SEQUENTIAL SATISFISATION FOR A PHOTOGRAMMETRIC BLOCK TRIANGULATION AND A GEODETIC CONTROL NETWORK

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ABSTRACT

Taking into account a generalized mixed model leading to the estimation of non stochastic unknown parameters and to the prediction of available stochastic prior information, a process is given to output an ideal dispersion matrix for the available control point coordinates that "satisfises" a criterion matrix for the photogrammetric unknowns. A second process is given to obtain a Second Order Design solution for the control network observations that also "satisfises" the criterion matrix for the photogrammetric unknowns. It is shown that the problem admits a solution if and only if the chosen criterion is "worse" than the unreachable dispersion matrix obtained in the hypothesis that available control point coordinates are exactly known, and "strictly worse" than it at least in the space of photogrammetric unknowns whose estimated value is affected by prior information; if not, a S.O.D. of the photogrammetric model is required. A method is then given, to test if the required precision is compatible with a requested value for the external reliability of the control network, and, if so, to obtain a S.O.D. solution that fulfills both requirements.

ANALITYCAL MODEL

Let a mixed linear model, coming from a Gauss-Markov model, be considered for a bundle method of block adjustment when stochastic prior information about some of the control point coordinates are available.

$$E(y) = y - v = A\,x = A1\,x1 + A2\,x2 \qquad \text{rank } A = q < m$$

$$E(x2) = 0$$

$$D(x2) = \Sigma_{ee} \qquad D(y) = \Sigma_{vv} + A2\,\Sigma_{ee}\,A2'$$

where:

x (m x 1) is the vector of increments of unknown object coordinates and exterior orientation parameters to their approximate values, which can be partitioned in a fixed part x1(m-r x 1) and a stochastic part x2 (r x 1) [where r > m-q, being m-q the column rank deficency];

x1 contains increments to approximate object coordinates and to orientation parameters;

x2 contains increments to available control point coordinates;

A(2n x m) is the design matrix which can be partitioned in two submatrices A1 (2n x m-r) and A2 (2n x r) according to the partition of vector x in its fixed part x1 and its stochastic part x2;

y(2n x 1) is the observation vector obtained as the difference between the vector of image coordinates (xi, yi, i = 1 ... n) and their approximate values;

v (2n x 1) is the vector of residuals to the image coordinates;

Σ vv (2n x 2n) is the positive definite dispersion matrix of the observations;

Σ ee (r x r) is the (semi)positive definite dispersion matrix of the prior information;

D( ) means dispersion; E( ) means expectation.

In order to find a solution for the vector of increments of unknown object coordinates and of exterior orientation parameters x we certainly have to proceed to the estimation of vector x1 and to the prediction of x2. In this work we only require the estimation of x1 and its dispersion matrix Σ x1 since we are interested in the formulation of criterion matrices for unknown parameters belonging exclusively to a photogrammetric model.

The BLUUE $\hat{x}1$ of x1 can be obtained by a least square method from:

$$\hat{x}1 = H1\ y \qquad (1)$$

$$H1: = [A1'(\Sigma vv + A2\ \Sigma ee\ A2')^{-1}\ A1]^{-1}\ A1'\ (\Sigma vv + A2\ \Sigma ee\ A2')^{-1}$$

and its dispersion matrix from

$$\Sigma x1: = [A1'\ (\Sigma vv + A2\ \Sigma ee\ A2')^{-1}\ A1]^{-1} \qquad (2).$$

while the prediction of x2 can be subsequently obtained by

$$\tilde{x}2 = \Sigma\ ee\ A2'(\Sigma\ vv + A2\Sigma\ eeA2')^{-1}\ (y - A1\ \hat{x}1)$$

THE RELATION "BETTER THAN"

A regular dispersion matrix C1 is "better than" an analogous matrix C2 if (equivalent conditions) [Van Mierlo 1982]:

$$f' C1 f \leqslant f' C2 f \qquad \forall f \qquad (3.1)$$

$$C1 - C2 \quad \text{seminegative definite} \qquad (3.2)$$

$$y' C1^{-1} y \geqslant y' C2^{-1} y \qquad \forall y \qquad (3.3)$$

In fact, if C1 and C2 are the dispersion matrices of two sets of estimators $\hat{k}1$ and $\hat{k}2$ of a vector k of unknowns, then (3.1) states that the precision of any f(k) estimated by $\hat{k}1$ will be better than the precision of $f(\hat{k}2)$.

