DISCRETE MATHEMATICAL TECHNIQUES IN THE ANALYSIS AND ADJUSTMENT OF HYBRID NETWORKS

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ABSTRACT:

Over the past decade, adjustment of hybrid networks —usually referred to as combined adjustment within the photogrammetric comunity— have deserved the attention of numerous researchers. In the software, when dealing with the heterogeneous data of hybrid networks, everything tends to be more complex. The paper shows how discrete techniques can help deal with this complexity. Two examples are discussed: the detection of a family of gross errors and the numbering of unknowns for fill-in reduction. The concept of discrete models for the network and standardized discrete kernels for the software are proposed.

KEY WORDS: discrete models, discrete techniques, graphs, matroids, graph filtering, hybrid networks.

1 INTRODUCTION

Over the past decade, hybrid networks have deserved the attention of numerous researchers. This is witnessed by the one time period 1984–1988 of WG III/1 (Working Group III/1: Accuracy Aspects of Combined Point Determination) of the ISPRS, the two time periods 1983–1987, 1987–1991 of SSG 1.73 (Special Study Group 1.73: Integrated Geodesy) of the IAG, and by the meetings organized, either separately or jointly, by the two organizations. Today, integrated geodesy and combined point determination are still active research fields [5]. (The trend towards combined approaches in geodesy and photogrammetry has been mainly influenced by three factors: the advent of satellite geodesy —in particular the Global Positioning System—, the development of comprehensive models including all type of data, and the availability of high-speed large-capacity computers [9].)

In general, combined solutions are expected to provide more accurate and reliable results. Not less important is that global approaches lead to a cost reduction in software development, maintenance and acquisition; that they promote closer collaboration and understanding between groups traditionally involved —as well as traditionally separated— in point determination tasks; and that, as a result, they introduce factors of rationality and coherence in the corresponding point determination projects.

The combined adjustment philosophy, however, has found small acceptance in practice.

In conventional adjustment problems, in the first step, the unknowns and their accuracy are determined. In the second step, it is common to detect poorly measured data subsets which can impair the quality of the global adjustment. When dealing with heterogeneous data sets everything tends to be more complex and even the first step may not be easy to carry out. Thus, for day-to-day practical projects, the magnitude and structure of system equations which result from the above general approaches may be brought up as an argument against their application. Indeed, this is not the point if a suitable numbering for the unknowns is computed.

The heyday of research and development in numbering of graphs associated to geodetic networks was the decade of the seventies and the early eighties. In addition to the availability of the obtained results,[1] the increasing computer capacity has contributed to some decay of the topic.

In combined networks, however, there are pathological structures which perturbate the regularity and locality that classical photogrammetric and geodetic networks have exhibited so far (Section 5). In order to apply the old good algorithms, those structures must be understood and characterized so they can be detected and eliminated. For that purpose, discrete mathematics seem to be the best tool.

Discrete techniques can also contribute to the structural analysis of networks, hybrid or not (see the related work in [7, 14]). These techniques, have already been [implicitely] used for the generation of initial approximations in the nonlinear cases and, in a much lesser extent, for the detection of what is known as gross errors. For instance, many times

[1] Three statements describe the situation well.

First, the rigorous solution of the problem is NP-complete. Secondly, there are many algorithms which perform well — even at best— under certain regularity conditions. Last, the problem has been somewhat closed since it has been proven that problems not satisfying those regularity conditions are not amenable to sparse gaussian elimination.

In what photogrammmetric and geodetic netwroks is concerned, the pure numbering policy —that is, abstract time and space complexity considerations— can be summarized as follows: if the network is medium-sized (up to 2000–3000 unknown groups or even less) use a sequential numbering algorithm, otherwise use nested dissection; and, of course, try to take advantage of any regular pattern which might occur (for instance in photogrammetric blocks).

identification errors are coped with pure statistical techniques like iterated reweighting. This approach sometimes leads to troublesome computation sessions and does not take into account that the error is of a structural nature and can be detected by structural means. This second aspect —structural analysis of networks— emerges in a natural way while investigating the pathologies mentioned above.

2 BASIC TERMINOLOGY AND RESULTS

In order to facilitate the understanding of the next sections some basic concepts from combinatorics are required. The concepts reviewed are graphs and their cycle structure, hypergraphs and matroids. Other more specific concepts will be introduced when required.

