A SOLUTION FOR SPACE RESECTION IN CLOSED FORM

T.Y. Shih and W. Faig
Department of Surveying Engineering
University of New Brunswick
P.O. Box 4400
Fredericton, N.B., Canada E3B 5A3
Commission 5

ABSTRACT

Conventionally, space resection has been solved by iterative means, although for many years photogrammetrists have attempted to find a solution in a closed form. Recent work with the projective transformation approach as utilized in the Direct Linear Transformation (DLT) formulation has led to a closed solution for a plane object. Tests of this solution are reported together with a critical review of other closed solutions to the space resection procedure.

INTRODUCTION

What is Space Resection?

According to Moffitt & Mikhail [1980]:

> The term space resection is the name given to the process in which the spatial position and orientation of photograph is determined based on photogrammetric measurements of the images of ground control points appearing on the photograph.

Thus, space resection in photogrammetry is an analogy to the space resection in surveying [Masry, 1979].

In essence, the space resection makes use of image coordinates and heavily weighted or fixed object space coordinates to determine the positional and rotational elements of a photograph, or of a camera. Following this basic definition, space resection of a single photo can be extended to include the interior orientation parameters, or it can be reduced to include positional elements only. Therefore, the space resection may have 3 parameters (X_C, Y_C, Z_C), 6 parameters (X_C, Y_C, Z_C, ω, φ, κ), or more.

What is a Closed Solution?

The collinearity equation model provides the most conventional solution. Six parameters could be solved for rigorously, when using this model. An extension could be made easily to include interior orientation and other parameters. However, this approach requires linearization, therefore, the convergence relies on the closeness of the initial approximation to the 'true' values.

Church, based on the image pyramid model, developed the well-known Church method [American Society of Photogrammetry, 1980] some 50 years ago. This model is an equivalent model to the collinearity equation model with a reduced parameter set. Here, only the positional elements are included. However, this model is also non-linear.

A lot of research effort has been devoted to methods that avoid the requirement for initial values. Such approaches are termed closed solution. Rampal [1979] formulated an approach based on the image pyramid model and utilized the distance relations, which provide a closed form. However, one condition is assumed: **the "object plane" is near parallel to the image plane.**

Hadem [1981] generalized Rampal's approach, but with different constraints: either one distance between an object point and the perspective center is approximately known, or numerical analysis techniques are used.

In contrast to the 3 parameter and 6 parameter space resections, an 11 parameter space resection was developed by Abdel-Aziz & Karara [1971]. This model is well-known as DLT (Direct Linear Transformation), including 11 algebraic parameters. Hadem [1981] and Okamoto [1981] indicated that these 11 DLT parameters are equivalent to 6 exterior orientation parameters, and 5 interior orientation parameters. Independent from these two studies, an equivalent physical model for the DLT model, in terms of conventional collinearity parameters was illustrated and tested by Shih & Faig [1987]. However, the DLT formulation represents a closed form solution for 11 parameters. The object has to have sufficient extension in all three dimensions in order to ensure a solution [Faig & Shih, 1986]. Although additional constraints can be added to reduce the number of unknown parameters, linearity of the model is lost along with these additions.

THE TWO-DIMENSIONAL DLT APPROACH

Based on the relations derived in Shih & Faig [1987], the two-dimensional DLT approach is formulated. A 2-D to 2-D perspective transformation is utilized for space resection, then transformed from the algebraic space into the physical space. This approach requires a nearly plane object. When the three-dimensionality increases, biases from relief displacements will negatively influence the solution. Therefore, this approach is a good supplement to the DLT approach with respect to the initial value problem. When the object has sufficient depth differences, the full DLT aproach should be used. When the object is flat, then the 2-D DLT is sufficient.

The 2-D to 2-D perspective transformation can be written in the following form:

$$x = \frac{a_1X + b_1Y + c_1}{a_3X + b_3Y + 1}$$

$$y = \frac{a_2X + b_2Y + c_2}{a_3X + b_3Y + 1}$$

In order to make it work, the coordinate component in one dimension of the object space should be constant or zero. For simplicity's sake, Z was selected, i.e., all points have Z=0. This could be achieved for a flat object simply by applying a similarity transformation with 2 rotations and 1 translation. An inverse transformation will have to be carried out after the space resection.

