line, and the coefficients with a prime represent the given line. If a similarity transformation is applied, then the observation equations can be derived from Eq. (9) as follows:

$$\begin{bmatrix} a'_k \\ b'_k \\ c'_k \\ d'_k \end{bmatrix} + \begin{bmatrix} v_{a'_k} \\ v_{b'_k} \\ v_{c'_k} \\ v_{d'_k} \end{bmatrix} = S \begin{bmatrix} a_k & b_k & -c_k & -d_k \\ b_k & -a_k & -d_k & c_k \\ c_k & d_k & a_k & b_k \\ d_k & -c_k & b_k & -a_k \end{bmatrix} \begin{bmatrix} \cos \theta \cos k\Delta t \\ \cos \theta \sin k\Delta t \\ \sin \theta \cos k\Delta t \\ \sin \theta \sin k\Delta t \end{bmatrix}. \quad (12)$$

In Eq. (12), the parameters, S , θ and Δt , are unknowns, and the equations are nonlinear. Combining the equations of all the harmonics from 1 to a maximum harmonic m , we obtain an redundant system of nonlinear equations. Least-squares adjustment can be used to solve the unknowns by minimizing the summation of squared residuals. An iterative approach of least-squares adjustment can be applied to solve such nonlinear equations with the given approximations of the unknowns.

For open-line matching, Eq. (12) can be simplified as

$$\begin{bmatrix} a'_k \\ c'_k \end{bmatrix} = S \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a_k \cos k\Delta t \\ c_k \cos k\Delta t \end{bmatrix}. \quad (13)$$

In order to linearize the equations, we let $e = S \cos \theta$ and $f = S \sin \theta$. In addition, the starting point is assumed to be at either end of the open line, so that $\Delta t = 0$ or π . When $\Delta t = 0$, the observation equations will be

$$\begin{bmatrix} a'_k \\ c'_k \end{bmatrix} + \begin{bmatrix} v_{a'_k} \\ v_{c'_k} \end{bmatrix} = \begin{bmatrix} a_k & -c_k \\ c_k & a_k \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}. \quad (14)$$

When $\Delta t = \pi$, they become

$$\begin{bmatrix} a'_k \\ c'_k \end{bmatrix} + \begin{bmatrix} v_{a'_k} \\ v_{c'_k} \end{bmatrix} = \begin{bmatrix} a_k(-1)^k & -c_k(-1)^k \\ c_k(-1)^k & a_k(-1)^k \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}. \quad (15)$$

The observation equations become linear in this case, so that unknowns can be solved without iteration. However, in order to know whether Eqs. (14) or (15) should be used, the parameter Δt should be determined in advance. The method to approach this will be described in section 4.4. The parameters S and θ can be derived from the solution of e and f by using

$$\begin{aligned} S &= \sqrt{e^2 + f^2}, \\ \theta &= \arctan(f/e). \end{aligned} \quad (16)$$

4.3 Weight Matrix and Mean-Square Error

Using least-squares adjustment, the mean value of the coordinate differences between two matched lines should be 0, and a mean square error can be calculated from the differences. If we treat the coordinates x' and y' as observations with a variance σ^2 , the mean-square error of the match is the best estimate of σ^2 . It is obvious that those properties of least-squares adjustment are defined in the spatial domain. Because the matching is performed in the frequency domain, two questions arise. First, what should the weight matrix for the observations a'_k , b'_k , c'_k and d'_k be? Second,

how is the mean square error from the residuals $v_{a'_k}, v_{b'_k}, v_{c'_k}$ and $v_{d'_k}$ calculated?

