COMPUTER AIDED MATHEMATICAL MODELLING OF A COMPLEX 3D-OBJECT

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ABSTRACT

In order to elaborate a project for a building, based on a model (a work of art), first it was necessary to measure it exactly. The object vas very complex and unsuitable for direct measurement. Therefore, to obtain the spatial coordinates of an appropriate number of points lying on the object's surface the close-range photogrammetry was being applied.

The project engineers required that every existing surface should be replaced by an appropriate analytically defined surface. They wanted it to pass as close to the measured points as possible, having at the same time the simplest possible form of equation.

The final solution consisted of 11 planes, 11 spheres and 5 ellipsoids. The r.m.s. deviation of the measured points from the adopted surfaces, being 3.2 mm, was quite acceptable compared with the dimensions of the object.

It should be observed that computations of the relevant intersecting curves, even of such simple surfaces, involved considerable numerical difficulties.

INTRODUCTION

To first elaborate a project of a building and then construct a model (a maquette) in order to facilitate the visualization of the object is an usual practice in construction engineering.

Nevertheless, it is possible to reverse the order, i.e. to make a maquette first and then project a building based on it. Such was the case described in this paper. The model - a work of art, was fashioned by the well-known Yugoslav sculptor Dušan Džamonja. It is a wooden maquette, of approximate size 1x1.8x0.45 m (see Fig. 1). Dimensions of the erected building were planned to be about 150x250x65 m.

The project engineers set us the task to model the maquette mathematically. To achieve that we had to digitize the maquette first. It is obvious (see Fig. 1) that due to its complex spatial structure the model is markedly unsuitable for any kind of conventional measuring. Therefore, the methods of close-range photogrammetry naturally offered themselves. In that way we obtained model coordinates of a set of points lying on the object's surface - a digital model.

Due to measuring method the coordinates of digitized points contained errors and a surface of the maquette (see Fig. 1) was also consideralby rough. On the other hand, one sees that each partial surface produces the impression of a regular surface. Therefore, the project engineers additionally requested us to filter the digital model in some way and to smooth it, i.e. to replace it by an analytical model which would not define particular surfaces in terms of sets of points, but would use analytic equations. That analytical model had to fulfill the following conditions. On the one hand the

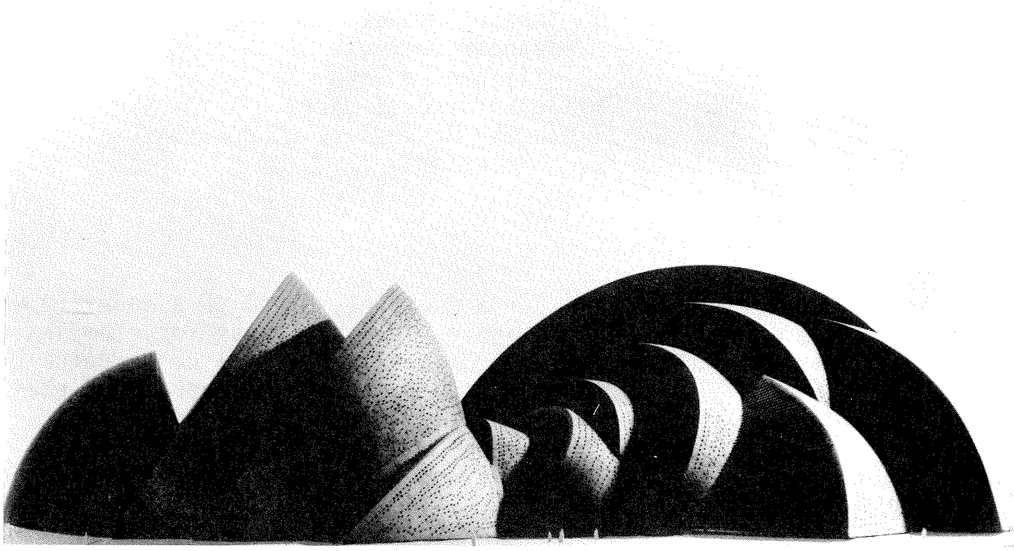

Fig. 1. View of the maquette

mathematical approximation of each real surface had to pass "close enough" to the measured points. "Close enough" is to be understood in the sense that all deviations had to remain within prescribed limits. On the other hand, there was a requirement that there should be as few different kinds of surfaces as possible. Also, the analytic equation of each surface had to be the simplest possible in order to meet possible future need for more easy modification of the analytical model, should the statical computations require that.

