SPATIAL RELATIONS BETWEEN SETS

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ABSTRACT:

In this paper, after an introduction to the basic ideas and notations of *metric topology*, a integrated theory of spatial relations (such as metric, order and topology) between sets is developed in which the relations are defined in terms of the intersections of the boundaries, interiors and exteriors of two dynamically generated sets based on the Hausdorff metric. Then some extended models are presented mainly for quantitatively deriving spatial relations between partially separated objects and objects in constrained spaces. Finally, examples for integrally reasoning different kind of spatial relations are given and some potential applications of presented theories in GIS area are also suggested.

1. INTRODUCTION

Conditions among spatial data are commonly expressed in terms of *spatial prepositions* or *spatial relations*. The spatial relations are often classified into *metric* (distances and directions), *order* (partial or total order) and *topology* three groups. Over the passed few years, the investigation of formal and sound methods of describing spatial relations have received unprecedented attention in the GIS area. Much progress has been made, particularly in the area of formalizing topological relations based on the mathematically well-defined 4/9-intersection model [Egenhofer and Franzosa, 1991, 1994; Egenhofer and Herring, 1991; Mark and Egenhofer, 1995]. In the meantime, many investigations also have been made for quantitatively deriving metric relations [Frank 1992; Peuquet and Zhang, 1987; Chen et al., 1995], and partial or total order relations [Kainz et al., 1993]. However, unlike the studies of topological relations, formalizations of metric and order relations are generally based on a diversity of models. How to integrally derive different kinds of spatial relations between sets (non-point-like) based on an mathematically well-defined unified algebra framework is still an open problem up to now. This lack of an integrated comprehensive theory of spatial relations has been a major impediment for solving many sophisticated problems in GIS, such as formally deriving complex spatial relations among spatial objects with multiple representations or uncertainties, integrally reasoning metric, order and topological spatial relations, and generation of the related standards for transferring spatial relations.

This paper focuses on the development of the unified algebra framework and associated models for deriving different kinds of spatial relations between sets. At first, after an introduction to the basic ideas and notations of metric topology, a integrated theory of spatial relations between sets is developed in which the relations are defined in terms of the intersections of the boundaries, interiors and exteriors of two dynamically generated sets based on the Hausdorff metric. Then some extended models are presented mainly for quantitatively determining spatial relations between partially separated objects and objects in constrained spaces. Finally, examples for integrally reasoning different kind of spatial relations are given and some potential applications of presented theory in GIS area are also suggested.

The remainder of this paper is structured as follows: Chapter 2 firstly reviews some related fundamental definitions. In chapter 3 a integrated theory of spatial relations is developed based on the metric topology theory and the dynamic 9-intersections. Chapter 4 contains some extensions of the presented theories and models. Practical algorithms and examples for integrally reasoning different kind of spatial relations are given in chapter 5. In the last chapter conclusions and outlook for further research are given.

2. THE FUNDAMENTAL DEFINITIONS

2.1. Partially Ordered Sets and Lattices

(a). Partially ordered sets: Let P be a set, a *partial order* on P is a binary relation $\leq$ on P such that, for every $x, y, z \in P$: (1). $x \leq x$ (*reflexive*); (2). if $x \leq y$ and $y \leq x$, then $x = y$ (*antisymmetric*); (3). if $x \leq y$ and $y \leq z$, then $x \leq z$ (*transitive*). A set with a reflexive, antisymmetric and transitive relation (order relation) $\leq$ is called a *partially ordered set* (or poset).

(b). Upper and lower bounds: Let P be a poset and $S \subseteq P$. An element $x \in P$ is an *upper bound* of S if $s \leq x$ for all $s \in S$. A *lower bound* is defined by duality. The set of all upper bounds of S is denoted by S^* and the set of all lower bounds is denoted by S_*. If S^* has a least element, it is called the *least upper bound* of S. By duality, if S_* has a largest element, it is called the *greatest lower bound* of S. A least upper bound or a greatest lower bound is always unique.