In order to answer the first question, variances and covariances between the Fourier coefficients should be analyzed. Let the covariance matrix of the x and y coordinates be an identity matrix multiplied by a unit weight variance σ_0^2 . From Eq. (1) we have the relationship between the coordinates and the coefficients as follows:

$$\begin{bmatrix} a_k \\ b_k \\ c_k \\ d_k \end{bmatrix} = \begin{bmatrix} \frac{1}{\pi} \int_0^{2\pi} x(t) \cos kt dt \\ \frac{1}{\pi} \int_0^{2\pi} x(t) \sin kt dt \\ \frac{1}{\pi} \int_0^{2\pi} y(t) \cos kt dt \\ \frac{1}{\pi} \int_0^{2\pi} y(t) \sin kt dt \end{bmatrix}. \quad (17)$$

One notable fact is that the off diagonal terms of the covariance matrix of the Fourier coefficients should all be zeros, because the functions used to calculate the coefficients are orthogonal. Also according to the error propagation law and Eq. (17), the diagonal terms (variances of the coefficients) can be calculated as follows

$$\begin{aligned} \sigma_{a_k}^2 &= \sigma_{c_k}^2 = \frac{\sigma_0^2}{\pi} \int_0^{2\pi} \cos^2 kt dt = \sigma_0^2, \\ \sigma_{b_k}^2 &= \sigma_{d_k}^2 = \frac{\sigma_0^2}{\pi} \int_0^{2\pi} \sin^2 kt dt = \sigma_0^2. \end{aligned} \quad (18)$$

It can, therefore, be concluded that the weight matrix is an identity matrix multiplied by σ_0^2 . This conclusion is also appropriate to the case of open-line matching.

The answer to the second question is that the mean-square error (MSE) is equal to the summation of the squared residuals in the frequency domain divided by 2. If the maximum harmonic is m , it can be expressed as

$$MSE = \frac{1}{2} \sum_{k=1}^m (v_{a'_k}^2 + v_{b'_k}^2 + v_{c'_k}^2 + v_{d'_k}^2). \quad (19)$$

Proof. The mean-square error is defined in the spatial domain as

$$MSE = \frac{1}{2\pi} \left(\int_0^{2\pi} v_x^2(t) dt + \int_0^{2\pi} v_y^2(t) dt \right).$$

By substituting Eq. (1) into above equation, the MSE can be expressed in the frequency domain as

$$MSE = \frac{1}{2\pi} \left[\int_0^{2\pi} \sum_{k=1}^m (v_{a_k} \cos kt + v_{b_k} \sin kt)^2 dt + \int_0^{2\pi} \sum_{k=1}^m (v_{c_k} \cos kt + v_{d_k} \sin kt)^2 dt \right].$$

Because of the orthogonal property, the formula becomes

$$MSE = \frac{1}{2\pi} \sum_{k=1}^m \left(\int_0^{2\pi} v_{a_k}^2 \cos^2 ktdt + \int_0^{2\pi} v_{b_k}^2 \sin^2 ktdt + \int_0^{2\pi} v_{c_k}^2 \cos^2 ktdt + \int_0^{2\pi} v_{d_k}^2 \sin^2 ktdt \right).$$

Because

$$\int_0^{2\pi} \cos^2 ktdt = \int_0^{2\pi} \sin^2 ktdt = \pi,$$

we obtain

$$MSE = \frac{1}{2} \sum_{k=1}^m (v_{a'_k}^2 + v_{b'_k}^2 + v_{c'_k}^2 + v_{d'_k}^2).$$

The theory and proof are also appropriate to the case of open-line matching, except there are no $v_{b'_k}$ and $v_{d'_k}$ terms.

4.4 First Approximations

The first approximations may be crucial for solving a set of nonlinear equations. With poor approximations, the computation may converge to a wrong solution or even be divergent. It is, therefore, important to provide good approximations for the adjustment computations.

In Eq. (12), let $k = 1$, then it seems possible to solve the approximations, S_0 , θ_0 and Δt_0 , from the four equations. Unfortunately, the parameters θ and Δt are dependent in each harmonic, so that θ_0 and Δt_0 solved from the first harmonic may be correct or incorrect with a difference of π . In order to assure the approximations are correct, the equations of the first two harmonics should be used.

