EVALUATION OF ALGORITHMS FOR SURFACE INTERPOLATION OVER TRIANGULAR PATCH

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ABSTRACT

Surface interpolation over a triangulation with scattered DEM may maintain the continuity and smoothness of surface while it is contributed by a condition that two triangles in each shaded quadrilateral must be related to the same planar affine map. In other words, two adjacent surface patches posses the same tangent plane at each point of their common boundary. According to this principle, two numerical experiments were used to examine three piecewise polynomial interpolants which are linear, cubic, and quintic polynomials. Five criteria including average absolute and relative accuracy, root mean square error, CPU time, and visualization were chosen to evaluate the tasks of surface construction. Conclusively, the best performance is the cubic polynomial interpolation.

KEY WORDS: DEM, surface interpolation, smoothness, triangulation.

1. INTRODUCTION

A smooth interpolatory surface from scattered DEM is especially desirable when a visual impression of the data is called for. This task may be handled by a method of surface interpolation on a triangular patch. Let the data set be comprised of a set of points (x_i,y_i), arbitrarily distributed in the x-y plane, with corresponding ordinates z_i, i = 1,...,n. These points are the components for a bivariate function F(x,y) which interpolates the data values, i.e., $F(x_j,y_j) = z_j$. This is denoted by bivariate interpolation and it should satisfy two requirements: first, the interpolant should be continuously differentiated everywhere in the underlying domain; second, the scheme should be local, i.e., evaluating the interpolant at a point within a specific triangle should require only function and gradient values at the vertices of that triangle. This approach, which has been employed by Lawson (1977) and Akima (1978); it consists of the following three procedures:

(1)Partition the convex hull of the set of points into triangles by connecting the points with line segments.
(2)Estimate partial derivatives of F with respect to x and y at each of the points using the data values on either a set of nearby points or all of the points.
(3)For an arbitrary point (x, y) in the convex hull of the set of points, determine which triangle contains the point, and compute an interpolated value F(x, y) using the data values and estimated partial derivatives at each of the three vertices of the triangle.

Later, the construction of triangulation from the scattered data is described briefly. Three different degree polynomials with gradient estimation were used to construct two different experimental surfaces. According to the error analysis and visualized judgement, three algorithms were evaluated.

2. TRIANGULATION IN A PLANE

Suppose that $P = \{p_i = (x_i, y_i)\}_1^n$ is a set of n distinct points in the plane. The set $T = \{(v1_i, v2_i, v3_i\}_1^m$ of triples of integers chosen from {1,..., n} is called a triangulation of P provided that the points $p_{v1_i}, p_{v2_i}, p_{v3_i}$ are the vertices of a triangle t_i for $1 \le i \le m$. Each triangle contains exactly three points of T and these are precisely the vertices of the triangle. The interior of the triangles $\{T_i\}_1^m$ are pairwise disjoint, i.e., no overlapping. The union of $\{T_i\}_1^m$ is equal to the convex hull of P.

In general, a given point set P can be triangulated in several ways and all triangulations of P can be shown by induction as the following relationship:

$$m = \text{number of triangles} = 2n - n_b - 2$$
$$n_e = \text{number of edges} = 3n - n_b - 3$$

where n_b denotes the number of boundary vertices. A triangulation is completely described by the set T of the integer triples giving the vertices of the triangles making up the triangulation. Hence, to store a triangulation in a computer, the 2n numbers describing P and the 3m integers describing T are needed. Before constructing the triangulation from a general set of points, it will be convenient to first consider the problem of triangulation a set of four points whose convex hull is a convex quadrilateral. In this case there are precisely two different triangulations which may denote by t_1 and t_2 (Figure 1).

