Orientation by Complex Transformation for Incomplete Model

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Abstract

Analytical relative orientation uses intrinsic condition without appealing to the embracing space of perspective center. Solution of problem is simply to find the correct Strain-Energy (parallax) function. Since it is analytic the process is a conformal transformation between image and object complex-planes. The general transformation function $w = \alpha z + z^2$ *comprises transformation for dx, dy, dz, d*κ *by first part and d*φ*, d*ω *by the second part. Extension of this results is applied to solve incomplete model or image-scene due to any reasons such as cloud or forest cover. Whatever the nature of incompleteness, conformal mapping can always map into a half plane and hence onto region into which any half plane can be transformed. Three cases of incomplete model are considered and hence deformation functions are derived.*

0. Introduction

Orientation theory on incomplete model is almost unknown to us and moreover, complete model orientation cannot be extended to solve the former. However we can not forget the existance and important of incomplete models due to cloud or forest covers. However non classical theory of orientation which will be treated extensively in this paper, have some concepts to solve incomplete models. Previous paper by the author[2] is on special theory of orientation showing that new formulation imposed by some conditions can derive the classical y-parallax equation. The present paper handles the orientation theory in more general approach. Construction of deformation tensor is treated here again from view point of differential vector calculus instead of previous approach from theory of elasticity.

1. Deformation is a second rank Tensor

Let position vector of a point P with respect to a given point Po be expressed as:

$$r = x.i + y.j + z.k \qquad \ldots (1)$$

During deformation, all points including P moves and rotates about Po. Thus final position of Po is P'o and P is P' with new position vector of r'. We are interested in the position of now P' with respect to now P'o. Let So be displacement vector suffered by Po and S displacement vector by P and with a relation:

$$S = So + dS = So + (r \cdot del)\, So \qquad \ldots (2)$$

where del = a vector operator.

Let $S = u(x, y, z).i + v(x, y, z).j + w(x, y, z).k$, and it can be shown that:

$$r' = r + S - So \qquad \ldots (3)$$

or $$r' = r + 1/2\,(del \times S) \times r + 1/2\, del\,(r.W) \qquad \ldots (4)$$

where $W = (r.\, del)\, So$

Since $1/2\,(del \times S) \times r$ represents a rigid-body motion about Po, we ignore this non deformation term and we have for deformation:

$$r' = r + 1/2\, del\,(r.W) \qquad \ldots (5)$$

Expansion of right hand side of equ. 5 can be shown in matrix form as:

$$r' = X + D.X \qquad \ldots (6)$$

where $X = (x\ y\ z)^T$ and D is a 3 x 3 matrix whose elements are partial differentials of u, v, w with respect to x, y, z and Jacobian $(uvw)/(xyz) \neq 0$. However the remarkable result here is that these nine components of matrix form a second rank deformation tensor say D_{ij} which is symmetric and can be expressed as:

$$D_{ij} = \begin{vmatrix} u_x & 1/2(u_y+v_x) & 1/2(w_x+u_z) \\ 1/2(u_y+v_x) & v_y & 1/2(w_y+v_z) \\ 1/2(w_x+u_z) & 1/2(w_y+v_x) & w_z \end{vmatrix} \qquad \ldots (7)$$

The D_{ij} can be expressed in a more familar form of translational and rotational components in two-dimension as:

$$D_{ij} = \begin{vmatrix} dx & d\kappa & d\varphi \\ d\kappa & dy & d\omega \\ d\varphi & d\omega & dz \end{vmatrix} \qquad \ldots (8)$$

where dx, dy, dz, $d\kappa$, $d\varphi$, $d\omega$ are displacements due to translation and rotation with obvious one to one equivalent to differential components of equ. (7). It must be noted that deformations in z-direction such as u_z and v_z for $d\varphi$ and $d\omega$ are zero.

2. One-to-one correspondance between deformation and parallax tensors

Affine orthogonal transformation of image point into a projection plane is carried out by a dyadic product of two vectors. This vector can be formed as follows:

We have in xy-plane u (x, y) = const and v (x, y) = const are orthogonal net work of curves forming like curvilinear coordinates. However at the neighbourhood of point P′ the orthogonal coordinates tangential to curves can be formed. This we can achieved as gradient or in short grad v (x, y) normal to curve v (x, y) = const and grad u (x, y) normal to curve u (x, y) and always satifying condition that grad v (x, y) grad u (x, y) = 0. These two normal vectors change directions with change of position of the point, however tensorial invarience remains unchained. Then a vector from perspective centre with focal length f to P′ can be written like this:

$$\text{grad } v(x, y).i + \text{grad } u(x, y).j + f.k \qquad \ldots (9)$$

Let the dyadic product be Pij and its nine components form a tensor like this:

$$P_{ij} = \begin{vmatrix} \text{Lap } v(x,y) & \text{grad } v(x,y).\text{grad } u(x,y) & f.\text{grad } v(x,y) \\ \text{grad } v(x,y).\ \text{grad } u(x,y) & \text{Lap } u(x,y) & f.\text{grad } u(x,y) \\ f.\ \text{grad } v(x,y) & f.\text{grad } u(x,y) & f.f \end{vmatrix} \qquad \ldots (10)$$

where Lap = Lapalacian = $\delta^2/\delta x^2 + \delta^2/\delta y^2$ and grad = $\delta/\delta x.i + \delta/\delta y.j$ or del of scalar function.