2.1 Graphs

Let V be a finite non empty set and $E \subset \{\{x,y\} : x \neq y; x,y \in V\}$ a collection of unordered pairs of elements of V. (V,E) will be called a *finite undirected graph with no loops and no multiple edges* or, in our context, simply a *graph*. If G is a graph then it will be written $G = (V(G), E(G))$. The elements of $V(G)$ are the *vertices* or *nodes* of G and the elements of $E(G)$ — unordered pairs of $V(G)$— are the *edges*. The amount $\#(V(G))$ is the *order* of the graph and $\#(E(G))$ is the *size*.

If $\{x_1, x_2\} \in E(G)$, then x_1, x_2 are *adjacent vertices*. Given a subset X of V, the *adjacent set* of X is defined as

$$Adj(X) = \{y \in V(G) - X : \exists x \in X; \{x,y\} \in E(G)\}.$$

$Adj(x)$ will also be used for $Adj(\{x\})$.

The *degree* of a vertex x, $d(x)$, is the number of vertices in $Adj(\{x\})$, i.e. $d(x) = \#(Adj(\{x\}))$. If $d(x) = \#(V(G)) - 1$ then x is said to be a *border vertex*.

The series $x_1\, a_1\, x_2\, a_2 \ldots a_{n-1}\, x_n$ $(n \geq 2)$ where $\{x_1, \ldots, x_n\} \subset V$, $\{a_1, \ldots, a_{n-1}\} \subset E$ and $a_i = \{x_i, x_{i+1}\}$ is a *path* of *length* $n-1$ which *connects* x_1 and x_n. It is said, then, that the path *contains* the above vertices and edges and that it is a path *through* them. A *path* of *length* 0 is a series x_1. A *closed path* is a path such that $x_1 = x_n$. A *chain* is a path with no repeated vertices. A *cycle* is a closed path with no repeated edges. A *simple cycle* is a closed path with no repeated vertices. Henceforth, if not otherwise stated, it will be assumed that when referring to cycle a simple cycle is meant.

A graph is *connected* if there is a chain which connects each pair of vertices.

The *distance* $d_G(x_1, x_2)$ —or simply $d(x_1, x_2)$— between vertices x_1 and x_2 of a connected graph G is the length of the shortest chain from x_1 to x_2.

2.2 Graph numberings and elimination graphs

A *numbering* of a graph $G = (V,E)$ is a one-to-one mapping $p : \{1, \ldots, \#(V)\} \longrightarrow V$. The graph G together with the numbering p is sometimes called an *ordered graph* and written as $G_p = (V,E,p)$. In this paper, the equivalent definition $p : V \longrightarrow \{1, \ldots, \#(V)\}$ will also be used.

For a given G_p the *elimination graph* is $(G, E \cup F)$, where the new edges in F are the *fill-ins*; the *fill-in factor* is $\#(E \cup F)/\#(E)$. Since the concept of fill-in is very well known in photogrammetry, only the *path theorem* [12] which characterizes them and which will be used in Section 4 as a definition is reviewed.

The path theorem[12]: Let G be a graph and p a numbering of G. Let $i > j$, $i = p^{-1}(a)$, $j = p^{-1}(b)$. Then $\{a,b\}$ is a fill-in if, and only if, there exists a path

$$a\, \{a, x_{p_1}\}\, x_{p_1} \ldots x_{p_m}\, \{x_{p_m}, b\}\, b$$

in the graph G^0, $p_1, \ldots, p_m < j$; where $p_r = p^{-1}(x_{p_r})$. □

Based on results derived from the above theorem [12][p.277], a fast fill-in generation algorithm (running time: $O(\#(V) + \#(E \cup F))$ can be devised.