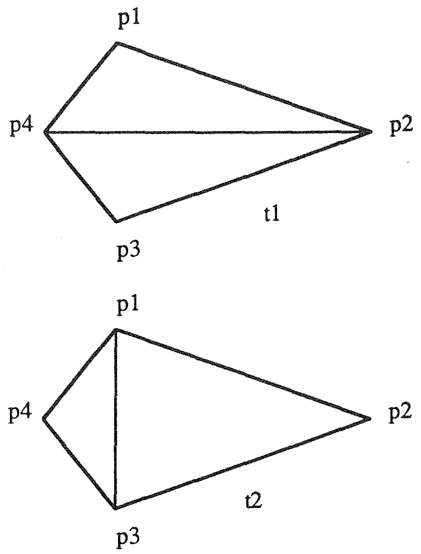

Figure 1: Triangulation of four points

A number of different criteria have been invented for choosing between t_1 and t_2. The simplest one is to let $p_1, \ldots, p_4$, be oriented as in Figure 1, and let d_1 = length of the diagonal p_2p_4, and d_2 = length of the diagonal P_1P_3. Then t_2 is a better triangulation of t_1 with respect to the shortest diagonal criterion which provided that $d_1 > d_2$. Although it is very easy to implement, the shortest diagonal criterion does not do a good job of avoiding thin triangle. The following max-min angle criterion contributed by Lawson (1977) is specially designed to avoid thin triangle. If there is a triangle t, let a(t) = minimum angle in t. Associated with the triangulation T, let a(T) = min $\{ a(t) : t \in T \}$. Then t_2 is better than t_1 with respect to the max-min angle criterion which provided that $a(t_2) > a(t_1)$. In Figure 1, $a(t_1) = 30^0$ while $a(t_2) = 46^0$, and thus according to the max-min angle criterion, triangulation t_2 is the better of the two.

Although the error in approximating a smooth function on a triangle can be estimated in terms of the largest angle in the triangle, Barnhill and Little (1984) suggest the following min-max angle alternative to the max-min angle criterion: if there is a triangle t, let a(t) = 1/(maximum angle in t). Associated with the triangulation T, let a(T) = min $\{ a(t) : t \in T \}$. Then t_2 is a better triangulation than t_1 with respect to the min-max angle criterion which provided that $a(t_2) > a(t_1)$. Regarding the optimal triangulations of general points sets, let Q be a criterion for choosing the optimal triangulation of a quadrilateral based on maximizing some measure a(t) of the thinness of the triangles. For each triangulation T of a point set P, let a(t) be the vector measure of goodness of T defined above. Then a triangulation t_2 is said to be an optimal triangulation of P with respect to Q provided that $a(t_2) > a(t_1)$ for all other triangulations t_1 of P.

Post optimized, iteratively built, and divide-and-conquer approaches are three rather different algorithms which can be adopted to work with the above swap criteria for constructing the optimal triangulations. The post optimized approach first constructs an initial triangulation, and then goes through the quadrilaterals and makes swaps where necessary. The iteratively built approach starts with one triangle and adds one point at a time, making sure that at each step the current triangulation is locally optimal. The divide-and-conquer approach divides the data up into pieces, finds locally optimal triangulations for each piece, and then merges these triangulations. Every triangulation approach requires at least nlog(n) operations (Lawson, 1977), where n is the number of points being triangulated.

3. GRADIENT ESTIMATION

Many techniques for producing a surface from scattered DEM require gradients at the data points. Typically, only positional data are known so the gradients must be estimated before the surface values can be computed. The quality of the surface depends on the estimated gradients so it is important to compute accurate estimates. For the planar interpolation problem, there are five common gradient estimation methods: Shepard, multiquadric, weighted quadric, weighted planar, and triangular Shepard methods. The Shepard method which interpolates only to positional data is defined by

$$GS(x, y) = \sum_{i=1}^{n} w_i(x, y)\, F_i\,(x, y) \tag{1}$$

where

$$w_i(x, y) = \frac{\prod_{j \neq i}^{n} d_j^2(x, y)}{\sum_{k=1}^{n} \prod_{j \neq k}^{n} d_j^2(x, y)}$$

and

$$d_j^2(x, y) = (x - x_i)^2 + (y - y_i)^2$$

GS(x, y) was localized in the following way. For each (x_i, y_i), GS is determined by the six data points closest to (x_i, y_i). Note that (x_i, y_i) has to be excluded from the computations; otherwise, the gradients at (x_i, y_i) would be zero.

The hyperbolic multiquadric method is defined by

$$GM(x, y) = \sum_{i=1}^{n} c_i \sqrt{d_j^2(x, y) + r} \tag{2}$$

where $d_j^2(x, y)$ is given above, r must be specified, and the coefficients c_i are found by solving the n by n system $GM(x_i, y_i) = F_i$, $i = 1, ..., n$. GM is localized as follows: for each data point (x_i, y_i), GM is determined by the nineteen points closest to (x_i, y_i) plus (x_i, y_i) itself.

The weighted quadratic least squares method GQ is determined by the ten points closest to (x_i, y_i) plus (x_i, y_i) itself, for each data point (x_i, y_i).The weights are

$$w_i(x, y) = (r_i - d_j^2(x, y))^2 \tag{3}$$

where r_i equals the distance between (x_i, y_i) and the most distant of the ten closest points. In addition, the weighted planar least squares method GP is determined by the eight points closest to (x_i, y_i) plus (x_i, y_i) itself, for each data point (x_i, y_i). The weights as above are used with r_i equal to the distance between (x_i, y_i) and the most distant of the eight closest points.