Assume that the functions u, v are <u>continuously differentiable</u> and satisfy <u>Cauchy-Riemann conditions</u> of $u_x = v_y$ and $u_y = -v_x$. We can also use one of invarient properties of a tensor of sum of diagonal elements equals zero and not forgetting that focal length is constant. Then, hence the term (f.f) can be expressed in term of u, v. Thus the final form of symmetric tensor becomes as:

$$\text{Pij} = \begin{vmatrix} v_{xx} + v_{yy} & v_{yy} + u_{yx} & f.v_x + f.v_y \\ v_{xy} + u_{yx} & u_{xx} + v_{yy} & f.u_x + f.\ u_y \\ f.v_x+f.v_y & f.u_x+f.u_y & -\ v_{xx} - u_{yy} \end{vmatrix} \ldots (11)$$

There is one-to-one correspondence between deformation (strain) tensor (equ. 8) and Parallax (stress) tensor (equ. 11). However, Pij can be seen as sum of two symmetric tensors namely in y-direction Pij (y) and in x-direction Pij (x) where

$$\mathrm{Pij}\ (y) = \begin{vmatrix} v_{xx} & v_{xy} & f.v_x \\ v_{xy} & u_{xx} & f.u_x \\ f.v_x & f.u_x & -v_{xx} \end{vmatrix} \qquad \ldots (12)$$

$$\mathrm{Pij}\ (x) = \begin{vmatrix} v_{yy} & u_{yx} & f.v_y \\ u_{yx} & v_{yy} & f.\ u_y \\ f.v_y & f.u_y & -\ u_{yy} \end{vmatrix} \qquad \ldots (13)$$

3. <u>Equation of Orientation as a Complex Transformation</u>

For condition of relative orientation the <u>inner product of deformation and parallax tensors</u> can be formed as:

$$U = D_{ij} \cdot [P_{ij}\ (y) + P_{ij}\ (x)] \qquad \ldots (14)$$

U is usually known as strain-energy and can be interpreted here as parallax. Since we are interested in parallax in y-direction Pij(x) tensor can be ignored. Thus we can show that the necessary and required condition for relative orientation in the neighbourhood of a point is that:

$$_B\ D_{ij} \cdot P_{ij}\ (y)\ dB = 0 \qquad \ldots (15)$$

where B is region around the point.

Since we are considering a point, the integration can be disregarded. On expansion of equ. 15 we have:

$$v_{xx}.dx + u_{xx}.dy - v_{xx}.dz + 2v_{xy}.d\kappa + 2f\ v_x.d\varphi + 2f\ u_x.d\omega = 0 \qquad \ldots (16)$$

It is obvious that problem of orientation is mainly a choice of function which is freely termed as parallax function.

Let us choose a complex function of the form $u + i.v \doteq 1/6\ (x + i.y)^3$ where $i = -\ 1$, with real and imaginary parts of function satisfying the Cauchy-Riemann conditions. Thus the analytic functions with non vanishing Jacobian (u, v)/(x, y) can perform conformal mapping or transformation.

We see that real function $u = - 1/6\ x^3 + 1/2\ xy^2$ and imaginary function $v = 1/2\ x^2\ y - 1/6\ y^3$ and thus equ. (16) becomes :

$$y.dx - x.dy - y.dz + 2\ x.d\kappa + 2f\ xy.d\varphi - 2f\ (x^2 - y^2).d\omega = 0 \qquad \ldots (17)$$

Then let $dy = - a$, $dz = - i.a$ and $dx = - b$, $2\ d\kappa = i.b$, then the first four terms is of equ. (18) is a complex transformation of the form:

$$w' = \alpha\ z, \text{ where } \alpha = a + i.b \text{ and } z = x + i.y. \qquad \ldots (18)$$

and if $d\varphi = i/f$ and $d\omega = 1/2f$ then remaining two terms of equ. (18) are of another complex transformation of the form;

$$w'' = z^2, \text{ where } z = x + i.y. \qquad \ldots (19)$$

Combined transformation of equs. 18, 19 gives:

$$w = \alpha z + z^2 \qquad \ldots (20)$$

From equ. 20, we can see that the equ. 17 is a complex equation of this form: $\alpha z + z^2 = 0$. Regular grid is deformed by orientation elements dx, dy, dz, $d\kappa$ into arbitrarily orientated regular grid (equ. 18) with stretching or contracting in size depending $a > 1$ or $a < 1$. However, that of $d\varphi$, $d\omega$ deform regular grid into orthogonal curvilinear grid (equ. 19) maintaining angle but not shape.

