CRITICAL CONFIGURATION OF OBJECT SPACE CONTROL POINTS FOR THE DIRECT LINEAR TRANSFORMATION

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ABSTRACT

The numerical conditions of the normal equation matrix for a Direct Linear Transformation is influenced by geometric parameters, both in image and object spaces. The rigorous compilation of the l -1 norm condition number provides some insight into the causes of numerical problems. A number of cases were evaluated yielding interesting results, and based on these for the depth of object problem, a numerical approach to overcome ill conditioning was developed.

INTRODUCTION

For its simple structure, the Direct Linear Transformation method (DLT) is a surprisingly efficient algorithm and has the potential to be utilized in a real-time sensing system associated with micro-electronics.

DLT is based on the concept of space resection, however, with the inclusion of interior orientation parameters, eleven unknowns have to be solved. Chen (1985) stated, when comparing DLT with space resection based on collinearity equations:

It needs more full control points, at least six, and the solution is sensitive to the configuration of the object-space control, so extra care should be taken to incorporate as much deviation from the planar configuration as depth of field consideration permits.

After loooking at the geometry of space resection, especially in view of "dangerous surfaces", numerical aspects of the DLT approach are investigated.

DANGEROUS SURFACES IN PHOTOGRAMMETRY

The subject has been treated extensively in the special literature, e.g., in Jordan et. al., (1972), Hofmann (1953), and even in photogrammetric textbooks (Schwidefsky and Ackermann, 1976; Wang, 1979), and thus is outlined only briefly.

A basic mathematical formulation for dangerous surfaces can be derived (Jordan et. al., 1972) as follows: Assuming that there are n functions F_j to determine n parameters X, then a solution

$$X^T = (x_1, x_2, ..., x_n) \qquad (1)$$

is obtained from

$$F_j\ (x_1, x_2, ..., x_n) = 0; \qquad j = 1, ..., n \qquad (2)$$

If there are other solutions, then we can obtain these from the first solution (1) by series development

$$F_j(x') = F_j(x) + \sum_{i=1}^{n} \partial F_j/\partial x_i\ dx_i = 0\ . \qquad (3)$$

Thus we have a linear homogeneous system of equations for dX which has non-zero solutions if the functional determinant of equation (2) is equal to zero:

$$D(x_1, x_2, \dots x_n) = \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & & \frac{\partial F_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & & & \frac{\partial F_n}{\partial x_n} \end{vmatrix} = 0$$

In the specific case of space resection, this determinant can be represented as a product of two determinants when expressing the position vector of the unknown camera station in terms of the spatial distances to the control points and the angles enclosed by these.

One of these determinants, when set to zero represents the coplanity condition, i.e., control points plus camera station lie in one plane, which does not apply for the space resection. Setting the second determinant to zero and selecting a suitable coordinate system, results in the *dangerous cylinder*, which is a circular cylinder containing the three control points in its base circle. If the camera station lies on this cylinder, the space resection becomes indeterminable.

Since the mathematical model is affected by the stochastic nature of the photogrammetric observations, the *dangerous space* will become larger. Using the concept of y-parallax (i.e., $p_y = y' - y''$, which is the difference in the y-coordinate for a point imaged in two photographs), then the dangerous space is bounded by surfaces which correspond to the standard deviation σ of p_y. These surfaces are regular orthogonal surfaces of second order, which can be generally represented by:

$$C_1Z + C_2XZ + C_3Y + C_4XY + C_5 (Y^2 + Z^2) = \pm \sigma$$

Intersection of such a surface with a normal section X = constant always produces a circle. Depending on its parameters, this surface can be a hyperboloid, a hyperbolic paraboloid, a cone, cylinder or a pair of planes.

For space resection (see also Wang (1979)), the dangerous cylinder is defined as already mentioned, namely with its base circle passing through the control points. All other cylinders or a sphere do not represent dangerous surfaces.

For the DLT solution, the equivalent manipulation of the functional determinant provides the plane as the highest order dangerous surface, i.e.;

$$C_1X + C_2Y + C_3Z + C_4 = 0$$

and the location of the camera station is of no practical significance. However, numerical considerations need to be investigated.

TESTING OF NUMERICAL CONDITIONS FOR THE DLT NORMAL EQUATION MATRIX

Mathematical Formulation of Test Parameters

The degree of ill conditioning of the normal equation matrix, whether caused by numerical reasons or the geometry, can be measured by its condition number.

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In this paper, the l -1 norm condition number was used as indicator (Forsythe, et. al., 1977):

$$COND(A) = ||A|| \, ||A^{-1}||$$

$$||A||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

The smaller this condition number, the better the condition.

