Plane A-angle Coordinate System And Its Applicatoins On Photogrammetry

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Abstract

In this paper, the concept of plane α-angle coordinate system is put forward. The characteristics of plane α-angle cordinate system and its relations to plane rectangular coordinate system are discussed . The expression forms of parameters as distance, area, azimuthal angle and angle of intersection in plane α-angle coordinate system are given. As its application, the effects on coordinates, distance, area, azimuthal angle and angle of intersection ,because of Digital Coordinate Instrument's x-axis and y-axis being not rectangular , are analyzed respectively . Also a simple and practical method for calculating the angle of Digital Coordinate Instrument's x-axis and y-axis is developed.

Introduction

Plane rectangular coordinate system is the most popular and convinient coordinate system both in theoretical studies and practical applications . But in some cases, it is difficult to get a standard rectangular coordinate system because of the limits of techenology . A typical example can be found in the Digital Coordinate Instument(DCI) . The DCI is a new instrument which is commonly used in the filed of topography, cadastre and photogrammetry in the last a few years. Its geometrical structureare mainly two guides , i.e. x-axis and y-axis , are perpendicular toeach other . The X and Y coordinates of the point at which the cross aims can be displayed in real time in the procedure of moving. Because of the limits oftechonology, the angle between x-axis and y-axis is not 90° but α ($\alpha \neq 90°$) . The angle can be made up to $90° \pm 45''$ by the current technology. In other words, the x-axis and y-axis do not form a strict plane rectangular coordinate system but a plane α-angle coordinate system . The main purposes of DCI are to measure the coordinates of the points on maps or images and in turn to determine the positions of the points on maps according to their coordinates. Because DCI's x-axis and y-axis is not rectangular , the measured coordinates are not in plane rectangular coordinate system but in plane α-angle coordinate system . It is commonly concerned how it influences on the coordinates of a point, the distance between two points, the area of a polygon and the angle between two straight lines. Therefore, it is both theoretically and practically significant to analyze the characteristics of plane α-angle coordinate system.

Plane α-angle Coordinate System

1. The Concept of Plane α-angle coordinate System

Passing a given point O, draw two axises between which there is a angle α ($0° < \alpha < 180°$). This two axises, which are with the same zero point and length scale , are called x-axis and y-axis respectively . In this way , a plane α-angle coordinate system XOY is established. The coordinates of an arbitary point M in plane α-angle coordinate system are measured by x and y (see Fig.1). The coordinates of point M are signed as M(x,y). Specially, if $\alpha = 90°$ then the plane α-angle coordinate system is plane rectangular coordinate system.

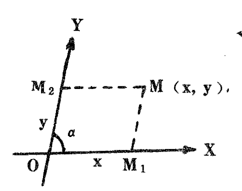

Fig.1

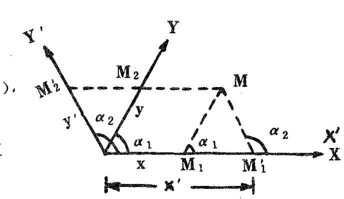

Fig.2

2. The Relations Between two Plane α-angle Coordinate System

In Fig. 2 , XOY is a plane α_1-angle coordinate system($\angle XOY = \alpha_1$) and X'OY' is a plane α_2-angle coordinate system ($\angle X'OY' = \alpha_2$). Suppose the coordinates of of point M are (x,y) in XOY and (x',y') in X'OY' respectively , the following formulas can be developed easily by using the theorem of sine.

$$\left.\begin{aligned} x' &= x + \frac{y}{\sin\alpha_2}\sin(\alpha_2 - \alpha_1) \\ y' &= \frac{y}{\sin\alpha_2}\sin\alpha_1 \end{aligned}\right\} \qquad (1)$$

or expressed in the form of matrix as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{\sin(\alpha_1 - \alpha_2)}{\sin\alpha_2} \\ 0 & \dfrac{\sin\alpha_1}{\sin\alpha_2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (2)$$

The relationship between the two plane α-angle coordinate systems is expressed by formula (1) or formula (2). Specially, if $\alpha_2 = 90°$ and $\alpha_1 = \alpha$,formula (2) becomes into

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (3)$$

Formula (3) is the formula with which the coordinates of a point in plane rectangular coordinate system can be got from the corresponding coordinates in plane α-angle coordinate system.