(c). Lattices: A *lattice* L is a poset in which every pair of elements has a least upper bound and a greatest lower bound. A lattice is called complete when a greatest lower bound and a least upper bound exist for every subset of the poset. It can be proven that every finite lattice is complete [Kainz et al., 1993].

2.2. Mathematical Morphology

Mathematical morphology (MM) is an approach to the analysis of structures based on set theoretic concepts. Let X be a given object set, B be a set of structure element, the two fundamental morphological operations on X are defined as follows:

Dilation: $X \oplus B = \bigcup_{b\in B} X_b$ [1]

Erosion: $X \ominus B = \bigcap_{b\in B} X_{-b}$ [2]

where X_b is defined as the translation of X by vector b, i.e., $X_b = \{x + b | x \in X\}$. From equations [1] and [2], we know that dilation is an expansion of the set and erosion is a shrinking of the set. The detail definitions of other morphological operators and their properties can be found in [Serra, 1982].

2.3. Metric Spaces

A *metric space* is a pair consisting of a set E and a mapping $(p_1, p_2) \to d(p_1, p_2)$ of $E \times E$ into R_+ ,having the properties: (1). $p_1 = p_2 \Leftrightarrow d(p_1, p_2) = 0$; (2). $d(p_1, p_2) = d(p_2, p_1)$ (*symmetry*); (3). $d(p_1, p_2) \le d(p_1, p_3) + d(p_3, p_2)$ (*triangle inequality*). The function d is called a *metric* and $d(p_1, p_2)$ is called the *distance* between the points p_1 and p_2. Distance between points $p_i(x_{i1}, x_{i2}, \dots, x_{in})$ in R^n is described in terms of the *Minkowski* d_t -metric:

$$d_t(p_1, p_2) = (|\sum_{j=1}^{n} x_{1j} - x_{2j}|^t)^{1/t} \quad [3]$$

Conventional *Euclidean distance* is defined by the d_2 -metric. Similarly, the *Manhattan distance* defined by the d_1 -metric, and the *maximum distance* defined by the d_∞ -metric. Some examples of different distances are shown in **Fig.1**.

2.4. Topological Spaces

A *topological space* is a pair consisting of a set E and a collection $\mathcal{A}$ of subsets of E called the *open sets*, satisfying the three following properties: (1). every union (finite or otherwise) of open sets is open; (2). every finite intersection of open sets is open; (3). the set E and the empty set $\cap$ are open.

One of the most important properties which a topological space can satisfy is that of *compactness*. A topological space E is said to be compact if it is separated and if from every open covering of E one can select a finite subcovering of E. Some other related definitions of topological concepts, such as interior, closure and boundary can be found in [Egenhofer and Franzosa, 1991].

2.5. The Hausdorff Metric

(a). Metric topology: A metric d on a set E includes a topology on E, called *metric topology* defined by d. This topology is such that $U \subset E$ is an open set if, for each $p_1 \in U$, there is an $\varepsilon > 0$ such that the d-ball of radius ε around p_1 is contained U . A d-ball is the set of points whose distance from p_1 in the metric d is less than ε , i.e. $\{p_2 \in E | d(p_1, p_2) < \varepsilon\}$. Notice that the metric topological spaces are *Hausdoff* and *separable*.

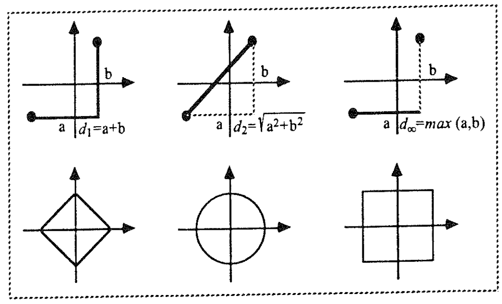

Fig.1. Three methods of measuring distance in the plane and the corresponding unit sets

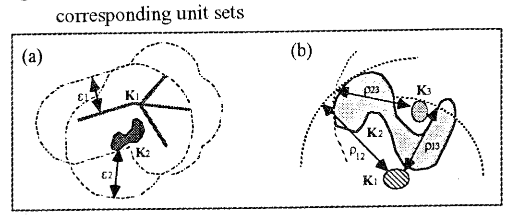

Fig.2. (a). the Hausdorff distance ε_2 ; (b).the distance properties.