The relation "better than" can be considered a purely mathematical operator released from the statistical meaning of matrices to which it is applied, and extended to any couple of symmetric square matrices of the same order, saying that A is (strictly) better than B, conversely that B is (strictly) worse than A in a given space S, if

$$f' A f \leqslant (<) \; f' B f \qquad \forall f \in S \,,\, f \neq 0$$

If A is (strictly) better than B in any suitable space, A can be said (strictly) better than B tout court.

The set of symmetric matrices of order n is not totally ordered with respect to the relation "better than": that is, "not better" does not imply "worse".

A number of useful theorems hold for this relational operator (see [Crosilla & Russo 1988]).

Assuming that $\bar{\Sigma}ee$ and $\bar{\bar{\Sigma}}ee$ are two different dispersion matrices of the prior information in the same network datum, let us call $\bar{\Sigma}x1$ and $\bar{\bar{\Sigma}}x1$ the dispersion matrices of the photogrammetric unknowns obtained replacing Σee in formula (2) by $\bar{\Sigma}ee$ and $\bar{\bar{\Sigma}}ee$ respectively.

Improving the precision of the prior information, the precision of the photogrammetric unknowns cannot be worsened: in fact, as it is shown in [Crosilla & Russo 1988],

$\bar{\Sigma}ee$ better than $\bar{\bar{\Sigma}}ee \Longrightarrow \bar{\Sigma}x1$ better than $\bar{\bar{\Sigma}}x1$

Moreover, improving the precision of all the prior information, the precision of the photogrammetric unknowns will be effectively improved in some way. More precisely, we can state that (see below)

$\bar{\Sigma}ee$ strictly better than $\bar{\bar{\Sigma}}ee \Longrightarrow \bar{\Sigma}x1$ strictly better than $\bar{\bar{\Sigma}}x1$ except that in the null space of L1', N(L1'), where L1 stands for $(A1' \Sigma vv^{-1} A1)^{-1} A1' \Sigma vv^{-1} A2$ (see below).

This space has an intuitive meaning. It can be written as $R^{\perp}(L1) = R^{\perp}(H1^{(o)}\, A2)$, where $H1^{(o)}$ is H1 given by (1) in the hypothesis that the coordinates of control points are exactly known (i.e., $\Sigma ee = 0$). A2 maps control point coordinates into a subspace R(A2) of coordinates of control point images; $H1^{(o)}$ maps this subspace into the subspace $R(H1^{(o)}\, A2)$ of unknowns of the photogrammetric model to be estimated. $R^{\perp}(L1)$ thus spans linear combination of unknowns whose estimation is not affected by values of control point coordinates.

Conversely, however, a better precision of the unknowns does not imply (of course) a better precision of ALL the prior information: this is why a S.O.D. of a network can lead to a redistribution of weights of observation rather than to an increase of them all.

FIRST STEP: AN IDEAL DISPERSION MATRIX FOR CONTROL POINT COORDINATES THAT "SATISFISES" A CRITERION MATRIX FOR THE PHOTOGRAMMETRIC UNKNOWNS.

Let us define $\Sigma x1^{(o)} = (A1'\, \Sigma vv^{-1}\, A1)^{-1}$. This is equivalent to assuming that all coordinates of control points are exactly known ($\Sigma ee = 0$).