In this study, the transformation from an arbitrary object plane to a horizontal plane is not included. Only the space resection itself is investigated. However, an APL version routine for this transformation is attached in the appendix.

The Formulation

From Shih & Faig [1987], the transformation from DLT to physical parameters are:

1. Station parameters (X_c, Y_c, Z_c):

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} = \begin{bmatrix} -b_{11} & -b_{12} & -b_{13} \\ -b_{21} & -b_{22} & -b_{23} \\ -b_{31} & -b_{32} & -b_{33} \end{bmatrix}^{-1} \begin{bmatrix} b_{14} \\ b_{24} \\ 1 \end{bmatrix}$$

2. Interior orientation and comparator parameters (x_p, y_p, f, a, b)

$$x_p = C^2 (b_{11} \cdot b_{31} + b_{12} \cdot b_{32} + b_{13} \cdot b_{33})$$

$$y_p = C^2 (b_{21} \cdot b_{31} + b_{22} \cdot b_{32} + b_{23} \cdot b_{33})$$

$$f^2 = C^2 (b_{11}^2 + b_{12}^2 + b_{13}^2) - x_p^2$$

$$y_p^2 + f^2/a^2 + b^2 f^2/a^2 = C^2(b_{21}^2 + b_{22}^2 + b_{23}^2)$$

$$x_p \cdot y_p - bf^2/a = C^2(b_{11} \cdot b_{21} + b_{12} \cdot b_{22} + b_{13} \cdot b_{23})$$

The last two equations provide a solution for a and b, while $C^{-2} = (b^2{}_{31} + b^2{}_{32} + b^2{}_{33})$

3. Rotational matrix (as a function of ω, φ, κ)

$$M_{31} = C \cdot b_{31}; \qquad M_{11} = (x_p \cdot M_{31} - C \cdot b_{11})/f;$$

$$M_{32} = C \cdot b_{32}; \qquad M_{12} = (x_p \cdot M_{32} - C \cdot b_{12})/f;$$

$$M_{33} = C \cdot b_{33}; \qquad M_{13} = (x_p \cdot M_{33} - C \cdot b_{13})/f;$$

$$M_{21} = (y_p M_{31} + b(f/a)M_{11} - Cb_{21})\,(f/a);$$

$$M_{22} = (y_p M_{32} + b(f/a)M_{12} - Cb_{22})\,(f/a);$$

$$M_{23} = (y_p M_{33} + b(f/a)M_{13} - Cb_{23})\,(f/a);$$

This rotational matrix can be decomposed into the actual rotations, following standard procedures.

Assuming that: a = 1,
b = 0, and
f is known, then:

$$x_p = C^2(b_{11} \cdot b_{31} + b_{12} \cdot b_{32} + b_{13} \cdot b_{33})$$

$$y_p = C^2(b_{21} \cdot b_{31} + b_{22} \cdot b_{32} + b_{23} \cdot b_{33})$$

$$x_p^2 = C^2(b_{11}^2 + b_{12}^2 + b_{13}^2) - f^2$$

$$y_p^2 = C^2(b_{21}^2 + b_{22}^2 + b_{23}^2) - f^2$$

$$x_p \cdot y_p = C^2(b_{11} \cdot b_{21} + b_{12} \cdot b_{22} + b_{13} \cdot b_{23})$$

Assuming that $x_p = y_p = 0$ and C^2 is always nonequal to zero, because of the nature of the rotational matrix (see the formulation in Shih and Faig [1987]), we have:

$$b_{11} \cdot b_{31} + b_{12} \cdot b_{32} + b_{13} \cdot b_{33} = 0$$

$$b_{21} \cdot b_{31} + b_{22} \cdot b_{32} + b_{23} \cdot b_{33} = 0$$

$$C^2 (b_{11}^2 + b_{12}^2 + b_{13}^2) = f^2$$

$$C^2 (b_{21}^2 + b_{22}^2 + b_{23}^2) = f^2$$

$$b_{11} \cdot b_{21} + b_{12} \cdot b_{22} + b_{13} \cdot b_{23} = 0$$

From these equations we can solve for b_{23}, b_{13}, b_{33}, and then, the collinearity parameters can be obtained by utilizing the DLT-to-Physical routine.