The triangular Shepard method is defined by the n data $\{(x_j, y_j, F_j)\}_{j=1}^{n}$ and a triangulation $\{(v1_i, v2_i, v3_i\}_{i=1}^{m}$, where m is the number of triangles in the triangulation. It is given by

$$GT(x, y) = \sum_{i=1}^{m} w_i(x, y)\, LF_i(x, y) \tag{4}$$

where

$$w_i(x, y) = \frac{1}{\prod_{j=1}^{3} d_{ij}^2(x, y)} \Big/ \sum_{i=k}^{m} \frac{1}{\prod_{j=1}^{3} d_{kj}^2(x, y)}$$

$LF_i(x, y)$ is the linear interpolant over the i-th triangle, and d_{ij} is the Euclidean distance from (x, y) to vertex j of the triangle i. Note that no parameters need be specified for LF, although it is dependent on the triangulation.

4. SURFACE INTERPOLANTS

Several interpolation schemes that assume prescribed values and their gradients on the boundary of a planar triangle have been developed. Barnhill, Birkhoff, and Gordon (1973) first developed such a method using lines parallel to the triangle sides, and Nielson (1979) has developed method for line segments joining vertices to their opposite sides. This method consists of using the planar interpolant on the underlying planar triangle with boundary data obtained by projecting values and gradients onto the plane.

For such an interpolation, let p be a point of the triangle with vertices, in counterclockwise order v1, v2, and v3, and let p' be the central projection of p onto the underlying planar triangle having the same vertices T = (v1,v2,v3), i.e., boundary points of the triangle project to boundary points of T. Denote by b1, b2, and b3 the barycentric coordinates of p' with respect to T. These are defined by

$$\sum_{i=1}^{3} b_i = 1 \tag{5}$$

and

$$\sum_{i=1}^{3} b_i v_i = p' \tag{6}$$

This is a basic scheme of piecewise linear surface interpolation. This scheme is given a triangulation of a set of points in the plane and computes the value at (x, y) of a piecewise linear surface. Equivalently, $b_i = A_i/A$, where A is the area of t and A_i is the area of the subtriangle (p', v_j, v_k) for $(i, j, k) \in s = \{(1,2,3), (2,3,1), (3,1,2)\}$. Consider the line defined by v1 and p'. This line intersects the edge opposite v1 at the point

$$q1' = \frac{b2\, v2 + b3\, v3}{b2 + b3} \tag{7}$$

The points q2' and q3' lying on the sides opposite v2 and v3, respectively, are defined similarly. The following side-vertex-based interpolant is derived by Lawson (1977). For a bivariate function f on T, the interpolant defines

$$F(p') = \sum_{i=1}^{3} w_i\, H_i(p') \tag{8}$$

where

$$w_i = \frac{b_j\, b_k}{b_i\, b_j + b_j\, b_k + b_k\, b_i}$$

for $(i, j, k) \in s$

and

$$H_i(p') = f(q_i') + b_i^2\,(3 - 2b_i)\,[f(v_i) - f(q_i')] + b_i\,(1 - b_i)\,[b_i < f(v_i), q_i' - v_i> - (1 - b_i) < f(q_i'), q_i' - v_i>]$$

that is, H_i is the Hermite cubic interpolation of the endpoint values and directional derivatives of f. Note that the weights w_i are not defined at the vertices where two of the barycentric coordinates are zero.

A similar scheme, the values of the function at point (x, y) in a triangle is interpolated by a bivariate fifth-degree polynomial in x and y; i.e.,

$$F(x, y) = \sum_{j=0}^{5} \sum_{k=0}^{5-j} q_{jk}\, x^j\, y^k \tag{9}$$

Note that there are twenty one coefficients to be determined. The values of the function and its first-order and second-order partial derivatives are given at each vertex of the triangle. This yields eighteen independent conditions. The partial derivative of the function differentiated in the direction perpendicular to each side of the triangle is a polynomial in the variable measured in the direction of the side of the triangle. Since a triangle has three sides, this yields three additional conditions and assures the smoothness of interpolated values.