$\Sigma x1^{(o)}$ represents an unreachable upper limit to the precision obtainable for the given photogrammetric unknowns, acting only on the precision of prior information: if a criterion matrix better than $\Sigma x1^{(o)}$ is to be satisfised, this cannot be done by improving the precision of control point object coordinates only: a S.O.D. of the photogrammetric block triangulation is required [see Crosilla, Forlani & Russo, 1986].

On the contrary, if a criterion matrix $\tilde{\Sigma} x1$ worse than $\Sigma x1^{(o)}$, and strictly worse than it at least in the ortogonal complement of N(L1') is chosen, it is always possible to satisfisy $\tilde{\Sigma} x1$ acting only on the precision of the control network. This means that, calling Ac the design matrix of the control network, it is always possible to compute as will be shown in the following a positive definite diagonal weight matrix Ps for the measurements of the control network such that the resulting variance covariance matrix for the photogrammetric unknowns

$$\Sigma x1 = \left\{ A1' \left[\Sigma vv + A2(Ac'\, Ps\, Ac)^{-1} A2' \right]^{-1} A1 \right\}^{-1}$$

is better than $\tilde{\Sigma} x1$.

Such a criterion can be built, for instance, computing $\Sigma x1$ with a given, feasible weight matrix P and improving then the difference $\Sigma x1 - \Sigma x1^{(o)}$ by contraction of its dominant eigenvalues.

The algorithm

Considering one of the equivalent forms for the inverse of a sum of matrices reported in Henderson Searle [1981, pg. 58], the dispersion matrix $\Sigma x1$ given by (2) can also be expressed as

$$\Sigma x1 : = [N11 - N12 \ (I + \Sigma ee \ N22)^{-1} \ \Sigma ee \ N12']^{-1}$$

where

$$N11 := A1' \ \Sigma vv^{-1} \ A1 \qquad (:= \Sigma x1^{(o)-1})$$

$$N12 := A1' \ \Sigma vv^{-1} \ A2$$

$$N22 := A2' \ \Sigma vv^{-1} \ A2$$

Developing the inversion of the square bracket, it follows that

$$\Sigma x1 : = N11^{-1} + \ldots$$

$$\ldots + N11^{-1} \ N12 \ [I-(I+\Sigma eeN22)^{-1} \ \Sigma ee \ N12' \ N11^{-1} \ N12]^{-1} \ldots$$

$$\ldots (I+\Sigma ee \ N22)^{-1} \ \Sigma ee \ N12' \ N11^{-1}$$

setting $L1 : = N11^{-1} \ N12$

$$L2 : = N12' \ N11^{-1} \ N12$$

it turns out that

$$\Sigma x1 := \Sigma x1^{(o)} + L1 \ [I-(I+ \Sigma ee \ N22)^{-1} \ \Sigma ee \ L2]^{-1} \ldots$$

$$\ldots (I+\Sigma ee \ N22)^{-1} \ \Sigma ee \ L1'$$

which can be rewritten as

$$\Sigma x1 - \Sigma x1^{(o)} := L1 \ M \ L1'$$

M is symmetric and null in the same space where Σee is null and only there. This form shows evidently that any feasible $\Sigma x1$ is equivalent to $\Sigma x1$ in $N(L1')$, and, if Σee is positive definite, strictly worse than it in $N^{\perp}(L1')$.

Let us now consider the equation (for M unknown)

$$\tilde{\Sigma} x1 - \Sigma x1^{(o)} = L1 \ M \ L1'$$

If col (A2) $<$ col (A1), this equation is inconsistent: otherwise, it admits exact solutions. In both cases, let us find its approximate l.s. solution or its minimum norm exact solution as

$$\tilde{M} = L1^{+} \ (\tilde{\Sigma} x1 - \Sigma x1^{(o)}) \ L1'^{+}$$

Since $\tilde{\Sigma} x1 - \Sigma x1^{(o)}$ is positive definite at least in the

ortogonal complement of N(L1'), and non-negative definite elsewhere, also

$$\bar{\Sigma}x1 - \Sigma x1^{(o)} = L1\ \bar{M}\ L1' = L1\ L1^{+}\ (\tilde{\Sigma}x1 - \Sigma x1^{(o)})\ L1'^{+}\ L1'$$