4. Application of conformal transformation on Incomplete Models

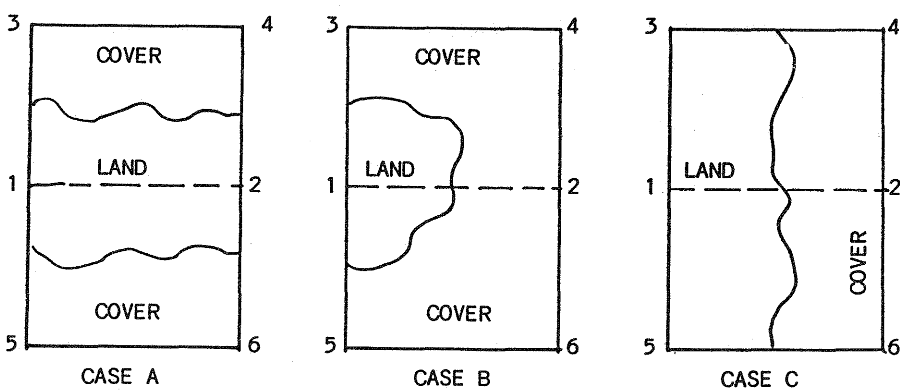

FIGURE 1

Study of case A

Consider a neat model where regions around point 1, 2 are available for orientation. This case can be consider as a plane with real points 1, 2 and points 3, 4, 5, 6 at infinity. For the sake of convenience we will consider half of this neat model, that is points 1, 2, 3, 4. This plane can be transformed into a half plane by transformation $w_1 = \sin(z)$. Then the equation for orientation will be $w_2 = a.w_1 + (w_1)^2$. On substitution of $\sin(z) = z - z^3/3! + z^5/5! - \ldots$ and considering first two terms we have:

$$w_2 = \alpha (z - z^3/6) + (z - z^3/6)^2 \qquad \ldots (21)$$

Expanding the above equation and neglecting sixth power term we have the similar form of transformation as complete model:

$$w2 = (1 - z^2)(\alpha z + z^2) \qquad \ldots (22)$$

Study on Case B

Consider the upper half of the neat model with cloud cover over the area except around point 1 and considering the shape of areaas a quadrant of a circle with centre at point 1. For this case three stages of transformation have to be performed to get a half plane. Firstly, quadrant of the circle can be transformed into a semi circle in a half plane by $w_1 = z^2$. Secondly, transformation $w_2 = (1 + w_1)/(1 - w_1)$ provides the quarter plane. Finally, the transformation $w_3 = w_2^2$ provides a half plane. Then the equation for orientation is achieved by the transformation $w_4 = \alpha\, w_3 + (w_3)^2$. Hence, the transformation we have is:

$$w_4 = \alpha Z + Z^2 \quad \text{where } Z = [(1 + z^2)/(1 - z^2)]^2 \qquad \ldots (23)$$

This is also similar form of transformation as complete model.

Study on Case C

Consider the upper half plane of the neat model with cloud cover at points 2, 4. This case can be seen as case A with origin shifted and axes rotated through 90 degrees. Thus we can transformed this to a half plane by transformation $w_1 = i.z^2$, where $i^2 = -1$. Then the equation for orientation will be $w_2 = \alpha\, w_1 + w_1^2$, which we can see as:

$$w_2 = \alpha\, i.z^2 - z^4. \qquad \ldots (24)$$

All three transformations (equ. 22, 23 & 24) uses fourth power of z or $(x + i.y)^4$ and thus we can choose a function $u + i.v = (x + i.y)^5$ to find the deformation functions by orientation elements such as omega- and phi-functions. Equating the real and imaginary part of the left and right sides of the above equation we will see that:

$$u = x^5 - x^3y^2 + xy^4 \quad \text{and} \qquad \ldots (25)$$

$$v = x^4y - x^2y^3 + y^5 \qquad \ldots (26)$$

From theory it is obvious that $\delta u/\delta x$ provides omega-function $F(\omega)$ or omega-component of the tensor and $\delta v/\delta x$ gives phi-function $F(\varphi)$ or phi-component of the tensor. These functions tell us the behaviour of deformation due to ω, φ. Thus we have:

$$F(\omega) = (x^2 - y^2)^2 \quad \text{and} \qquad \ldots (27)$$

$$F(\varphi) = + 2xy\ (x^2 - y^2) \qquad \ldots (28)$$

The above equations are the higher order form of those for phi and omega in equation 17.

5. Conclusion

Whatever the nature of incompleteness of the model, conformal transformation can always map into a half plane. The three cases shown are just to show examples of this transformation from incomplete to complete models and transformations used are among common transformations. However, it is also possible to find a particular transformation for a particular type and shape of incomplete model. Deformation function for the other orientation elements can be work out from the equation of orientation. Real parts of coefficients b, a are related to dx, dy and imaginary parts of coefficient a, b are related to dz, $d\kappa$ respectively.

6. References

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(3) Marsden, J. E. and Hoffman, M. J. 1987. *Basic Complex Analysis*. W. H. Freeman & Company, New York, USA.