(b). The Hausdorff metric: Hausdorff's metric is defined on the space $\mathcal{K}$ where each point is a non empty compact set of R^2 . If K_1 and K_2 denote two non empty compact set in R^2 (or equivalently two points in $\mathcal{K}$), and $B(\varepsilon)$ is the closed ball with a radius ε , then the quantity:

$$\rho(K_1, K_2) \equiv \inf\{\varepsilon : K_1 \subset K_2 \oplus B(\varepsilon), K_2 \subset K_1 \oplus B(\varepsilon)\} \quad [4]$$

defines a metric ρ on $\mathcal{K}$, known as the Hausdorff metric. From equation [4], ρ is the radius of the smallest closed ball B such that both K_1 is contained in the set $K_2 \oplus B(\varepsilon)$ generated by dilation and K_2 is contained in the dilated set $K_1 \oplus B(\varepsilon)$. It can be proven that the Hausdorff distance $\rho(K_1, K_2)$ satisfies all the properties of distance functions [Serra, 1982]. **Fig.2** illustrates the notation and properties of Hausdorff distances.

In particular case, when K_1 and K_2 are reduced to two points, the Hausdorff distance $\rho(K_1, K_2)$ coincides with the Euclidean distance.

3. SPATIAL RELATIONS BETWEEN SETS

3.1. Topological Relations between Sets

Topological relations are spatial relations that are preserved under such as rotation, scaling, and rubber sheeting. The model for binary topological relations is based on the usual concepts of point-set topology with open and closed sets [Egenhofer et al., 1994]. The binary topological relations between two objects, K_1 and K_2, in $I\!R^2$ is based upon the intersection of K_1's interior (K_1^o), boundary (∂K_1),and exterior (K_1^-) with K_2's interior (K_2^o), boundary (∂K_2), and exterior (K_2^-). A 3×3 matrix $\Im_9$, called the 9-intersection as follows:

$$\Im^9 = \begin{bmatrix} K_1^o \cap K_2^o & K_1^o \cap \partial K_2 & K_1^o \cap K_2^- \\ \partial K_1 \cap K_2^o & \partial K_1 \cap \partial K_2 & \partial K_1 \cap K_2^- \\ K_1^- \cap K_2^o & K_1^- \cap \partial K_2 & K_1^- \cap K_2^- \end{bmatrix} \quad [5]$$

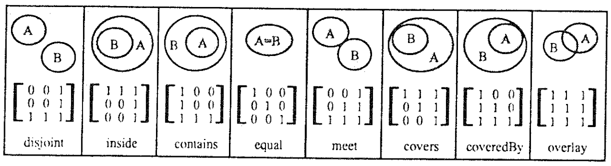

Fig.3. The eight topological relations between two regions in IR^2 .

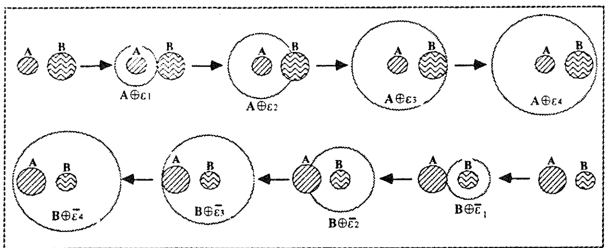

Fig.4. The dynamic topological relations derived by the dynamic 9-intersections.