The effected measurements of the maquette, as well as the subsequent mathematical-analytical modelling will be described on the following pages.

TAKING OF PHOTOGRAPHS AND MEASUREMENTS OF MODELS

It is evident from Fig. 1 that the maquette was not suitable for the direct surveying. Fast, economical and precise definition of the maquette geometry is still possible to be carried out by means of the terrestrial photogrammetry. Since for the definition of each particular surface it was necessary to determine the great number of points, which were not specially marked, we have chosen to make a classical analogue photogrammetric measurement of each stereomodel. Having completed these measurements of all three stereomodels, they were transformed by spatial transformation in a common coordinate system of the maquette.

The photographs of the maquette were taken in Džamonja's atelier with Carl Zeiss Jena UMK terrestrial camera equipped with 100 mm Lamegon-wide angle focal lenght lens. This lens has the possibility of changing the principal distance setting, which permits a range of focus from 1.4 m to infinity without disturbing the interior orientation.

In order to cover the maquette entirely it was necessary to take photographs from three bases disposed around the maquette. In consideration of objective possibilities the bases were placed so that the base-distance ratio was 1:3 to 1:5 with the mean distance of 3 m. Under these circumstances the scale of the photographs was from 1:20 to 1:40.

The measurements were made on a mechanical projection instrument Wild Autograph A7. 497 points laying on the surface of the maquette were measured from the three stereo pairs. With the help of 11 pass points disposed all over the maquette, the model coordinates of each separate stereomodel were spatially transformed into the common coordinate system of the maquette. The right-handed Cartesian coordinate system of the maquette was set so that the X-Y axes were parallel with the base of the maquette and the Z-axis was perpendicular to it.

On the basis of estimated accuracy, after the spatial transformation of model coordinates in the maquette coordinate system of the pass points, the position r.m.s. was found to be $m_p = \pm 0.011$ mm in the scale of the maquette 1:1.

The spatial coordinates of each point were digitized with WILD EK 22 and entered into the desktop computer HP 9845 S. After the spatial transformation of model coordinates into the coordinate system of the wooden maquette they were stored on the magnetic tape for further data processing.

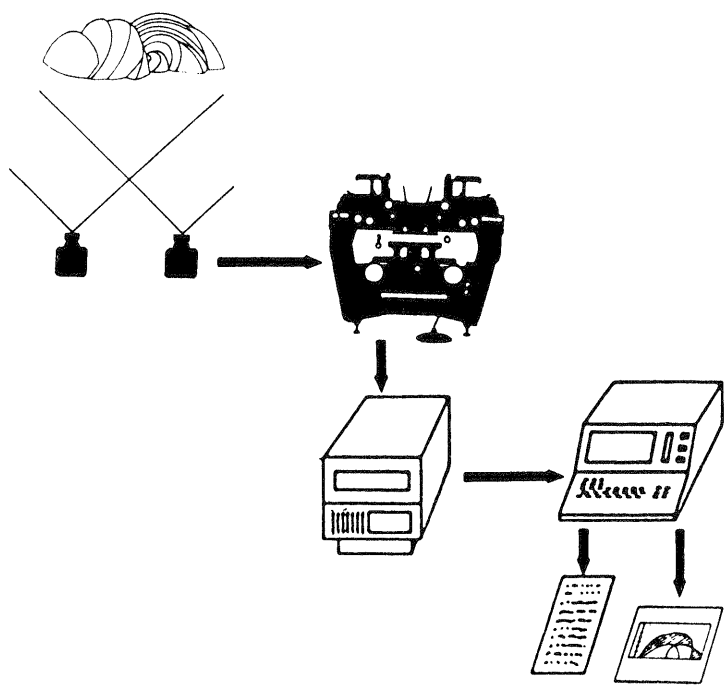

Fig. 2. The hardware configuration

MATHEMATICAL MODELLING

To describe the object mathematically, every existing surface on the maquette had to be replaced by a mathematically defined one. At first, it might seem that the mathematical part of the problem is a trivial one. However, it should be noted that we had to fulfill the following requirements:

a) the adopted analytic surfaces had to represent the real ones in the best possible way, i.e. they had to deviate from the measured points as little as possible;
b) the analytic surfaces had to be as simple as possible;
c) the number of different kinds of analytic surfaces had to be minimal.

The question arises: how to formulate those conditions mathematically and obtain a formulation usable for subsequent treatment. The third requirement was particularly problematic. Furthermore, the first of the three contradicts the remainig two. It means, we had to find ont the most acceptable compromise. As it is often customary, and in this case also in accordance with the nature of the problem, we used the term "deviation of the surface from the measured points" in the sense of $S = \Sigma d_i^2$. Here d_i ($i=1,\dots,n$) denote Euclidean distances between points and surface. When the kind of surface is chosen (e.g. plane), one has to determine its parameters in such a way as to make S to achieve minimum value.

Bearing in mind that the choice surface type depends on all three mentioned requirements, it is impossible to make the final choice in advance. A general procedure for solving the problem, without making even a preliminary choice of surfaces to be considered, is unfortunately impossible. Therefore, we decided to start with a larger choice of surfaces, to perform relevant computations and after that make a final choice, taking into account all the three aforesaid requirements.

We decided to consider the following surfaces:

1. plane,
2. sphere,
3. quadric surfaces,
4. polynomials in two variables, up to the fourth degrees.

This choice seems to fulfill the requirement b). However, as will be discussed briefly later on, even such uncomplicated surfaces can, in the course of further treatment, cause considerable numerical difficulties. Therefore it makes sense, although it may at first seem strange, to consider some special cases separately. From Fig. 1 it is obvious that such a special case of the quadric surface, which should be considered separately, is a sphere. In fact, it happened with several existing surfaces that some sphere fits just slightly worse to the measured points than another more complicated quadric.

Taking into account all the three requirements, it was natural to choose the sphere in such a situation.

As regards the plane, the problem is to find the plane for which the sum of squared Euclidean distances between given points and the plane becomes minimal. The problem can be solved exactly and is reduced to finding the least eigenvalue of a symmetric (3x3)-matrix.

As for the sphere there is no exact solution. Indeed, in theory it can be computed with any desired accuracy by solving the following nonlinear system of equations iteratively

$$\Sigma \left(r_i - \frac{\Sigma r_i}{n}\right) \frac{x_i - x_0}{r_i} = 0$$

$$\Sigma \left(r_i - \frac{\Sigma r_i}{n}\right) \frac{y_i - y_0}{r_i} = 0$$

$$\Sigma \left(r_i - \frac{\Sigma r_i}{n}\right) \frac{z_i - z_0}{r_i} = 0$$

with $r_i = \sqrt{(x_i-x_0)^2+(y_i-y)^2+(z_i-z_0)^2}$

However, this solution has some disadvantages, such as its sensitivity to the choice of initial approximation. Therefore, we decided to use an approximate solution. Instead of sum S of squared Euclidean distances between given points and the sphere, we minimized

$$S' = \Sigma \left((x_i-x_0)^2+(y_i-y_0)^2+(z_i-z_0)^2-R^2\right)^2,$$

where (x_0,y_0,z_0) was the center and R the radius of the sphere. The problem was reduced to a system of linear equations. After solving it, we also computed particular perpendicular distances d_i and then S, which we took as a criterion for the quality of adjustment (clearly, it is a more natural criterion thas S').