2.3 Cycle bases

Let $G = (V,E)$ be a graph and p a one-to-one map $p : \{1, \ldots, \#(E)\} \longrightarrow E$; to each cycle c of G the element $\mu(c) = (u_1, \ldots, u_{\#(E)})$ of $Z_2^{\#(E)}$ is associated, defined as

$$u_i = \begin{cases} 1 & \text{if } p(i) \text{ is an edge of } c, \\ 0 & \text{otherwise.} \end{cases}$$

Let $c_1, \ldots, c_n$ be n cycles of G and their respectives $\mu(c_1), \ldots, \mu(c_n)$ as defined above. $c_1, \ldots, c_n$ are said to be *dependent* if there is a non empty subset $Q \subset \{1, \ldots, n\}$ such that

$$\sum_{i \in Q} \mu(c_i) = \mathbf{0},$$

where, recall, the summation is taken in Z_2. Otherwise they are called *independent*. [2]

A *cycle basis* of G is a set $\{c_1, \ldots, c_k\}$ of independent cycles such that any cycle c of G can be written

[2] Usually, it is also written $\sum_{i \in Q} c_i$ instead of the formally correct $\sum_{i \in Q} \mu(c_i)$.

as

$$c = \sum_{i \in Q} c_i$$

for some $Q \subset \{1, \ldots, k\}$,$Q \neq \emptyset$. The cycles of the basis will be called *basic cycles.*

The number k above is constant for all cycle basis and is called the *cyclomatic number* of G. If the graph has p connected components the cyclomatic number equals $\#(E) - \#(V) + p$. It is denoted by $\nu(G)$.

2.4 Hypergraphs

If V is a a finite non empty set and E, $E \subset \mathcal{P}(V)$, a family of non empty subsets of V such that

$$\bigcup_{E_i \in E} E_i = V,$$

then the couple H, $H = (V, E)$, is called a *hypergraph.* The elements of V are referred to as the *vertices* of the hypergraph. The *edges* or *hyperedges* are the elements of E.

2.5 Matroids

In 1935, H.Whitney introduced the matroid concept in order to investigate linear independence in an abstract way. Let $A = \{a_1, \ldots, a_n\}$ be a finite set and $\mathcal{F} \subset \mathcal{P}(A)$. $(A, \mathcal{F})$ is defined to be a *matroid on* A if, and only if,

1. $F \in \mathcal{F} \Rightarrow F \neq \emptyset$,
2. $\{a_i\} \in \mathcal{F}$, for $i = 1, \ldots, n$,
3. $F \in \mathcal{F}$, $F' \neq \emptyset$, $F' \subset F \Rightarrow F' \in \mathcal{F}$,
4. and for each $S \subset A$, the members of $\mathcal{F}$ that are maximal in S have the same cardinality.

The elements of A are called the *elements of the matroid* $(A, \mathcal{F})$ and the elements of $\mathcal{F}$ are called the *independent sets* of the matroid. Maximal independent sets are called *matroid bases* and minimal dependent sets are called *circuits.*

It is very easy to show [3][Chapter 2] that, if

$$C = \{H \in E(G) : H \text{ is a cycle}\},$$

and

$$F = \{I \in \mathcal{P}(C) : I \text{ is an independent set of cycles}\},$$

then (C, F) is a matroid on C whose bases contain $\nu(G)$ elements.

If a weight $w(c)$ for each element c of C is defined —for instance, $w(c)$ could be taken as $l(c)$; the length of c in G— then the weight $w(B)$ of a basis B, $B \in F$, is defined as

$$w(B) = \sum_{c \in B} w(c).$$

For an elementary introduction to the subject see [2].

2.6 NP-completeness

The class NP contains the problems solvable by nondeterministic polynomial algorithms. The subclass NP-complete of NP contains the hardest problems of NP in the sense that a solution for a NP-complete problem is a solution to any other problem in NP through a transformation by a polynomial time algorithm.

A nondeterministic algorithm is an algorithm that at each step has several choices for the next step. A polynomial or polynomial time algorithm is an algorithm that gives a result after a number of steps which is bounded by a polynomial function.

3 NETWORK DISCRETE MODELS

In [4] the practical convenience that network discrete models be available is discussed in connection with the many algorithms of a discrete nature involved in the software; the conclusion is that comprehensive discrete network models and their corresponding discrete software modules are missing concepts in our systems.

A widely long since accepted abstraction is the correspondence between a network NE and a graph $G(NE)$: to each unknown parameter group[3] a graph vertex is associated; there is an edge connecting two vertices if their corresponding parameters are involved in a same observation (see [6]).