will result positive definite in the ortogonal complement of N(L1'), and null in N(L1'). It will therefore be possible, finding the maximum general eigenvalue λmax of $L1\ \bar{M}\ L1'$ with respect to $\tilde{\Sigma}x1 - \Sigma x1^{(o)}$ in this space, to obtain

$$\bar{\bar{M}} = 1/\lambda max\ \ \bar{M}$$

so that

$$\bar{\bar{\Sigma}}x1 = \Sigma x1^{(o)} + L1\ \bar{\bar{M}}\ L1'$$

satisfises $\tilde{\Sigma}x1$.

> The general eigenvalues of A with respect to B in N (B) are the eigenvalues of Vab:
>
> $Vab := G'^{+}\ A\ G^{+}$
>
> where G is given by the spectral decomposition of B:
>
> $B = Ub\ Lb^{\frac{1}{2}}\ Lb^{\frac{1}{2}}\ Ub' := G'\ G.$

Now as M is given by

$$M = [I - (I+\Sigma ee\ N22)^{-1}\ \Sigma ee\ L2]^{-1}\ (I+\Sigma ee\ N22)^{-1}\ \Sigma ee,$$

it is legitimate to multiply on the left both the terms by

$$(I + \Sigma ee\ N22)\ [I - (I+\Sigma ee\ N22)^{-1}\ \Sigma ee\ L2]$$

obtaining

$$\Sigma ee\ [I + (L2 - N22)\ M] = M$$

if the square bracket turns out to be regular, $\bar{\bar{\Sigma}}ee$ can be computed as

$$\bar{\bar{\Sigma}}ee = \bar{\bar{M}}\ [\ I + (L2 - N22)\ \bar{\bar{M}}]^{-1}\ ,$$

such that

$$\bar{\bar{\Sigma}}x1 = [A1'\ (\Sigma vv + A2\ \bar{\bar{\Sigma}}ee\ A2')^{-1}\ A1]^{-1}\ .$$

> (Should the square bracket be singular, it would be enough to divide $\bar{\bar{M}}$ by a number slightly greater than 1 and not equal to the module of any singular value of $(L2 - N22)\bar{\bar{M}}$ to obtain a new $\bar{\bar{M}}$ that satisfises $\tilde{\Sigma}x1$ and makes the square bracket invertible).

$\bar{\bar{\Sigma}}ee$ turns out symmetric, positive definite in the same space where $\bar{\bar{M}}$ is positive definite, and null where $\bar{\bar{M}}$ is null, namely in N(L1). Also this space has an intuitive meaning: it can be rewritten as $N(H1^{(o)}\ A2)$, and is the space of coordinates of

control points whose value do not affect the estimated value of any photogrammetric unknown. For example, if a control point do not appear in any photo, its coordinates will fall in N(A2), and hence in N(L1). If an arbitrary matrix positive definite in N(L1) and null in its ortogonal complement is added to $\tilde{\Sigma}ee$ to obtain a positive definite matrix $\tilde{\Sigma}ee$, $\tilde{\Sigma}x1$ remains unchanged.

SECOND STEP: S.O.D. FOR THE CONTROL NETWORK

$\tilde{\Sigma}ee$ can be used as a criterion matrix to be satisfised by the control network. As is well known, the literature reports many approximating algorithms that lead eventually to a diagonal weight matrix P such that

$$(Ac' \; P \; Ac)^{-1} \doteq \tilde{\Sigma}ee$$

or $Ac' \; P \; Ac \doteq \tilde{\Sigma}ee^{-1}$.