The procedure for quadric surfaces was a generalization of the described procedure for the sphere. We minimized

$$S''=\Sigma(A_{11}x_i^2+A_{12}x_iy_i+A_{13}x_iz_i+A_{14}x_i+A_{22}y_i^2+A_{23}y_iz_i+A_{24}y_i+A_{33}z_i^2+A_{34}z_i+A_{44})^2$$

which also led to some system of linear equations. For a general quadric surface, the Euclidean distances d_i between particular points and the surface cannot be computed exactly. Therefore, it was achieved by using numerical methods, and so $S = \Sigma d_i^2$ was obtained as a criterion for the quality of adjustment.

For polynomials in two variables of an arbitrary degree the problem cannot be solved exactly. Therefore, instead of S, we tried to minimize

$$S''' = \Sigma \left(f(x_i',y_i') - z_i'\right)^2$$

where x_i', y_i', z_i' were coordinates of given points in some appropriate coordinate system a position of which, relative to the considered surface, was favourable for solving the given problem. Unfortunateley, it came out that for some existing surfaces there was no rational choice of the coordinate system for fitting polynomials. Also, the polynomials of the third and the fourth degrees can sometimes be undesirably undulating, thus behaving in a way opposed to the nature of the problem (see Fig. 1). Therefore, we hoped that we might succesfully come to a solution employing only planes, spheres and eventually other quadric surfaces.

After the completion of all described adjustments, in accordance with the requirements a), b) and c), we chose 11 planes, 11 spheres and 5 ellipsoids. In doing so, neither the look nor dimensions of the object were significantly distorted. The root mean square deviation between the given points and the adopted surfaces was 3.22 mm (for particular surfaces it was ranging from 0.2-6.2 mm, maximal deviations amounting to 0.3-13.6 mm, see Tables 1 and 2). Compared to the dimensions of the object those deviations are quite acceptable. Namely, bearing in mind the roughness of the surface of the maquette (Fig. 1), even the highest maximal deviation of 13.6 mm does not seem exaggerated particularly when we know that the corresponding point is situated on a sphere having diameter of nearly 1 m.

Planes			Spheres			Ellipsoids		
No.of pts.	Max. dev. mm	rms. dev. mm	No.of pts.	Max. dev. mm	rms. dev. mm	No.of pts.	Max. dev. mm	rms. dev. mm
25	6.8	4.0	45	6.6	2.9	34	7.5	3.6
24	7.2	4.1	23	6.5	2.8	49	7.9	2.9
8	3.4	1.8	9	3.6	2.1	23	5.7	2.3
21	3.2	1.6	15	4.3	2.0	37	10.0	3.0
15	2.5	1.5	20	6.3	3.1	18	5.7	2.7
21	7.0	3.0	19	6.7	3.4			
13	4.3	2.4	22	13.6	6.2			
20	3.3	1.5	31	3.0	1.3			
12	4.4	2.1	117	9.6	4.3			
26	4.6	1.7	6	1.7	1.1			
16	4.2	2.1	6	0.3	0.2			

Table 1. Maximal and r.m.s. deviations for each particular surface

	Total no. of points	maximal deviation	r.m.s. deviation
Planes	201	7.2 mm	2.7 mm
Spheres	313	13.6 mm	3.6 mm
Ellipsoids	161	10.0 mm	3.0 mm

Table 2. Maximal and r.m.s. diviations for particular kinds of surfaces

FURTHER TREATMENT OF THE MATHEMATICAL MODEL

As described, the maquette was replaced by a mathematical model suitable for computer aided tretment. It resulted, for instance, in a ground plan (Fig. 3.), shape of supporting archs (Fig. 4.), and so on. It is, of course, more important that the equations of all surfaces, as well as of relevant intersecting curves, were placed at disposal of project engineers, to be used in further computations according to their needs and conception.