It is also known how an undirected graph $G(N)$ describes the zero-nonzero sparsity pattern of a block symmetric matrix and that $G(NE) = G(N)$ if N is the normal equations matrix of NE.

This, however, represents only a part of the problem.

To improve the situation consider the network adjustment [sparse] design matrix $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ and the hypergraph H such that $\#(V(H)) = n$, and a numbering p of the vertices

[3] A parameter group stands for a set of parameters related in the obvious way. In a photogrammetric block, the three coordinates of a point constitute a parameter group, also the six orientation elements of an image; the set of selfcalibration parameters constitutes as well another parameter group.

$V(H)$. Then, consider the map ϵ

$$\begin{aligned} \epsilon : \{1,\ldots,m\} &\longrightarrow E(H) \\ i &\longrightarrow E_i \end{aligned}$$

where $\epsilon(i) = E_i$ if $E_i = \{p(i_1),\ldots,p(i_q)\}$ and the only non zero elements of row i in A are $a_{ii_1},\ldots,a_{ii_q}$. ϵ may be non bijective.

H does contain all structural information of the network, including the observations. The problem of finding an optimal sequence for loading the partial normal equations —if required— can be formulated in terms of finding a proper numbering for the hyperedges of H.

$G(NE)$ can be obtained from $H(NE)$ as follows

$$H(NE) \xrightarrow{\alpha} H(NE)^\star \xrightarrow{\beta} L(H(NE)^\star), \qquad (1)$$

where $G(NE) = L(H(NE)^\star)$, $L(H(NE)^\star)$ is the *representative graph* of $H(NE)^\star$ and $H(NE)^\star$ is the *dual hypergraph* of $H(NE)$.

If $H = (\{v_1,\ldots,v_n\},\{E_1,\ldots,E_m\})$ is a hypergraph, the dual hypergraph H^*

$$H^* = (\{e_1,\ldots,e_m\},\{V_1,\ldots,V_n\})$$

is defined through the relation

$$e_i \in V_j \Leftrightarrow v_j \in E_i$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. It is apparent that $(H^*)^* = H$.

The representative graph $L(H)$ of H is a graph of order m -i.e., $V(L(H)) = \{x_1,\ldots,x_m\}$- defined with the equivalence

$$\{x_i,x_j\} \in E(L(H)) \Leftrightarrow E_i \cap E_j \neq \emptyset,$$

for $1 \leq i,j \leq m$.

In the above two definitions, for the sake of a simpler notation, the existence of a one-to-one correspondence between sets of same cardinality has been highlighted with subindices.

In transformation (1), step α is a *discrete transpose*, step β is a *discrete —symbolic— product of discrete matrices*. The composition of them is a *discrete transpose product*. (There are algorithms available that given $H(NE)$ and a numbering p of its vertices, directly generate the elimination graph of $G_p(NE)$.)

$H(NE)$ can be introduced as the *network discrete model* —for hystorical reasons one could call $G(NE)$ the network [associated] graph— although it is an open question whether one should allow for multiplicities in the edges of the hypergraph $H(NE)$. If so, then the map in 1 would be one-to-one.

A last observation is that network discrete models provide the skeleton for the network abstract data types to be used in the adjustment systems. Moreover, since graphs, hypergraphs, lists and sets are basic mathematical objects, the fundamental operations and algorithms on them can be borrowed from standard discrete mathematical software packages.

4 A REMARK ON CLASSICAL PHOTOGRAMMETRIC NETWORKS

Graphs associated to photogrammetric blocks, for both bundle or independent model methods, are bipartite. A graph G is called *bipartite* if $V(G)$ can be partitioned into two subsets —parts— in a way that an edge always connects vertices of different parts. In the case of a bundle network, one part has as many vertices as points are in the network and the other part has a number of vertices that equals the number of images.

n-partite graphs are defined in a similar way. For a given graph G, its *chromatic number* $\chi(G)$ is the smallest n for which G is n-partite. In general, computing $\chi(G)$ is NP-complete. Note that a bundle network with additional selfcalibration parameters can be associated to a tripartite graph.

For photogrammetric blocks, in the frame of a general network adjustment program, the traditional steps of forming the reduced normal equations and numbering of their group unknowns, can be put in an abstract form as follows:

- check the network graph for bi- or tripartiteness.