5. DESCRIPTION OF EXPERIMENTS

Two numerical experiments were provided to examine three surface interpolation algorithms: linear, cubic, and quintic polynomials. The first experiment is a set of hundred simulated DEM points. Their x and y coordinates are generated by random number generator. The z coordinates are calculated according to an arbitrary function as the following and the diagram is shown in Figure 2. The second experiment has fifty scattered DEM points, feature points of terrain, which come from a field project (Figure 3).

$$\begin{aligned} z(x, y) = {} & 0.75 \exp\left[\frac{-(9x-2)^2 - (9y-2)^2}{4}\right] \\ & + 0.75 \exp\left[-\frac{(9x+1)^2}{49} - \frac{(9y+1)}{10}\right] \\ & + 0.5 \exp\left[\frac{-(9x-7)^2 - (9y-3)^2}{4}\right] \\ & - 0.2 \exp\left[-(9x-4)^2 - (9y-7)^2\right] \end{aligned}$$

6. CRITERIA OF EVALUATION

The purpose of this evaluation was to examine the potential of three interpolation algorithms: linear, cubic, and quintic polynomials. The following criteria were used to determine the relative goodness of the three algorithms:

(1)Mean absolute error (MAE)

$$MAE = \frac{\sum_{i=1}^{n} (z^* - z)}{n} \tag{10}$$

where z* are interpolated values and z are function values.

(2)Mean relative error (MRE)

$$MRE = \frac{\sum_{i=1}^{n} \frac{(z^* - z)}{z}}{n} \tag{11}$$

(3)Root mean square error (RMSE)

$$RMSE = \frac{\sum_{i=1}^{n} (z^* - z)^2}{n} \tag{12}$$

(4)Running-time of the central processing unit

(5)Visual inspection of the surfaces

7. ANALYSIS OF RESULTS

The task of surface interpolation was done by three different degree polynomials over triangular patch algorithms. There are one evaluation table and two sets of diagrams for analyzing the potential of these three algorithms. For the first criterion, the values of mean absolute error gives information about the error range of interpolation. In this case, the quintic approach has the smallest error; the linear approach has the largest error; and the cubic approach has an error below that of the linear and above that of the quintic approach. Based on mean relative error, the situation is similar to that of the first criterion. The quintic approach has the smallest relative error; the linear approach, the largest relative error; and the cubic approach, a relative error below that of the linear and above that of the quintic approach. As a matter of fact, the relative error of the cubic approach is much closer to that of the quintic approach. The root mean square error carries one of the most important messages in this evaluation. A little change occurs in this portion. The cubic approach gives the best result, while the linear approach gives the worst case. For the running-time of the central processing unit (CPU), in this research, three algorithms were run in the IBM compatible 486 personal computer. The consuming time is proportional to the degree of polynomial. The quintic approach requires more time and the linear approach requires less time. Table 1 displays above four items. In the second experiment, CPU running-time of three algorithms are 28.12 seconds in the linear approach, 67.94 seconds in the cubic approach, and 83.27 seconds in the quintic approach. Regarding the visualization, i.e., the smoothness of interpolated surface, on inspection of the pictures of simulated DEM in Figure 2 and scattered DEM in Figure 3, the performances of three algorithms are attractive. They are very smooth and their shapes are very good. With regard to the smoothness of the surface, the surfaces from the cubic approach have the best appearance.

According to above five criteria of evaluation, the general idea about these three algorithms is that the linear approach is time-saving and less accurate; the cubic approach has good accuracy and may generate smooth surface; and the quintic approach also has good accuracy, but consuming-time. Hence, the cubic approach has the great potential for surface interpolation in the scattered DEM.

8. CONCLUSIONS

Three-dimensional measurements passing through a surface are often taken by scientists and engineers. The methods of linear, cubic, and quintic polynomial on the triangular patch have application in finite-element analysis and computer-aided geometric design, as well as in the scattered DEM interpolation problem treated here. According to evaluation in this research, conclusively, the cubic approach is recommended to do surface interpolation in the scattered DEM.

Two related problems which were encountered during the research will be studied in the next phase. The first problem is the error bounds. The classical error bounds for approximating a smooth function on a triangle by a polynomial depend explicitly on the size of the smallest angle in the triangle. Thus, if the error bounds for a precise polynomial are defined over the two triangles forming a triangulation of a convex quadrilateral, the triangulation produced by the max-min angle criterion can get better bounds. If the Delaunay triangulation is constructed, the criterion for detecting the error bounds is worth consideration. The second problem is the fitness of triangulation to terrain, especially in the peak, pit, hole, island, and so forth.