Test strategy

In order to evaluate the numerical conditions, the following aspects were investigated:

- number of object points
- translation (origin of the coordinate systems)
- ratio of focal length to object distance
- exterior orientation rotations
- field angle
- depth of object

It was decided to utilize computer simulation for this study. Thus test data were generated by projecting object coordinates into image space with specified interior and exterior orientations.

Two test patterns were selected, consisting of two layers with 18-, and three layers with 27 regularly distributed object points.

Test Results

a) Number of object points.

An increased number of object points does not improve the condition number, in fact when going from 18 to 27 points it became slightly worse. It appears that when distribution, pattern and geometrical configuration remain unchanged, rounding off errors affect the condition number.

This clearly shows that the numerical condition, representing the stability of inversion of the normal equation matrix has nothing to do with statistical reliability of the actual solution, which improves with larger overdetermination. Since we are testing the numerical condition, all subsequent tests were carried out with 18 points.

b) Translation

object space			image space			
x	y	cond. no.	X_0	Y_0	Z_0	cond. no.
0	0	0.7×10^2	0	0	0	0.4×10^2
-0.2	-0.2	0.5×10^2	1	1	1	0.3×10^3
10	10	0.2×10^7	10	10	10	0.3×10^7

In both domains, a large translation results in a large condition number, i.e. a poor numerical condition. However, it is interesting to note that the best location of the image coordinate origin is at the centre of gravity of the image coordinates (-0.2, -0.2), rather than at the principal point.

Numerically, an increase in the size of the coordinate figures leads to larger normal equation coefficients, thus it is efficient. Thus it is advisable to shift the origin of the coordinate system into the centre of gravity, which was done for the subsequent tests.

c) Ratio of focal length to object distance

image scale (enlarged) scale ratio	image/object numbers	condition
0.1	0.02	0.4×10^4
1	0.2	0.5×10^2
5	1	7.1
10	2	0.4×10^2
100	20	0.3×10^3
1000	200	0.2×10^6

The pattern shows that the best condition is achieved when the scale ratio is unity, and that the condition number increases with the second power ($CN_i \approx CN_o \cdot s^2$).

Thus, if we chose the units carefully, the numerical condition will improve, e.g. for 1:10 use mm (image) and cm (object); for 1:1000 use mm (image) and m (object). If both scales are equally expanded, then the actual size of the figures governs the condition number (i.e. 0.4×10^2 for multiplication by 10; 1.0×10^6 for multiplication by 100).

d) Exterior orientation rotations

ω	φ	α (in degrees)	cond. number
0	0	0	0.46×10^2
30	0	0	0.53×10^2
0	30	0	0.53×10^2
0	0	30	0.50×10^2
45	0	0	0.51×10^2
0	45	0	0.51×10^2
0	0	45	0.51×10^2
60	0	0	0.53×10^2
0	60	0	0.53×10^2
0	0	60	0.50×10^2
90	0	0	0.46×10^2
0	90	0	0.46×10^2
0	0	90	0.46×10^2
30	30	30	0.58×10^2
45	45	45	0.40×10^2

The rotations have no significant influence on the numerical condition.

e) Field angle (for constant scale)

focal length	cond. number
1	0.482×10^2
10	0.482×10^2
100	0.482×10^2
1000	0.482×10^2

This shows that the field angle, unlike for space intersection, has no influence on the numerical condition.

f) Depth of object

depth/planimetry ratio	cond. number
1: 1	0.4×10^2
1:10	0.2×10^4
1:100	0.2×10^6

X/Y/Z ratio	cond. number
1.0: 1.0:1	0.4×10^2
0.5: 0.5:1	0.5×10^5
0.5: 1.0:1	0.2×10^5
1.0: 0.5:1	0.2×10^5
0.1: 1.0:1	0.5×10^5

While the factor for the ratio between depth and planimetry is quite indicative, the mutual ratio between the magnitudes along the three axis provides good insight. It should be noted, that the problem of depth lies not in the relation between object and image, i.e. is independent of the geometrical arrangement that includes the camera station, but solely depends on the dimension ratios of the object itself.

Analysis of These Test Results

It is well-known that the block adjustment problem is a datum problem, where 7 known coordinates are the minimum needed for a 3-D solution. The algebraic problem is internally solved. In other words, the relations between image points complete the algebraic requirements, and the only external information required is the datum definition. While the space-resection is an algebraic problem, all unknowns must be solved by the relations of the image points and their corresponding object points.

DLT is basically a resection/intersection approach. The resection deals with one photo at a time. thus, no help can be obtained from the geometrical configuration constraints from multi-bundles of one object point, convergent photography, and/or large base/height ratios. In this study, it was shown that:

1. The increase in the number of points does not improve the numerical condition, but it does improve the reliability condition statistically.