Similarly, if $\alpha_1=90°$ and $\alpha_2=\alpha$, formula (2) becomes into

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -\mathrm{ctg}\alpha \\ 0 & \csc\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (4)$$

Formula (4) is the formula with which the coordinates of a point in plane α-angle coordinate system can be got from the corresponding coordinates in plane rectangular coordinate system.

In fact, formula (2) can be infered from formulas (3) and (4). Firstly, the coordinates in plane α-angle coordinate system are transformed into rectangular coordinates by using formula (3).

$$\begin{pmatrix} x_r \\ y_r \end{pmatrix} \begin{pmatrix} 1 & \cos\alpha_1 \\ 0 & \sin\alpha_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (5)$$

And then the rectangular coordinates are be transformed into ones in plane α-angle coordinate system by using formula (4).

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -\mathrm{ctg}\alpha_2 \\ 0 & \csc\alpha_2 \end{pmatrix} \begin{pmatrix} x_r \\ y_r \end{pmatrix} \qquad (6)$$

Considered formulas (5) and (6), the following formula is obtained:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -\mathrm{ctg}\alpha_2 \\ 0 & \csc\alpha_2 \end{pmatrix} \begin{pmatrix} 1 & \cos\alpha_1 \\ 0 & \sin\alpha_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \dfrac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_2} \\ 0 & \dfrac{\sin\alpha_1}{\sin\alpha_2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

It is evident that the above formula is as same as formula (2).

3. Translation and Rotation of Plane α-angle coordinate System

i) Translation

Suppose XOY is a plane α-angle coordinate system with zero point O and X'O'Y' is the same coordinate system but with zero point O', then the relation between (x,y) and (x',y') is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad (7)$$

See Fig.3. Here (x,y) and (x',y') are the coordinates of point M in XOY and X'O'Y' respectively, and (x_0, y_0) are the coordinates of O' in XOY.

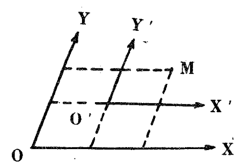

Fig.3

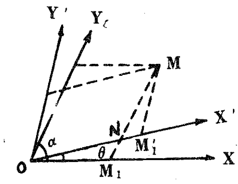

Fig.4

ii) Rotation

Suppose X'O'Y' is a plane α-angle coordinate system got by rotating the XOY with a angle of θ and the coordinates of point M in XOY and X'O'Y' are (x,y) and (x',y') respectively. In Fig.4,

$$x' = OM'_1 = ON + NM'_1$$

In $\triangle ONM_1$,

$$\angle M_1ON = \theta, \quad \angle OM_1N = 180° - \alpha, \quad \angle ONM_1 = \alpha - \theta$$

Then the following formulas can be obtained by using the theorem of sine.

$$ON = \frac{OM_1}{\sin(\alpha-\theta)}\sin(180°-\alpha) = \frac{x}{\sin(\alpha-\theta)}\sin\alpha$$

$$M_1N = \frac{OM_1}{\sin(\alpha-\theta)}\sin\theta = \frac{x}{\sin(\alpha-\theta)}\sin\theta$$

$$MN = MM_1 - M_1N = y - \frac{x}{\sin(\alpha-\theta)}\sin\theta$$

$$NM'_1 = \frac{MN}{\sin(180°-\alpha)} \cdot \sin\theta = \frac{MN}{\sin\alpha}\sin\theta$$

finally fomula (8) is obtained.

$$\left.\begin{aligned} x' &= \frac{\sin^2\alpha - \sin^2\theta}{\sin\alpha\sin(\alpha-\theta)}x + \frac{\sin\theta}{\sin\alpha}y \\ y' &= -\frac{\sin\theta}{\sin\alpha}x + \frac{\sin(\alpha-\theta)}{\sin\alpha}y \end{aligned}\right\} \qquad (8)$$

Formula (8) can also be expressed in the form of matrix as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \dfrac{\sin^2\alpha-\sin^2\theta}{\sin\alpha\sin(\alpha-\theta)} & \dfrac{\sin\theta}{\sin\alpha} \\ -\dfrac{\sin\theta}{\sin\alpha} & \dfrac{\sin(\alpha-\theta)}{\sin\alpha} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (9)$$