If the answer is yes, then proceed and

1. generate the two or three partitions,
2. generate a partial elimination graph R,
3. number the vertices of a suitable subgraph of R.

Testing whether a graph is bipartite and generating the vertex parts is easy (the well known equivalence between bipartite graphs and graphs with no cycles of odd length must be used). For tripartite graphs the generation of the parts is more involved but it still can be done [3][Chapter 3]. This solves step 1.

For step 2, consider the following less restrictive definition of graph numbering than the one given in Section 2.2: any map $p : V \longrightarrow \{1,\ldots,\#(V)\}$. If p is taken as $p(\text{any point}) = 1$ and $p(\text{any image}) = 2$, then the elimination graph R generated by applying the path theorem to p corresponds to the graph of the intermediate symmetric matrix obtained after factorization of the point unknowns [3][Chapter 2].

For the last step, it suffices to extract the subgraph of R corresponding to the image unknowns and apply your favorite numbering algorithm to it.

```
F\C  12345678901234567890123456789 01

01   +.....+.....+.....+.....+.....+
02   ..............................**..
03   ......................**...*..*.
04   ......................*..*.*....*
05   ......................*..*.*....*
06   ......................**...*..*.
07   +.....+.....+.....+.....+..**.+
08   ......................**...*..*.
09   ................**...*..*.*....*
10   ..............*..*.*....*......
11   ..............*..*.*....*......
12   ................**...*..*.*....*
13   +.....+.....+.....+..**.+.*..*+
14   ................**...*..*.*....*
15   ..........**...*..*.*....*......
16   ........*..*.*....*............
17   ........*..*.*....*............
18   ..........**...*..*.*....*......
19   +.....+.....+..**.+.*..*+*....*
20   ..........**...*..*.*....*......
21   ...**...*..*.*....*............
22   ..*..*.*....*..................
23   ..*..*.*....*..................
24   ...**...*..*.*....*............
25   +.....+..**.+.*..*+*....*.....+
26   ...**...*..*.*....*............
27   ..*..*.*....*..................
28   .*....*........................
29   .*....*........................
30   ..*..*.*....*..................
31   +..**.+.*..*+*....*.....+.....+
```

Figure 1: A cut subset for *SQ* 31 *LD* 6 (automated nested dissection).

In general, all the different policies of dealing with the "sorting of unknowns" in photogrammetric blocks can be reduced to numbering alternatives of bipartite graphs. Conversely, the extremely efficient methods devised by photogrammetrists for their blocks over the past two decades can be transferred to other sparse gaussian elimination problems in other fields where bipartite matrix graphs appear.

5 TROUBLESOME ASPECTS OF HYBRID NETWORKS

Compared to classical networks, hybrid networks may be troublesome because their local and regular connectivity structure is lost. In order to illustrate this statement, an example (Figure 1) will be given before generalizing.

Figure 1 depicts a cut subset generated in the first step of a nested dissection graph numbering algorithm for arbitrary networks [8]. [4] The elements of the cut

[4]The nested dissection algorithm has been selected since it is the algorithm to be used in large problems [11]. If the hybrid network is medium sized or small any sequential algorithm, for instance the banker's [15], will do the job reasonably well.

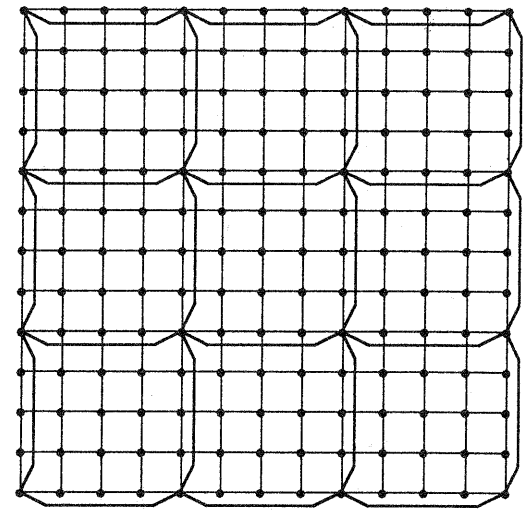

Figure 2: A graph with a 4-distance connected subgraph.