Three problems arise:

1) $(Ac' \; P \; Ac)^{-1}$ is not ensured to be better than $\tilde{\Sigma}ee$. On the contrary, in the general case, since $(Ac' \; P \; Ac)^{-1} - \tilde{\Sigma}ee$ is just the residual whose norm has been minimized, "its elements should show an irregular pattern around zero" [Schaffrin 1983], and it will be neither positive nor negative definite. But, computing the greater eigenvalue of $(Ac' \; P \; Ac)^{-1}$ with respect to $\tilde{\Sigma}ee$, say λmax, then $(Ac' \; \lambda max \; P \; Ac)^{-1}$ turns out automatically better than $\tilde{\Sigma}ee$, so that the latter is satisfised.

2) It is not ensured that P is positive definite. If some components of P are negative, however, leading them to zero or to an arbitrary positive value the result will be even better than the previous one [Russo 1988]. The second choice is preferable since it does not imply a First Order reDesign.

3) The obtained solution $\lambda max \; P$ is to be considered as a set of minimum values that satisfisy the imposed precision requirements: any diagonal $\bar{P}$ with $\bar{P}ii \geq \lambda max \; Pii$, satisfises as well the imposed criterion. However, high values of Pii should be avoided, not only for economical consideration (or tecnical infeasibility), but also because high values of Pii are critical for the reliability of the network.

FULLFILLMENT OF AN EXTERNAL RELIABILITY CONDITION

As is well known, the external reliability of a network can be measured [Baarda 1977] by

$$\bar{\delta}_o^2 := \max \left(\bar{\delta}_i^2 \right) = \max \left(\frac{\sigma_{\hat{y}_i}^2 \; \delta_o^2}{\sigma_{v_i}^2} \right) \qquad (4)$$

[Forstner 1979] where:

$\sigma \hat{y}i$ are the standard deviation of adjusted observations
σvi are the standard deviation of the residuals
δo^2 is the non-centrality parameter of the non-central Fisher distribution when α probability for a first kind error and β probability for second kind error are chosen.

If a maximum acceptable value $\bar{\delta}^2 o$ is imposed to $\bar{\delta}_i^2$, then from (4), substituting $\sigma \hat{y} i^2$ and σvi^2 by $(Ac\ \Sigma ee\ Ac')ii$ and by $(P^{-1} - Ac\ \Sigma ee\ Ac')ii$ respectively, the weights of observation must satisfy the following conditions:

$$Pii \leq \frac{1}{[Ac\ (Ac'\ P\ Ac)^{-1}\ Ac']ii} \cdot \frac{\bar{\delta}_o^2}{\delta_o^2 + \bar{\delta}_o^2} \qquad (5)$$

The problem is hence to find a diagonal matrix P that fulfills (5) and contemporaneously satisfises a given criterion matrix $\tilde{\Sigma}ee$.

It is not said a priori that this problem admits solutions. It can be noted that, if (5) admits a solution Pr (as would be in the case where δo^2 has been computed for a given, feasible weigth matrix), then ωPr, where ω is any positive scalar, is still a solution: so that it would be always possible to scale Pr so to obtain a set of weights that satisfises also the precision requirements. This is of little use, both because the obtained solution could be extremely unrealistic, and mainly because there is no simple way to obtain a set of weights that satisfies (5), if not previously known.

To search more realistically for a solution, $\tilde{\Sigma}ee$ must not only be satisfised, but also approximate as closely as possible.

Let us assume now that P is a diagonal weight matrix that satisfises a given criterion $\tilde{\Sigma}ee$, that is, $(Ac'\ P\ Ac)^{-1}$ is better than $\tilde{\Sigma}ee$. Then, condition (5) is ensured if

$$Pii \leq \frac{1}{[Ac\ \tilde{\Sigma}ee\ Ac']ii} \cdot \frac{\bar{\delta}_o^2}{\delta_o^2 + \bar{\delta}_o^2} := Pii(lim) \qquad (6)$$

In fact, being $Ac\ \tilde{\Sigma}ee\ Ac'$ worse than $Ac\ (Ac'\ P\ Ac)^{-1}\ Ac'$, each diagonal term of the former is greater (or equal) than the corresponding one of the latter, so that each rigth-hand side term of (6) turns out smaller (or equal) than the corresponding term of (5).