$$b_{23}^2 = -(K) \pm (K^2 + 4(b_{11}.b_{21} + b_{12}.b_{22}))^{\frac{1}{2}}$$

$$\text{where} \qquad K = b_{21}^2 + b_{22}^2 - b_{11}^2 - b_{12}^2$$

$$b_{13} = (b_{11}.b_{12} + b_{12}.b_{22})/b_{23}$$

$$b_{33} = (b_{21}.b_{31} + b_{22}.b_{32})/b_{23}$$

Because of the high order equations that were used, the algebraic sign of b_{13}, b_{23}, b_{33} is not defined. The result is that the camera station can be placed on either side of the image. This is understandable because a horizontal plane object cannot define a three-dimensional datum. If there is a slight deviation from the object plane, then the sign can be defined from the relief displacements. Although three points uniquely define a plane, this dual solution problem does not happen in an iterative space resection approach with collinearity equations, because the initial values have already specified the side.

The Test

Several numerical tests were carried out to illustrate this and successful results were achieved. APL versions of the test programs are included as Appendix.

The object coordinates which were used in these tests are

	X	Y	Z
1	1	1	0
2	1	2	0
3	3	4	0
4	3	5	0
5	5	7	0

while the image coordinates were generated with the given orientation parameters.

Test 1

The given parameters are:

$(X_C, Y_C, Z_C, \omega, \phi, \kappa, x_o, y_o, p_d) = (2\ \ 2\ \ 10\ \ .1\ \ .2\ \ .3\ \ 0\ \ 0\ \ 3)$

The generated image coordinates are:

	x	y
1	0.1047951028	-0.6677451877
2	0.1981304989	-0.3763549021
3	0.9711205896	0.01903496351
4	1.056831476	0.306672862
5	1.85609465	0.7205987849

Performing the 2-D DLT, i.e., the analytical rectification, the 8 parameters were obtained as:

$(b_{11}, b_{12}, b_{14}, b_{21}, b_{22}, b_{24}, b_{31}, b_{32}) =$

(0.2822043472 0.09433758404 -0.2728083865 -0.08729603439 0.2847389063 -0.858423905 -0.01996003638 0.009830192291)

(b_{13}, b_{23}, b_{33}) can be obtained with the proposed formulation and given (x_o, y_o, p_d) values:

(-0.0480275476 0.04635381611 -0.09797403118).

Transforming the 11 DLT parameters into physical parameters, the following values were obtained:

The Perspective Centre

$(X_C, Y_C, Z_C) = (2, 2, 10)$

$(x_o, y_o, p_d) = (-6.052831834E-13,\ \ 1.66487247E-16,\ \ 3)$

Affinity:

$(a, b) = (1, -1.119688121E-29)$

The Rotational Elements:

$(\omega, \phi, \kappa) = (0.1,\ \ 0.2,\ \ 0.3)$

This represents an exact recovery.

Test 2

The given parameters are:

$(X_C, Y_C, Z_C, \omega, \phi, \kappa, x_o, y_o, p_d) = (-1\ -2\ 10, 1, 2, 3, 0, 0, 3)$

The generated image coordinates are:

	x	y
1	1.369732908	0.2229279666
2	1.452328617	0.5114946584
3	2.268391639	0.9341209602
4	2.342757286	1.218577493
5	3.186814278	1.66095499

Performing the 2-D DLT, the 8 parameters were obtained as:

$(b_{11}, b_{12}, b_{14}, b_{21}, b_{22}, b_{24}, b_{31}, b_{32}) =$

(0.2881280702 0.09631781474 0.9711205896 -0.08912845663 0.290715832 0.01903496351 -0.02037901549 0.01003653686)

(b_{13}, b_{23}, b_{33}) can be obtained with the proposed formulation and given (x_o, y_o, p_d) values:

(-0.049035690 0.04732682438 -0.1000305942).