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Table 1: Evaluation of three algorithms using simulated DEM

	Algorithm		
Criterion	Linear	Cubic	Quintic
MAE	0.00480	-0.00052	-0.00216
MRE	0.18911	0.09437	0.08900
RMSE	0.03526	0.02131	0.02535
Speed (second)	70.25	151.87	244.20
visualization	good	excellent	good

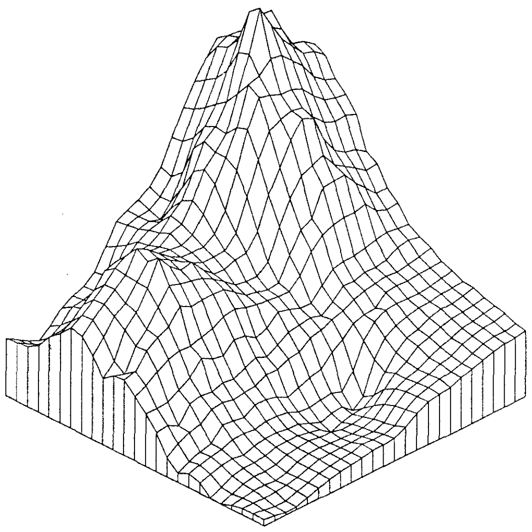

Figure 2a: Simulated DEM

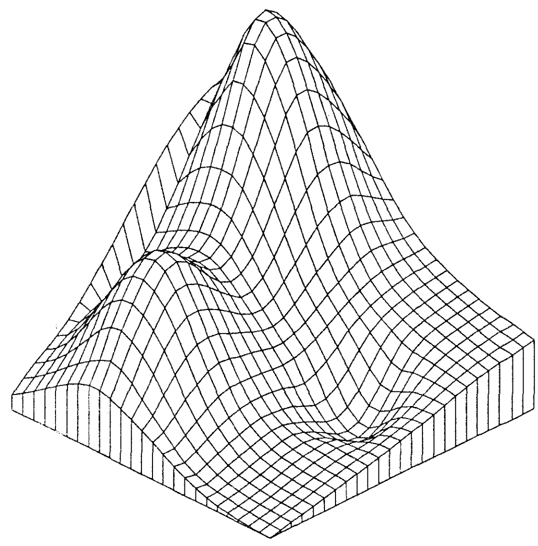

Figure 2c: Cubic surface interpolation