subset are marked with the character *. The graph —SQ 31 LD 6—is based on a regular grid graph — SQ 31— whose vertices are connected to their four N, E, S and W neighbors. Thus, SQ 31 is of order 961 and size 1860. SQ 31 LD 6 is SQ 31 plus a 6-distance connected subgraph LD 6 (see Section 6.3 and Figure 2 with a SQ 13 LD 4 graph); it is of order 961 and size 1920. In other words, SQ 31 LD 6 is a simplification of a regular graph perturbed with the long edges of LD 6. The simplification aims at being representative of a photogrammetric block which is adjusted together with the terrestrial control network or, also, of a conventional geodetic network readjustment that brings together a main and a densification network.

The fill-in factor obtained after applying nested dissection to SQ 31 is 4.67; for SQ 31 LD 6 is 6.88 and for a graph of the type SQ 31 LD 6 LD 3 is 9.66 [3][Chapter 5]. Note that for either cases SQ 31 LD 6 and SQ 31 LD 6 LD 3, numberings do exist which lead to fill-in factors very close to 4.67! The problem behind is the inability of the algorithms to produce a clean cut subset in the presence of the perturbing subgraphs LD 6 and LD 3 (Figure 1). Of course, this depends on the particular numbering algorithm (see [11]); this point is discussed in [3].

In [3], more cases of hybrid troublesome networks are analysed. In general, it can be stated that graphs of hybrid networks have a dominant structure of the classical type plus some perturbing edges; for instance in aerial triangulation, edges induced by drift correction parameters if aerial GPS control is used or edges induced by the terrestrial control network which destroy bipartiteness.

To face those problems, rather than to establish brand new numbering algorithms, it seems wiser to develope network analysis tools for the detection and isolation of the perturbing edges as pursued in the following section. Once this is done, appropiate action can be taken before the actual numbering algorithms be applied to the unperturbed underlying graph. These tools are of interest even for conventional networks since [structural] gross errors, for instance point numbering errors, modify the network graph connectivity in a similar way as observations between distant parameters do.

6 SOME DISCRETE ANALYSIS TOOLS

The concepts developed in this section can be found in more detail in [3][Chapter 6]. They are here introduced from less to more difficult, starting with the almost trivial task of analysing the connectivity propagation.

6.1 Superconnectivity

For any $x \in V(G)$ consider the sequence

$$d_0(x)\, d_1(x)\, \ldots\, d_r(x),$$

where $d_i(x) = \#(L_i)$, $i \in \{0, \ldots, r\}$ and where $L_0 = \{x\}$, $L_1 = Adj(x)$ and $L_i = Adj(L_{i-1}) - L_{i-2}$ for $i \geq 2$.

A complete analysis of the sequences $d_0(x)\, d_1(x) \ldots d_r(x)$ requires too big a computational effort. The computation of a restricted number of elements of the above sequences —the first elements provide the most relevant information—, however, is almost as helpful as the whole computation. This is specially true if the very pathological vertices like border ones are removed from the graph as soon as they are detected.

6.2 Nonlocality

Let $G = (V, E)$ be a graph and $e = \{a, b\}$ one of its edges such that $\#(E) \geq 3$ and $(V, E - \{e\})$ is connected. The *nonlocality* of e, $r(e)$ is defined as

$$r(e) = d_{(V,E-\{e\})}(a, b).$$

$r(e) + 1$ is, obviously, the length of the shortest cycle through the edge e. The concept of nonlocality, however, fits the intuitive idea of *discrete distance* between graph —network— vertices better and can be further extended by considering $d_{(V,E-F)}(a, b)$ for some subset F of edges.

Nonlocality has some limitations. For instance, for any edge e of P in Figure 3 it is $r(e) = 6$. This allows the detection and isolation of P. On the contrary for the graph of Figure 2 the detection of the perturbing subgraph is not possible through the analysis of nonlocality since $r(e) = 3$ for any edge. This question is dealt with in the next section.