Topological invariance, applicable to the 9-intersection, are the content (i.e. emptiness or non-emptiness) of a set, the dimension, and the number of separations [Egenhofer and Franzosa, 1994]. The contents invariant is the most general criterion as other invariant can be considered refinements of non-empty intersections, and is only invariant discussed in this paper. By considering the values empty (0) and non-empty (1), one can distinguish between $2^9=512$ binary topological relations in which only a small subset can be realized when the objects of concern are embedded in IR^2 . Egenhofer and Herring (1991) showed that, for two regions with connected boundaries embedded in IR^2, the 9-intersection distinguishes just 8 different relations, i.e. *disjoint, contains, inside, equal meet, covers, coveredBy,* and *overlap* [see **Fig.3**]. However, when we apply the 9-intersection model to describing topological relations between other types of spatial objects, such as point-objects and line-objects, as well as binary topological relations combining different types of spatial objects such as a line and a region, a point and a line, or a point and a region, the situation will be more complicated. According to the results of Mark et. al. (1995), for two simple lines 33 different spatial relations are possible, and for a line and a region, 19 are possible. For the detail descriptions of topological relations, please find in [Egenhofer and Franzosa, 1991, 1994; Mark et. al., 1995].

3.2. Metric Relations between Sets

3.2.1. Dynamic 9-intersection

According to the topological properties of morphological dilation [Serra, 1982], if the set K_i and the structure element set $B(\varepsilon_t)$ are both closed sets, then the dilated set $K_i \oplus B(\varepsilon_t)$ is also the closed set. Based on this result, we extent the general 9-intersection to the dynamic intersections as follows:

$$\mathfrak{I}^9_{(1,2)}(\varepsilon_i)=\begin{bmatrix} [K_1\oplus B(\varepsilon_i)]^\circ\cap K_2^\circ & [K_1\oplus B(\varepsilon_i)]^\circ\cap\partial K_2 & [K_1\oplus B(\varepsilon_i)]^\circ\cap K_2^- \\ \partial[K_1\oplus B(\varepsilon_i)]\cap K_2^\circ & \partial[K_1\oplus B(\varepsilon_i)]\cap\partial K_2 & \partial[K_1\oplus B(\varepsilon_i)]\cap K_2^- \\ [K_1\oplus B(\varepsilon_i)]^-\cap K_2^\circ & [K_1\oplus B(\varepsilon_i)]^-\cap\partial K_2 & [K_1\oplus B(\varepsilon_i)]^-\cap K_2^- \end{bmatrix} \quad [6]$$

$$\mathfrak{I}^9_{(2,1)}(\varepsilon_j)=\begin{bmatrix} K_1^\circ\cap[K_2\oplus B(\varepsilon_j)]^\circ & K_1^\circ\cap\partial[K_2\oplus B(\varepsilon_j)] & K_1^\circ\cap[K_2\oplus B(\varepsilon_j)]^- \\ \partial K_1\cap[K_2\oplus B(\varepsilon_j)]^\circ & \partial K_1\cap\partial[K_2\oplus B(\varepsilon_j)] & \partial K_1\cap[K_2\oplus B(\varepsilon_j)]^- \\ K_1^-\cap[K_2\oplus B(\varepsilon_j)]^\circ & K_1^-\cap\partial[K_2\oplus B(\varepsilon_j)] & K_1^-\cap[K_2\oplus B(\varepsilon_j)]^- \end{bmatrix} \quad [7]$$

where the K_1 and K_2 are given two closed sets, the $K_i \oplus B(\varepsilon_t)$ means relevant morphological dilation by the closed ball B with radius ε_t , and the $\mathfrak{I}^9_{(i,j)}(\varepsilon_t)$ means dynamic 9-intersection with parameter ε_t from K_i to K_j . Based on equations [6] and [7], we can derive dynamic topological relations by using the different parameter ε_t [see **Fig.4**].

In particular case, when $\varepsilon_t=0$, the structure element $B(\varepsilon_t)$ is reduced to the original point $\{o\}$, according to the algebraic properties of morphological dilation [Serra, 1982], we have $K_i \oplus B(\varepsilon_t) = K_i \oplus \{o\} = K_i$; then the dynamic 9-intersections $\mathfrak{I}^9_{(i,j)}(\varepsilon_t)$ defined in equations [6] and [7] coincide with the general 9-intersection $\mathfrak{I}_9$ in equation [4].