In formula (9), the matrix

$$A = \begin{pmatrix} \dfrac{\sin^2\alpha-\sin^2\theta}{\sin\alpha\sin(\alpha-\theta)} & \dfrac{\sin\theta}{\sin\alpha} \\ -\dfrac{\sin\theta}{\sin\alpha} & \dfrac{\sin(\alpha-\theta)}{\sin\alpha} \end{pmatrix}$$

is the rotation trasformation matrix of plane α-angle coordinate system. It is easily proved that

$$|A| = 1$$

$$A^{-1} = \begin{pmatrix} \dfrac{\sin(\alpha-\theta)}{\sin\alpha} & \dfrac{\sin\theta}{\sin\alpha} \\ -\dfrac{\sin\theta}{\sin\alpha} & \dfrac{\sin^2\alpha-\sin^2\theta}{\sin\alpha\sin(\alpha-\theta)} \end{pmatrix}$$

Specially, if $\alpha=90°$, formula (9) becomes into

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (10)$$

It is well-known that formula (10) is rectangular coordinate rotation formula. Consider formulas (7) and (9) at the same time , the formula which includes both translation and rotationcan be expressed as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \dfrac{\sin^2\alpha-\sin^2\theta}{\sin\alpha\sin(\alpha-\theta)} & \dfrac{\sin\theta}{\sin\alpha} \\ -\dfrac{\sin\theta}{\sin\alpha} & \dfrac{\sin(\alpha-\theta)}{\sin\alpha} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad (11)$$

The Expression Forms of Parameters

In plane rectangular coordinate system ,the parameters like distance , area, azimuthal angle and angle of intersection are commonly used . How about their expression forms in plane α-angle coordinate system?

1.Distance

Suppose the coordinates of points A and B are (x_1, y_1) and (x_2, y_2) respectively , See Fig.5,the distance from A to B is

$$s=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+2(x_2-x_1)(y_2-y_1)\cos\alpha} \qquad (12)$$

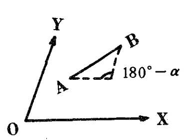

Fig.5

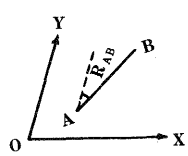

Fig.6

2.Area

Suppose the vertexes coordinates of a polygon which is compose of n sides are (x_1, y_1) , (x_2, y_2) , ……, (x_n, y_n) respectively in plane -angle coordinate system and their coresponding rectangular coordinate are (x_{r1}, y_{r1}) , ……, (x_{rn}, y_{rn}) ,the area of the polygon is

$$S=\frac{1}{2}\text{ABS}\left(\begin{vmatrix} x_{r1} & y_{r1} \\ x_{r2} & y_{r2} \end{vmatrix} + \cdots + \begin{vmatrix} x_{rn-1} & y_{rn-1} \\ x_{rn} & y_{rn} \end{vmatrix} + \begin{vmatrix} x_{rn} & y_{rn} \\ x_{r1} & y_{r1} \end{vmatrix}\right) \qquad (13)$$

consider formula (5) and formula (13),

$$\begin{vmatrix} x_{ri-1} & y_{ri} \\ x_{ri+1} & y_{ri+1} \end{vmatrix} = \begin{vmatrix} x_i+y_i\cos\alpha & y_i\sin\alpha \\ x_{i+1}+y_{i+1}\cos\alpha & y_{i+1}\sin\alpha \end{vmatrix}$$

$$=x_i y_{i+1}\sin\alpha+y_i y_{i+1}\sin\alpha\cos\alpha-x_{i+1}y_i\sin\alpha-y_i y_{i+1}\sin\alpha\cos\alpha$$

$$=x_i y_{i+1}\sin\alpha-x_{i+1}y_i\sin\alpha=\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix}\sin\alpha$$

in these formulas,i=1,2,...,n,and suppose $x_{n+1}=x,\ y_{n+1}=y,\ x_{rn+1}=x_{r1},\ y_{rn+1}=y_{r1}$ then

$$S=\frac{1}{2}\text{ABS}\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right)\cdot\sin\alpha \qquad (14)$$

Suppose

$$S_0=\frac{1}{2}\text{ABS}\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right)$$

It is clear that S_0 is the area calculated by using the formula of computing area in rectangular coordinate system .Therefore, formular (14) can also be written as

$$S=S_0\sin\alpha \qquad (16)$$

In fact,formula (16) is also correct for polygons with curve sides.