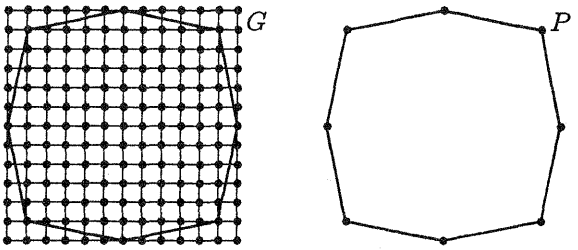

Figure 3: A graph G with a 6-distance connected subgraph P.

6.3 Graph filtering

First, the concept of n-distance connected subgraph is introduced. Let n be a positive integer, $G = (V, E)$ a graph and P a connected subgraph such that $\#(V(P)) \geq 2$ and that $d_I(u, v) \geq n$, $\forall u, v \in V(P)$ where $I = (V, E - E(P))$. P is then called a *n-distance subgraph of G.*

Figure 2 shows a 4-distance subgraph and Figure 3 a 6-distance subgraph (P).

A graph filter f is a graph operator; i.e. given a graph G, $f(G)$ is a subgraph of G. Throughout the section, B will stand for a minimal weight (length) cycle basis of G and q for an integer $q \geq 3$.

Filter $\mathbf{h^q}$. $h_B^q(G)$ is the subgraph of G defined as

$$h_B^q(G) = G(\{v \in V(G) : \exists c_j \in B, w(c_j) < q, \; v \text{ is a vertex of an edge of } c_j\}).$$

$h_B^q(G)$ may be nonconnected or the empty set. Based on the fact that (C, F) is a matroid on C (Section 2.5), it can be proven that $h_B^q(G)$ does not depend on the particular base B.

Filter $\mathbf{l_B^q}$. In a similar way the subgraph $l_B^q(G)$ may be defined as

$$l_B^q(G) = G(\{v \in V(G) : \exists c_j \in B, l(c_j) > q, \; v \text{ is a vertex of an edge of } c_j\}).$$

Again $l_B^q(G)$ may be nonconnected or even the empty set but in this case $l_B^q(G)$ is clearly dependent on B. In spite of this dependency, under certain conditions, some edges of G belong to $l_B^q(G)$ no matter which particular minimal basis is chosen. This property is useful in practice and formalized as follows: if P is a q-distance connected subgraph, for any minimal cycle basis B, it holds that $P \subset l_B^q(G)$.

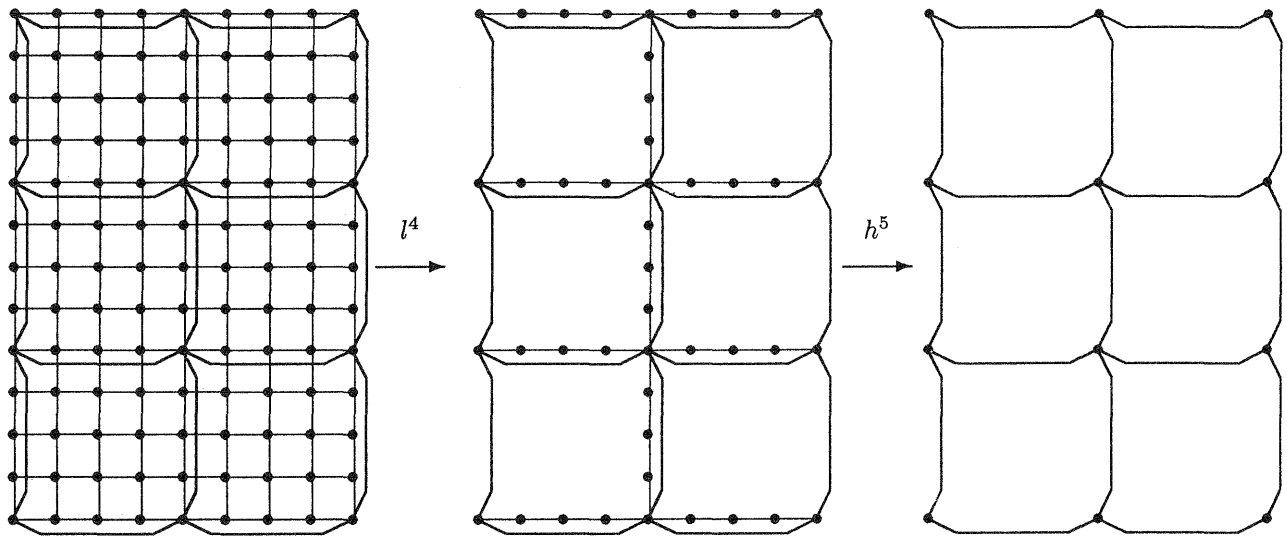

Figure 4: Detection and isolation of a 4-distance connected subgraph.