> This can be easily verified substituting f in (3.1) by a vector whose i-th component is 1 and all the remaining are zero.

This allow us to perform a test that may ensure the existence of a solution for our problem before starting to search it. Computing

$$\Sigma ee\ (lim) = (Ac'\ P(lim)\ Ac)^{-1}$$

if $\Sigma ee(lim)$ turns out better than $\tilde{\Sigma} ee$, i.e., its maximum general eigenvalue with respect to $\tilde{\Sigma} ee$, $\lambda lmax$, is less than or equal to 1, P(lim) itself can be considered a solution, and $\lambda lmax$ P(lim) a more economical one. If this test fails, however, a solution cannot be said to exist, but could still exist: we can only state that, to satisfisy $\tilde{\Sigma} ee$, some of the weights should assume a value larger than the corresponding Pii(lim), but condition (5) could still be satisfied.

Let us assume that the test suceeds. To obtain a better approximation of $\tilde{\Sigma} ee$, the solution Ps of a S.O.D. approximating algorithm is needed. Furthermore, to obtain a solution satisfisying $\tilde{\Sigma} ee$, Ps must be multiplied by the maximum general eigenvalue $\lambda smax$ of $(Ac' \; Ps \; Ac)^{-1}$ with respect to $\tilde{\Sigma} ee$.

If $\lambda smax$ is close to 1, the solution $\lambda smax$ Ps that satisfises $\tilde{\Sigma} ee$ showns to be also a well approximating solution.

It is possible, however, that some components of $\lambda smax$ Ps do not satisfy (5), and hence the reliability requirements are not fulfilled. This is easy to obtain with very redundant networks, since S.O.D. algorithms tend to exalt the weight of only one or some of a group of related observations, decreasing the weight of the remaining ones (possibly to negative values).

In this case, it is possible to fix the weights that exceed the corresponding right-hand side of (5) to a value that ensures the requested reliability, and do not prevent satisfisation of $\tilde{\Sigma} ee$. Possible choiches are Pii(lim) (in which case, the reliability requirement (6) will be "just" fulfilled for that observation), or $\lambda lmax$ Pii(lim) (in which case, it will be the precision requirement to be "just" satisfised, while the reliability requirement will be even better fulfilled), or an intermediate value.

New values for the weights not fixed in this way can be obtained by a "static solution" of the S.O.D. problem (already described in the literature [Illner 1988]), so as to approximate again $\tilde{\Sigma} ee$, and then multiplying them by a new computed maximum general eigenvalue to satisfisy it.

Partitioning P in Pu (unknown) and Pf (fixed), the S.O.D. problem can be written as:

$$(Au' \; Pu \; Au + Af' \; Pf \; Af)^{-1} \doteq \tilde{\Sigma} ee$$

$$Au' \; Pu \; Au + Af' \; Pf \; Af \doteq \Sigma ee^{-1}$$

$$Au' \; Pu \; Au \doteq \tilde{\Sigma} ee^{-1} - Af' \; Pf \; Af$$

$$(Au' \; Pu \; Au)^{-1} \doteq (\tilde{\Sigma} ee^{-1} - Af' \; Pf \; Af)^{-1}$$

If again some of the new weights do not fulfill (5), the process can be iterated. Since at each iteration some more weights are fixed, this method will always end up with a set of weights (at worse equal to Pii(lim)), that will satisfisy $\tilde{\Sigma} ee$ and fulfill the reliability condition (5). If some of them

turn out negative, they can be set with no damage to any suitable positive value lower than the corresponding Pii(lim).

If the initial test fails, i.e. λ lmax > 1, it is possible to fix only for the next iteration each weight that do not fulfill (5) to the corresponding right-hand side term. The process should stop when (5) is met by all the weights (solution) or by none of them (infeasibility); however, convergence is not ensured.

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