We firstly linearize Eq. (12) by letting

$$\begin{aligned} cc1 &= S \cos \theta \cos \Delta t; & cc2 &= S \cos \theta \cos 2\Delta t; \\ cs1 &= S \cos \theta \sin \Delta t; & cs2 &= S \cos \theta \sin 2\Delta t; \\ sc1 &= S \sin \theta \cos \Delta t; & sc2 &= S \sin \theta \cos 2\Delta t; \\ ss1 &= S \sin \theta \sin \Delta t; & ss2 &= S \sin \theta \sin 2\Delta t. \end{aligned} \quad (20)$$

Then they can be solved by using the following formulas:

$$\begin{bmatrix} cc1 \\ cs1 \\ sc1 \\ ss1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & -c_1 & -d_1 \\ b_1 & -a_1 & -d_1 & c_1 \\ c_1 & d_1 & a_1 & b_1 \\ d_1 & -c_1 & b_1 & -a_1 \end{bmatrix}^{-1} \begin{bmatrix} a'_1 \\ b'_1 \\ c'_1 \\ d'_1 \end{bmatrix}; \quad (21)$$

$$\begin{bmatrix} cc2 \\ cs2 \\ sc2 \\ ss2 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & -c_2 & -d_2 \\ b_2 & -a_2 & -d_2 & c_2 \\ c_2 & d_2 & a_2 & b_2 \\ d_2 & -c_2 & b_2 & -a_2 \end{bmatrix}^{-1} \begin{bmatrix} a'_2 \\ b'_2 \\ c'_2 \\ d'_2 \end{bmatrix}.$$

According to Eq. (21), we define

$$\begin{aligned} C1 &= cc1 - ss1 = S_0 \cos(\theta_0 + \Delta t_0); \\ S1 &= cs1 + sc1 = S_0 \sin(\theta_0 + \Delta t_0); \\ C2 &= cc2 - ss2 = S_0 \cos(\theta_0 + 2\Delta t_0); \\ S2 &= cs2 + sc2 = S_0 \sin(\theta_0 + 2\Delta t_0). \end{aligned} \quad (22)$$

Then the approximations can be calculated as

$$\begin{aligned} S_0 &= \sqrt{C1^2 + S1^2}, \\ \theta_0 &= 2 \arctan(S1/C1) - \arctan(S2/C2), \\ \Delta t_0 &= \arctan(S2/C2) - \arctan(S1/C1). \end{aligned} \quad (23)$$

In fact, there are other combinations of Eq. (20) to solve the approximations. However, in practice, it is not necessary to elaborate the computation of the approximations.

For open-line matching, although the system is linear, the parameter Δt should be determined in advance. The observation equations of the first two harmonics are also required to solve the problem. According to Eq. (14), there are 2 unknowns and 2 equations, so that one can solve

2 approximate rotation angles from the equations of each harmonic. Let θ_1 and θ_2 denote the rotation angles solved when $k = 1$ and $k = 2$ respectively, then Δt can be determined by using the following algorithm:

if $|\theta_1 - \theta_2|$ close to 0, $\Delta t = 0$;
 if $|\theta_1 - \theta_2|$ close to π , $\Delta t = \pi$.

5. EXPERIMENTS

5.1 Matching Closed Lines

Fig. 3 shows a digitized, closed line (A). The line B is a candidate to match line A. The starting points are indicated by solid circles. In order to check the computed transformation parameters we copied line A to line B by the following transformation:

translation $\Delta x = 150$
 $\Delta y = 100$
 scale $S = 0.6$
 rotation $\theta = 45^\circ$

After the transformation, we shifted the starting point to the 10th node and added Gaussian noise ($\mu = 0, \sigma = 3$).

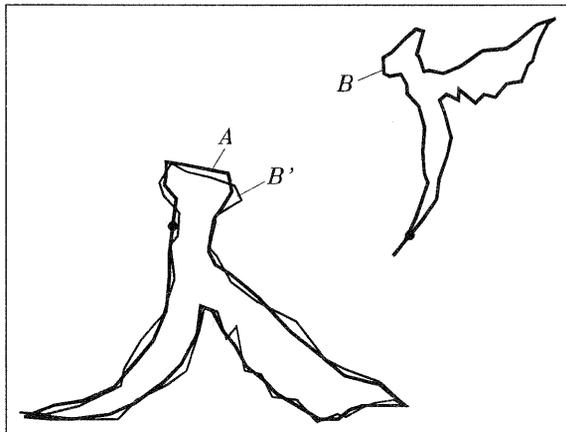


Fig. 3. Example of matching closed lines.