An example of detection/isolation of a 4-distance connected subgraph with the composed filter $h^5 \circ l^4$ is given in Figure 4.

6.4 On algorithms for minimal weight bases

A major drawback of the discrete filters proposed in the former section is the lack of fast algorithms for finding cycle bases of minimal length. In 1987, J.D.Horton published the first algorithm for constructing minimal weight cycle bases [10]. Although it is a very expensive algorithm —it takes time $O(m^3 n)$, for a graph of order n and size m— it has the benefit of showing that the problem of finding a minimal cycle basis is not NP-complete.

As Horton writes in his paper, *there is considerable room for improvement on this problem.* In particular, it remains an open question whether a faster algorithm for [sparse] graphs do exists.

7 HINTS FOR NUMBERING ALGORITHMS

A practical way to deal with the numbering of complex graph structures is to build up the numbering map $p : V(G) \longrightarrow N$ in n steps, which is equivalent to build q where

$$q : V(G) \longrightarrow N \times \overset{n}{\cdots} \times N, \qquad (2)$$

and then identify the natural order in N with the lexicographic order in N^n.

If $q(v) = (q_1(v), \ldots, q_n(v))$, the sets I_i, $I_i = \{q_i(v)\}_{v \in V(G)}$ are computed succesively for $i = 1, \ldots, n$ according to different criteria set for each step. An example for such criteria sequence could be as follows.

1. Compute the connected components $C_1, \ldots, C_m$ of G; $q_1(v) = k$ if v is a vertex of the connected component C_k.

2. For each $i \in I_1$ consider the subgraph

$$G(\{v : q_1(v) = i\});$$

set $q_2(v) = 2$ if v is a border vertex (Section 2.1) or almost (a vertex with too many adjacent vertices according to some relative threshold).

3. Consider the family of subgraphs

$$G(\{v : q_1(v) = i, q_2(v) = j\}),$$

for all $i \in I_1$ and $j \in I_2$. Try to apply some of the techniques described; for instance, check for bipartiteness, look for high nonlocality values, etc.

4. Consider the family of subgraphs

$$G(\{v : q_1(v) = i, q_2(v) = j, q_3(v) = k\}),$$

for all $i \in I_1$, $j \in I_2$ and $k \in I_3$. Apply to each subgraph nested disection or any algorithm you might prefer.

5. ...

Note that the former sequence is just an example for the sake of illustrating the idea, and that it can be further refined if one has the algorithmic tools to do it. A possibility is to devote some step m to assign values to $\{q_m(v)\}_{v \in V(G)}$ according to *a priori* known information on the network which might come from a previous adjustment, from approximate algorithms

or directly from the human being in charge of the computation.

If the complexity of the situation so requires or if one is trying to detect structural gross errors, the above procedure could be done even interactively.

8 CONCLUSIONS AND OUTLOOK

From Section 3, Section 4 and from [3] it seems possible to set up a discrete model for the classical [least squares] adjustment of general networks. All the information required for the model is contained in the hypergraph associated to the functional model design hypermatrix (block matrix). In particular, operations like formation of reduced normal equations, formation of nested dissection blocks and partial elimination of unknown groups can be formulated as pure [generalized] numbering/elimination operations on graphs.

It is quite clear that for some of the concepts and the results presented here to become practicable (recall Section 6.4) key problems are still to be solved; considerable research is still to be done both in the theoretical and applied sides. This is, therefore, just an intermediate paper though some of its ideas have been already applied at the *Institut Cartogràfic de Catalunya* in the development of the GeoTeX system [4]. (More details, practical motivation and proofs to all statements made here can be found in [3].)

Last but not least, it will be more than enough if the paper contributes to the growing feeling that techniques from discrete mathematics can be of help for a new generation of photogrammetric/geodetic procedures and software, even in the almost old-fashioned field of network adjustment.

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