3.Azimuthal angle

Because the azimuthal angle varies very complicately in quadrants,only the angle of quadrant is discussed here.Suppose the coordinates of point A and B are (x_1, y_1) (x_2, y_2) in plane α-angle coordinate system and their rectangular coordinates are (x_{r1}, y_{r1}) and (x_{r2}, y_{r2}) ,see Fig.6,the angle of quadrant

$$R_{AB}=\text{arctg}\left|\frac{x_{r2}-x_{r1}}{y_{r2}-y_{r1}}\right| \qquad (17)$$

Consider formulas (5) and (17),

$$R_{AB}=\text{arctg}\left|\frac{x_2+y_2\cos\alpha-x_1-y_1\cos\alpha}{y_2\sin\alpha-y_1\sin\alpha}\right| =\text{arctg}\left|\frac{x_2-x_1}{y_2-y_1}\cdot\frac{1}{\sin\alpha}+\text{ctg}\alpha\right| \qquad (18)$$

4.Angle of Intersection

Suppose the coordinates of points A,B and C are (x_1, y_1) 、(x_2, y_2) and (x_3, y_3) in plane α-angle coordinate system , according to formula (18), we have

$$R_{AC}=\text{arctg}\left|\frac{x_3-x_1}{y_3-y_1}\cdot\frac{1}{\sin\alpha}+\text{ctg}\alpha\right|$$

$$R_{AB}=\text{arctg}\left|\frac{x_2-x_1}{y_2-y_1}\cdot\frac{1}{\sin\alpha}+\text{ctg}\alpha\right|$$

If the azimuthal angles coresponding to R_{AC} and R_{AB} are α_{AC} and α_{AB} respectively ,then

$$\angle BAC=\alpha_{AC}-\alpha_{AB} \qquad (19)$$

Error Analysis

Generally speaking,it is difficult to ensure DCI's x-axis and y-axis being strictly rectangular. When DCI is used for digitizing,the coordinates obtained are in plane α-angle coordinate system. If they are regarded as rectangular coordinates , errors surely exist in parameters like coordinates ,distance , area and azimuthal angle.This kind of errors are systematic ones.

1.Influence on Coordinates

Suppose the coordinates of a point measured on DCI are (x,y) and the coresponding rectangular coordinates are (x_r, y_r), The errors of coordinates can be got by using formula (5).

$$\begin{cases} \Delta x = x - x_r = x - (x + y\cos\alpha) = -y\cos\alpha \\ \Delta y = y - y_r = y - y\sin\alpha = y(1 - \sin\alpha) \end{cases} \qquad (20)$$

For $0° < \alpha \le 90°$, the signs of Δx, Δy vary with quadrants as what are showed in Tab.1. For $90° \le \alpha < 180°$, the signs of Δx, Δy vary with quadrants as what are showed in Tab.2.

Tab.1

	Δx	Δy
quadrant I	<0	>0
quadrant II	<0	>0
quadrant III	>0	<0
quadrant IV	>0	<0

Tab.2

	Δx	Δy
quadrant I	>0	<0
quadrant II	>0	<0
quadrant III	<0	>0
quadrant IV	<0	>0

2.Influence on Distance

$$\Delta s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} - \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\alpha}$$

$$= -2(x_2 - x_1)(y_2 - y_1)\cos\alpha\ y_1)\cos\alpha / (\sqrt{(x_2 - x_1)_2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\alpha})$$

for DCI, $\alpha \approx 90°$, so

$$\Delta s = -\frac{\Delta x\, \Delta y \cos\alpha}{s} \qquad (21)$$

For $0° < \alpha \le 90°$, if Δx, Δy have the same sign, $\Delta s \le 0$; if Δx, Δy have the contrary sign, $\Delta s \ge 0$. For $90° \le \alpha \le 180°$, if Δx, Δy have the same sign, then $\Delta s \ge 0$; if Δx, Δy have the contrary sign, then $\Delta s \le 0$.

3.Influence on area

$$\Delta S = S - S\sin\alpha = S(1 - \sin\alpha)$$

$$\frac{\Delta S}{S} = 1 - \sin\alpha \qquad (22)$$

For $\alpha \in (0°, 180°)$, $\Delta S > 0$ is always correct. This means that area measured by DCI is always larger than its true value.