Transforming 11 DLT parameters into physical parameters, the following values were obtained:

The Perspective Centre

$(X_C, Y_C, Z_C) = (-1,\ -2,\ 10)$

$(x_o, y_o, p_d) = (-2.56992288E\text{-}11,\ 1.236479095E\text{-}16,\ 3)$

Affinity:

$(a,\ b) = (1,\ -3.530728797E\text{-}28)$

The Rotational Elements:

$(\omega, \phi, \kappa) = (0.1,\ 0.2,\ 0.3)$

Test 3

The given parameters are:

$(x_c, y_c, z_c,\ \omega, \phi, \kappa, x_o, y_o, p_d) = (2\ \ 2\ \ -10\ \ 1\ \ 2\ \ 3\ \ 0\ \ 0\ \ 3)$

The generated image coordinates are:

	x	y
1	0.8836717793	-0.2744352479
2	0.7944316451	-0.5739169782
3	0.009587324038	-0.9649809951
4	-0.08754975043	-1.268240005
5	-0.8460891729	-1.640984055

Performing the 2-D DLT, the 8 parameters were obtained as:

$(b_{11}, b_{12}, b_{14}, b_{21}, b_{22}, b_{24}, b_{31}, b_{32}) =$

(-0.2941219867 -0.09832151032 1.285444758 0.09098259229 -0.2967635816 -0.0715516452 0.02080295932 -0.010245432653)

(b_{13}, b_{23}, b_{33}) can be obtained with the proposed formulation and given (x_o, y_o, p_d) values:

(-0.05005577644 0.04831136238 -0.01024532653).

Transforming 11 DLT parameters into physical parameters, the following values were obtianed:

The Perspective Centre

$(X_C, Y_C, Z_C) = (2, 2, 10)$

$(x_o, y_o, p_d) = (-2.698161447E-12, 8.899471175E-17, 3)$

Affinity:

$(a, b) = (1, -2.668023336E-29)$

The Rotational Elements:

$(\omega, \phi, \kappa) = (-0.1, -0.2, -2.841592654)$

The last test illustrates a problem with a possible dual solution.

CONCLUDING COMMENTS

This approach explains the 2-D to 2-D perspective transformation from a conventional collinearity equation's point of view. Eight parameters are explicitly interpreted. Compared with the physical interpretation through a rectifier, as conventionally used in textbooks, further appreciation of the algebraic form of the perspective transformation can be achieved.

This discussion also explains why "calibration" is possible with a "plane" object. There are 8 algebraic parameters, and there are 8 equivalent physical parameters. The limitation imposed on the physical model but not on the algebraic model, is the rotational matrix in this case. This also explains why an absolutely vertical photograph cannot be calibrated for its focal length with a flat object. The "calibration" is still there, but has to correspond to a different physical meaning.

The dual solution of this approach can be geometrically explained. It is caused by the fact that the positive direction of normal vector of the object plane is not defined. With relief displacement, the dual solution problem can be solved to some extent by evaluating the height difference vector and the radial displacement vector.

Practically, this approach provides a method to calculate initial values for a planar object. A rigorous calibration could be done as well. However, in order to avoid using numerical analysis techniques, the more familiar collinearity equation model may be better suited.

Recalling that Rampal's approach [Rampal, 1979] provides a practical closed solution for an object plane which is nearly parallel to the image plane case, the 2-D DLT approach requires that the rotation matrix is not equal to identity. This limitation is caused by the current algorithm which is used to recover b_{13}, b_{23}, b_{33}. When the rotation matrix is close to identity, all associated terms approach zero. This means that Rampal's approach and the approach can be coupled for practical applications on flat objects, while the DLT takes care of the truly spacial objects.

Finally, it should be noted that:

1. the numerical condition and problems associated with critical configurations for the developed approach require further research,

2. the general solution of a closed form space resection with 3 or 6 parameters is still lacking. However, keeping in mind that three points uniquely define a plane, 4 control points will be the minimum requirement for any general solution without initial values, provided that these 4 points are not lying on the same plane.