According to the matching process described in section 4, line B is transformed to fit line A. Through the matching process, the mean-square error (intrinsic measure) as well as the transformation parameters and phase shift (extrinsic measure) are calculated. The mean-square error is expected to be about 3, and the calculated transformation parameters should be the inverse-transformation parameters used to copy line B.

The following results are obtained:

Intrinsic measure - mean-square error = 3.44
Extrinsic measure - translation $\Delta x = -149.3$
 $\Delta y = -99.9$
 scale $S = 1.65$
 rotation $\theta = -44.9^\circ$
 phase shift $\Delta t = -86.4^\circ$

The intrinsic measure is close to the number we expected. Line B' in Fig. 3 is the transformed line B using the estimated transformation parameters. Note that the area between line A and B' is minimized by the proposed matching process.

5.2 Matching Open Lines

Here, we repeat the procedure for open lines. Line D in Fig. 4 is a copy of line C, obtained with the following transformation:

translation $\Delta x = -150$
 $\Delta y = -100$
 scale $S = 0.5$
 rotation $\theta = 180^\circ$

Before the transformation we added Gaussian noise ($\mu = 0, \sigma = 3$) to C.

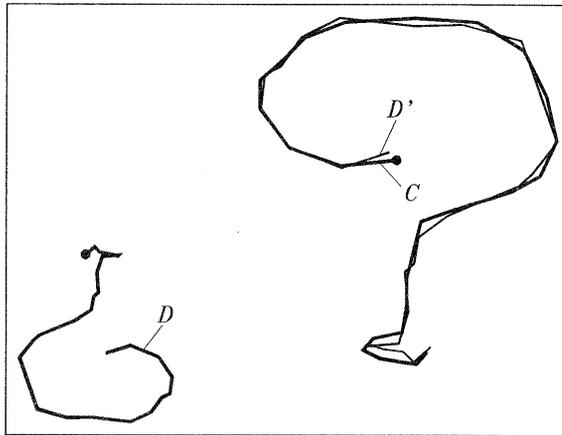


Fig. 4. Example of matching open lines.

We performed two experiments. First, the starting points of line C and D are at the same end. Second, we changed the starting point of line D to the other end. In both cases we obtained the same results. Line D' shown in Fig. 4 is the transformed version of line D. The matching results are listed as follows

Intrinsic measure - mean-square error = 2.6
Extrinsic measure - translation $\Delta x = 150.5$
 $\Delta y = 99.8$
 scale $S = 1.976$
 rotation $\theta = -179.4^\circ$

They correspond with the transformation applied to generate line D.

CONCLUSION

The mean-square error obtained in our matching approach is an ideal intrinsic measure to the goodness of the match. This quantity is obviously more transparent than the use of a table of deviation on shape invariants [Granlund, 1972; Lin and Hwang, 1989].

The quantities of extrinsic measure may be useful in some aspects. For example, in the case of recognizing handwriting of digital numbers, the digits of "2" and "5" as well as "6" and "9" are similar in shape but different in orientation. With intrinsic measure alone, they can hardly be distinguished [Pavlidis, 1980]. Checking the rotation parameter of the extrinsic measure will be helpful in this case. Matching edges of a pair of stereo images [Schenk, Li and Toth, 1991], is another example. One of the edges in the left image may be similar in shape to more than one edge in the right image. In this case, the extrinsic measure provides a global criterion.

In this study we match lines globally, that is, as whole entities. Consequently, lines which only match in parts cannot be dealt with. Since in many applications lines only match in parts, we are presently extending our approach to cope with this situation.

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