4.Influence on Azimuthal Angle

$$\Delta R = \operatorname{arctg}\left|\frac{\Delta x}{\Delta y}\right| - \operatorname{arctg}\left|\frac{\Delta x}{\Delta y} \cdot \frac{1}{\sin\alpha} + \operatorname{ctg}\alpha\right| \qquad (23)$$

For $0° < \alpha \le 90°$, if

$$-\frac{\cos\alpha}{1 - \sin\alpha} \le \frac{\Delta x}{\Delta y} \le -\frac{\cos\alpha}{1 + \sin\alpha}$$

then $\Delta R \ge 0$; if

$$\frac{\Delta x}{\Delta y} < -\frac{\cos\alpha}{1 - \sin\alpha} \text{ or } \frac{\Delta x}{\Delta y} > -\frac{\cos\alpha}{1 + \sin\alpha}$$

then $\Delta R < 0$.

For $90° \le \alpha < 180°$, if

$$-\frac{\cos\alpha}{1 + \sin\alpha} \le \frac{\Delta x}{\Delta y} \le -\frac{\cos\alpha}{1 - \sin\alpha}$$

then $\Delta R \ge 0$; if

$$\frac{\Delta x}{\Delta y} < -\frac{\cos\alpha}{1 + \sin\alpha} \text{ or } \frac{\Delta x}{\Delta y} > -\frac{\cos\alpha}{1 - \sin\alpha}$$

then $\Delta R < 0$.

In cadastre and photogrammetry, coordinate errors of x and y are required less than 0.1mm on map, error of distance less than 0.2mm on map and the relative error of area less than 1/1000. The maximum error allowance values of x,y, distance and area can be acquired by using fomulas (20), (21) and (22). The results are showed in Tab.3.

Tab.3

	allowance	$\lvert 90° - \alpha \rvert_{max}$
x	0.1mm	41"
y	0.1mm	1° 09'45"
distance	0.2mm	1'58"
area	1/1000	2° 43'45"

As showed in Tab. 3, x is the most sensitive paremeter to the angle between x-axis and y-axis. It requires that $\lvert 90° - \alpha \rvert$ <41". Area is the most insensitive one. Even if $\lvert 90° - \alpha \rvert$ is up to 2 43 45", the area measured is still less than the value of allowance. If all of the parameters are considreed at the same time, it required that $\lvert 90° - \alpha \rvert$ <41". This is nearly identical to the technical specification of 45".

An Algorithm of Calculating A-angle

The angle between x-axis and y-axis is one of the most important specifications of DCI. It is also very useful to know the value of in the process of measure with DCI. A algorithm of calculating α is developed as fllows.

Determine two points A and B(line AB is not parallel to either x-axis or y-axis) and measure their coordinates (x_1, y_1) and (x_2, y_2) on a piece of paper. See Fig.7(a). Then rotate the paper about 90°. In this position, point A and B become A'and B' with coordinates of (x_1', y_1') and (x_2', y_2'), See Fig.7(b). Then for Fig.7(a),

$$S^2 = (x_2 - x_1)_2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\alpha$$

for fig.7(b),

$$S^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2 + 2(x_2' - x_1')(y_2' - y_1')\cos\alpha$$

so

$$\cos\alpha = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2}{2[(x_2' - x_1')(y_2' - y_1') - (x_2 - x_1)(y_2 - y_1)]} \qquad (24)$$

In fact, α-angle can be calculated by several groups of data in order to get a more accurate value. Tab.4 and Tab. 5 gives an example to calculate α-angle.

Conclusion

It is showed in this paper that plane α-angle coordinate system is the

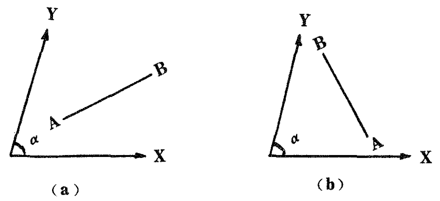

Fig.7

Tab.4

	x	y
A	84.25	282.96
B	347.59	422.05
C	222.26	120.86
A'	185.77	105.37
B'	46.38	368.17
C'	347.77	244.00

Tab.5

	AB	BC	AC
Dx	263.34	-125.33	138.01
Dy	139.09	-301.19	-162.10
Dx'	-139.39	301.32	161.93
Dy'	262.80	-124.17	138.63
α	90 04 42"	90 04 40"	90 04 28"
average value of α			90 04 40"

α-angle coordinate system is the expasion of plane rectangular coordinate system and plane rectangular coordinate system is the special case of plane α-angle coordinate system.The conclusions about plane α-angle coordinate system drawed in this paper are effective for plane rectangular coordinate system. It is believed that the conclusions will be used in other fileds of science.