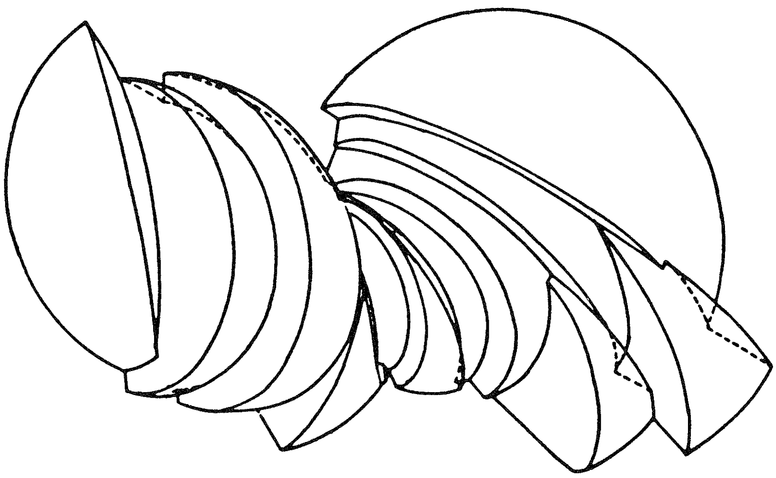

Fig. 3. Analytically obtained ground plan of the object

It would be interesting to mention numerical difficulties we were faced with when constructing figures 3 and 4. For example, the intersection of two ellipsoids is a curve of the fourth degree. To compute its coefficients, one has to solve a fourth degree equation. It is well-known that, in theory, there is an exact solution to the problem. In practice, on the contrary, it turned out that 12 significant digits, offered by the applied desktop computer, were far from being sufficient to ensure acceptable solutions, more precisely, the obtained solutions deviated very much from the real ones. Opposite to that, numerical iterative methods proved to be quite usable for finding the coordinates of desired number of points lying on the intersecting curve. A detailed discussion of those problems was given in LAPAINE, PETROVIĆ, FIEDLER 1985.

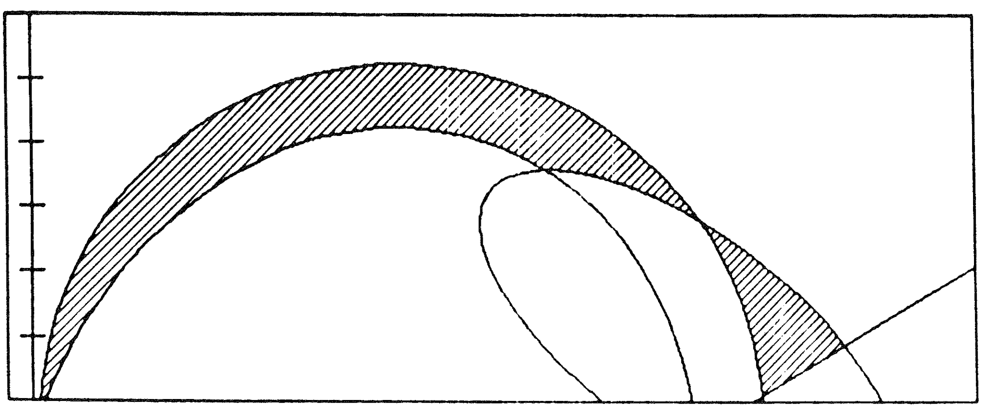

Fig. 4. Computer constructed section of the object by means of an inclined plane

Our analytical model served as a basis for further treatment conducted by the project engineers. It is still not completed. As an illustration, Fig. 5 presents a perspective view of the object they constructed using our equations.

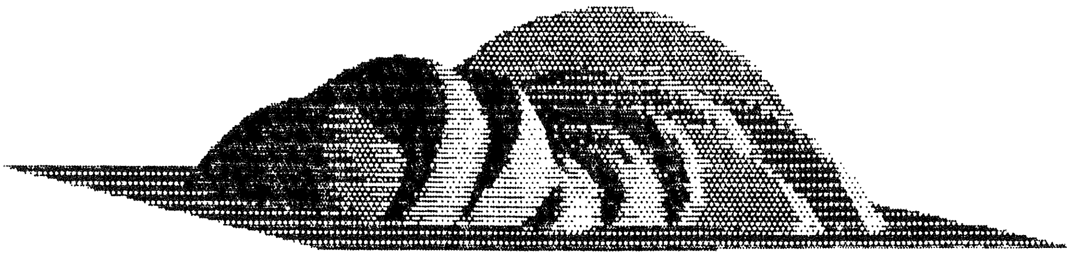

Fig. 5. Computer designed perspective view of the object

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