ACKNOWLEDGEMENTS

The authors wish to acknowledge the support by the National Science and Engineering Research Council of Canada and also want to thank Dr. C. Armenakis for his constructive suggestions and help rendered.

REFERENCES

1. Abdel-Aziz, Y.I. & H.M. Karara, 1971. "Direct Linear Transformation from Comparator Coordinates into Object Space Coordinates in Close-range Photogrammetry." *Proceedings of the ASP/UI Symposium on Close-Range Photogrammetry*, Urbana, Illinois, pp. 420-475.

2. American Society of Photogrammetry, 1980. Manual of Photogrammetry. 4th Edition (C.S. Slama, Editor), Falls Church, VA., Chap. 2.2.4.2 & 16.3.3.2.

3. Faig, W., & T.Y. Shih, 1986. "Critical Configuration of Object Space Control Points for the Direct Linear Transformation." International Archives of Photogrammetry and Remote Sensing, 26(5): 23-29.

4. Hadem, I., 1981. "Bundle Adjustment in Industrial Photogrammetry." *Photogrammetria*, 37 (1981): 45-60.

5. Masry, S.E., 1977. "Basics of Instrumental and Analytical Photogrammetry." Lecture Notes No. 33, Department of Surveying Engineering, University of New Brunswick, Fredericton, N.B., Canada.

6. Moffitt, F.H. & E.M. Mikhail, 1980. "Photogrammetry." 3rd Ed. Harper & Row, Inc. N.Y., U.S.A.

7. Okamoto, A., 1981. "Orientation and Construction of Models," Part I: "The Orientation Problem in Close-Range Photogrammetry." *Photogrammetric Engineering and Remote Sensing*, 47(10): 1437-1454.

8. Rampal, K.K., 1979. "A Closed Solution for Space Resection." *Photogrammetric Engineering and Remote Sensing* 45(9): 1255-1261.

9. Shih, T.Y., and W. Faig, 1987. "Physical Interpretation of the Extended DLT-model." *Proceedings of ASPRS Fall Convention*, Reno, Nevada, 385-394.