2. The larger the translation from the center of gravity to the origin, the larger the condition number.

3. The absolute photo scale does not have any effect. The best case is numerically with scale equal to one, and both image and object having values close to unity. However, when the scale is small enough to make the relief parallax insignificant, i.e. practically to zero, numerical problems will arise. In other words, the absolute scale determines the resolution of the the image space, and thus affect the precision and resolution of the whole system.

4. Rotations have little effect on the condition of the coefficient matrix of the normal equations.

5. The field angle has no effect at the resection stage.

6. The depth problem should be understood as the mutual ratio between the 3-D magnitude of object. The cube is the optimum configuration.

Practical Test

A test with real data was performed with a non-metric photograph, digitized on the Wild A-10. Twenty-seven points were used. In this data set, the height-planimetry ratio is 1/10.

case	0	1	2	3	4	5
power of condition number	very large	26	14	4	3	2

where:
case 0: original data, with center of gravity close to (5000., 5000), i.e., the initial value assigned during photo digitization;
case 1: photo coordinates reduced to their center of gravity as origin;
case 2: Case 1 plus object coordinates reduced as well;
case 3: Case 2 plus scaling of object coordinates by 1/1000, and of photo coordinates by 1/100;
case 4: Case 2 plus scaling object coordinates by 1/10000, and of photo coordinates by 1/1000;
case 5: Case 4 plus scaling algorithm (referred to in the next section).

It is obvious, that the only significant physical factor is the depth. In most cases of daily life, numerical causes for ill conditionaling are larger and more frequent than the critical configurations. Therefore, it is recommended to:

1. Reduce the size of figures in the coefficient matrix by transforming object points to a coordinate system with its origin at the centre of gravity and scaling observations by using proper units. This is helpful in the formation stage of normal equations as well.

2. Keep the mutual 3-D magnitude ratio in object space as close to 1 as possible.

Improving the Algorithm

Following Uotila's (1974) suggestion, an improved algorithm was tested:

$$X = -[A^T (\Sigma_{Lb} + ASA^T)^{-1} A]^{-1} A^T (\Sigma_{Lb} + ASA^T)^{-1} L \ .$$

In this algorithm, matrix S plays the key role. Σ_{Lb} is the weight matrix.

An algorithm of scaling the column elements by dividing with the corresponding diagonal elements was also studied:

$$\begin{bmatrix} N_{11}^{-1} & & 0 \\ & N_{22}^{-1} & \\ & & \\ & & \\ 0 & & N_{um}^{-1} \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} & N_{1u} \\ & N_{22} & \\ & & \\ & & \\ N_{u1} & & N_{un} \end{bmatrix} = \begin{bmatrix} N_{11}^{-1} & & \\ & N_{12}^{-1} & \\ & & \\ & & \\ 0 & & N_{uu}^{-1} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \\ \\ U_u \end{bmatrix}$$

A test with simulated data was performed using both algorithms. This data set has 27 points, Z/XY ratio is 1/100.

case	0	1	2	3	4
power of condition number	6	5	4	2	2

Case 0: data with the translation to their centre of gravity and proper scaling;
Case 1: Case 0 plus Uotila's algorithm with both weight matrix and S being identity matrices;
Case 2: Case 0 plus Uotila's algorithm with an identity weight matrix and diagonal elements of S as 10 to the power: (3., 2., 1., 0., 3., 2., 1., 0., 3., 2., 1.)

Case 3: Case 0 plus Uotila's algorithm with diagonal elements of the weight matrix equal to 0.5, the diagonal elements of S as 10 to the power:
(2., 2., -1., 2., 2., 2., -1., 2., 2., 2., -1.)

Uotila's algorithm has the ability to reduce the condition number for the matrix inversion. However, an extra matrix inversion has to be performed, and the selection of S needs human interference. The condition of the extra inverse has to be considered as well. This causes some practical difficulties.

The scaling algorithm is straightforward. However, once scaled, it lost its symmetry, and specified inversion algorithms for symmetric matrices can no longer be used.

CONCLUDING REMARKS

Like other methods, the evaluation of photogrammetric data using the DLT approach will be affected by both geometric and numerical conditions.

Geometric conditions together with stochastic influences govern the precision of the estimated parameters. It is thus very important to avoid the proximity of dangerous surfaces.

Numerical conditions affect mainly the inversion of the normal equation matrix, and thus the ability to arrive at a numerical solution. Both simple steps, like translations and the manipulation of units, and more involved algorithms can improve the numerical conditions. One physical aspect has to be considered here, namely, the depth of the object. Numerically, a unit ratio between coordinate differences in all three directions in object space provides the best solution. While this may not always be possible to achieve, e.g., because of optical and other physical restrictions, an effort should be made to approximate this because such an arrangement is also geometrically strong.

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