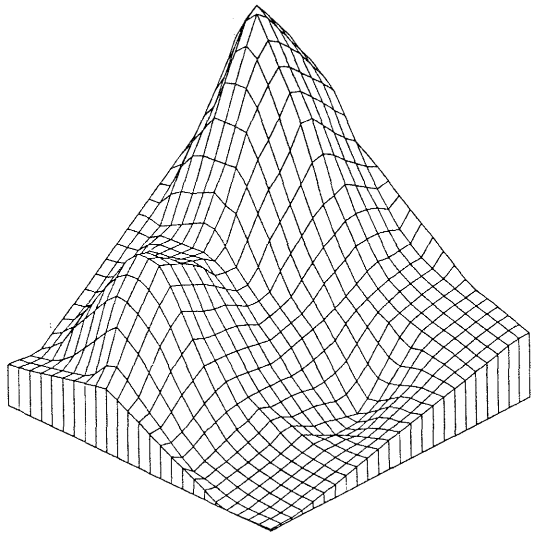

Figure 2b: Linear surface interpolation

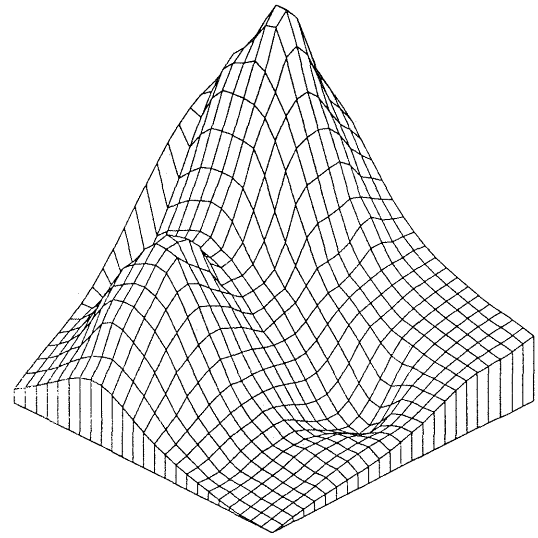

Figure 2d: Quintic surface interpolation

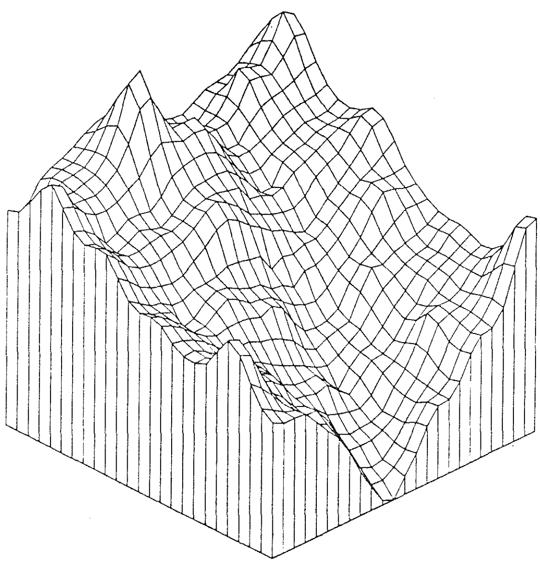

Figure 3a: Scattered DEM

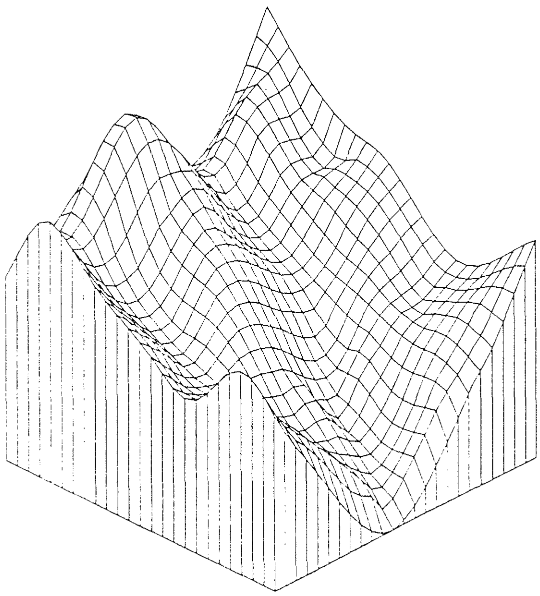

Figure 3c: Cubic surface interpolation

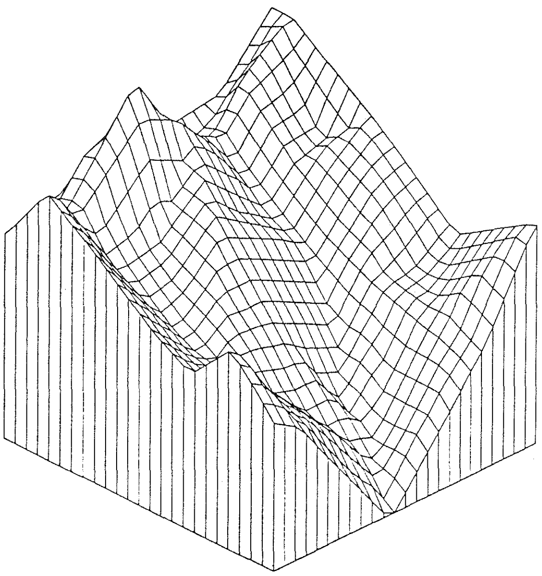

Figure 3b: Linear surface interpolation

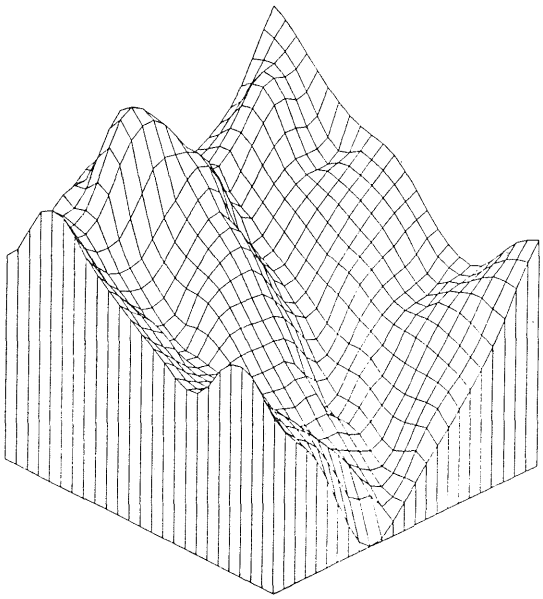

Figure 3d: Quintic surface interpolation