APPENDIX A

THE 2-D DLT APPROACH

```
    ∇DLT∆TO∆PHYSICAL[⎕]∇
    ∇ J←DLT∆TO∆PHYSICAL BIJ;PC;XP;YP;PD;C;J1;J2;M;W∆P∆K;AB;A;B
   ⍝
   ⍝ TRANSFORM THE 11 DLT PARAMETERS TO EQUIVALENT
   ⍝ PHYSICAL PARAMETERS
   ⍝ ===============================================
    J← 3 4⍴BIJ,1
    J1← 0 ¯1 ↓J
    J2← 3 ¯1 ↑J
    PC←-J2⌹J1
    SKIP 1
    ' THE PERSPECTIVE CENTER '
    PC
    XP←(+/J1[1;]×J1[3;])÷C←+/J1[3;]*2
    YP←(+/J1[2;]×J1[3;])÷C
    PD←(((+/J1[1;]*2)÷C)-XP*2)*0.5
    SKIP 1
    ' XP, YP, PD'
    XP,YP,PD
   ⍝
    AB←-(((+/J1[1;]×J1[2;])÷C)-XP×YP)÷PD×PD
    A←(PD×PD)÷(((+/J1[2;]*2)÷C)-(YP*2)+(AB*2)×(PD*2))
    A←A*0.5
    B←AB×A
    SKIP 1
    'AFFINITY : A , B'
    A,B
   ⍝
    M← 3 3 ⍴0
    C←C*¯0.5
   ⍝              HEADACHE : HOW TO DEFINE THE SIGN ?
    M[3;]←J1[3;]×C
    M[1;]←-((C×J1[1;])-XP×M[3;])÷PD
    M[2;]←((YP×M[3;])+(AB×PD×M[1;])-C×J1[2;])÷(PD÷A)
    SKIP 1
    ' THE ROTATION MATRIX'
    M
    '(⍉M)+.×M'
    (⍉M)+.×M
   ⍝
    W∆P∆K←R∆ELE M
    (W∆P∆K←R∆ELE-M) IF(+/+/M-ROTATION∆1 W∆P∆K)≥0.001
    SKIP 1
    ' THE ROTATION ELEMENTS : W , P , K'
    W∆P∆K
    J←(,PC),W∆P∆K,XP,YP,PD,A,B
    ∇
```

```
    ∇REC∆BZ[⎕]∇
    ∇ J←F REC∆BZ A;JB;JC;JD;JE;JF
   ⍝
   ⍝  RECTIFICATION, FROM 8 B TO 11 B
   ⍝================================
    J← 1 1 0 1 1 1 0 1 1 1 0 \A
    JB←(J[1]×J[9])+J[2]×J[10]
    JC←(J[5]×J[9])+J[6]×J[10]
    JD←(J[1]*2)+J[2]*2
    JE←(J[5]*2)+J[6]*2
    JF←(J[1]×J[5])+J[2]×J[6]
    J[7]←(|ROOT 1,(JE-JD),-JF×JF)*0.5
    J[11]←-JC÷J[7]
    J[3]←-JF÷J[7]
    ∇
```

```
      ∇ J←P∆G DLT∆REC NTE;A;X;W;I;N;U;Q;R;P∆G∆O
  ⍝
  ⍝                 RECTIFICATION
  ⍝  FORM THE DESIGN MATRIX
  ⍝ ========================
   J←8⍴0
   R←((2×1↑⍴P∆G),1)⍴0
   P∆G∆O←P∆G
   I←0
  LOOP:
   I←I+1
   A←P∆G DLT∆A∆REC J
  ⍝ ====================================================
   W←(P∆G DLT∆W∆R J)
  ⍝ ================================================
   N←(⍉A)+.×A
   U←(⍉A)+.×W
   X←-(Q←⌹N)+.×U
   R←-(A+.×X)+W
   P∆G[;1]←P∆G∆O[;1]+((2×1↑⍴P∆G)⍴ 1 0)/,R
   P∆G[;2]←P∆G∆O[;2]+((2×1↑⍴P∆G)⍴ 0 1)/,R
   J←J+,X
   ⎕←I,J
   →0 IF 1E¯13≥⌈/|X
   →LOOP IF I<NTE
      ∇
```

APPENDIX B

TRANSFORMING A PLANE TO A HORIZONTAL ONE

```
      ∇PLANE∆FIT[⎕]∇
      ∇ J←PLANE∆FIT A;N;R
  ⍝
  ⍝  FIT A(X,Y,Z) TO AX + BY + CZ = 1
  ⍝ ====================================
   N←(⍴A)[1]
   J←(N⍴1)⌹A
   ⎕←' FITTING ERROR'
   ⎕←R←(N,1)⍴(N⍴1)-A+.×J
   ⎕←' R.M.S.E.'
   ((+/,R×R)÷(N-3))*0.5
   SKIP 1
      ∇

      ∇TO∆HORIZON[⎕]∇
      ∇ J←TO∆HORIZON A;R;C
  ⍝
  ⍝  TRANSFORM ANY PLANE TO A HORIZONTAL PLANE
  ⍝         (PARALLAL TO X-Y PLANE)
  ⍝   A IS THE COEFFICIENT OF THE PLANE FUNCTION
  ⍝     WHICH CAN BE OBTAINED FROM  THE FUN. PLANE∆FIT
  ⍝   OUTPUT  IS AN  ORTHONORMAL TRANSFORMATION  MATRIX
  ⍝        FOLLOW THE ALGORITHM GIVEN BY DR. TINGLEY, 1988
  ⍝ ====================================================
   J←R←IDENTITY 3
  ⍝
   R[1;1]←R[3;3]←A[3]÷(+/A×A)*0.5
   R[1;3]←-R[3;1]←(1-R[1;1]*2)*0.5
  ⍝
   J[1;1]←J[2;2]←A[1]÷C←(+/A[1 2]*2)*0.5
   J[2;1]←-J[1;2]←A[2]÷C
  ⍝
   J←(⌹J)+.×R+.×J
      ∇
```