3.2.2. Distance relations between sets

Distance relations are spatial relations that are defined under different distance functions, such as the d_1 , d_2 and d_∞ -metrics between spatial points, as well as the Hausdorff distance ρ between spatial objects. Since spatial points are the special spatial objects with simple structures, in general cases only the Hausdorff distance ρ is discussed in this paper.

According to the derived dynamic topological relations by the dynamic 9-intersections of equations [6] and [7] with different parameter ε_t, such as dynamic equal, dynamic covers (or dynamic corveredBy) and dynamic contains (or dynamic inside), we can simply get the Hausdorff distance $\rho(K_1,K_2)$ between two closed sets K_1 and K_2 by calculating the minimum and maximum dilated distances based on equation [4]:

$$\left.\begin{array}{l} \rho(K_1,K_2)=\max\{\min(\varepsilon_i),\ \min(\varepsilon_j)\};\ \text{when} \\ \mathfrak{I}^9_{(1,2)}(\varepsilon_i)=\begin{bmatrix}100\\010\\001\end{bmatrix},\begin{bmatrix}000\\010\\001\end{bmatrix},\begin{bmatrix}111\\001\\001\end{bmatrix},\begin{bmatrix}011\\001\\001\end{bmatrix},\begin{bmatrix}001\\011\\001\end{bmatrix},\begin{bmatrix}001\\111\\001\end{bmatrix},\begin{bmatrix}111\\101\\001\end{bmatrix},\begin{bmatrix}1{*}1\\{*}11\\001\end{bmatrix}; \\ \mathfrak{I}^9_{(2,1)}(\varepsilon_j)=\underbrace{\begin{bmatrix}100\\010\\001\end{bmatrix},\begin{bmatrix}000\\010\\001\end{bmatrix}}_{equal},\underbrace{\begin{bmatrix}100\\100\\111\end{bmatrix},\begin{bmatrix}000\\100\\111\end{bmatrix},\begin{bmatrix}000\\010\\111\end{bmatrix}}_{contains},\underbrace{\begin{bmatrix}010\\010\\111\end{bmatrix},\begin{bmatrix}110\\100\\111\end{bmatrix},\begin{bmatrix}1{*}0\\{*}10\\111\end{bmatrix}}_{convers}. \end{array}\right\} \quad [8]$$

where " * " means either empty (0) or non-empty (1). The binary distance relations derived by equation [8] are suitable for different types of spatial objects, such as point-objects, line-objects and region-objects, as well as combining different types of spatial objects such as a line and a region, a point and a line, or a point and a region. Some examples are shown in **Fig.5**.

3.2.3. Directional relations between sets

The Hausdorff metric between sets is effected by the choice of that metric functions. Directional relations between sets can be defined by the Hausdorff metric of *angular bearings*. The computation of direction from one spatial object to another is identical to that for metric function except that the angular bearing is computed for each ordered pair in the Cartesian product. The angular bearing is measured in the sense of navigation bearings (i.e. increasing clockwise from north).

For calculation of the directional relation $\varphi(K_i,K_j)$ between two non empty compact sets K_1 and K_2 in R^2 , we select the angular bearing set $R(\alpha_t)$ as instead of closed ball $B(\varepsilon_t)$, then we can define the Hausdorff directions and the dynamic 9-intersections for directional relations as follows:

$$\left.\begin{aligned}\varphi(K_1,K_2) &\equiv \inf\{\alpha : K_1 \subset K_2 \oplus R(\alpha \pm \pi), K_2 \subset K_1 \oplus R(\alpha)\}\\ \varphi(K_2,K_1) &\equiv \varphi(K_1,K_2) \pm \pi\end{aligned}\right\} \quad [9]$$

$$\Re^9_{(1,2)}(\alpha_i) = \begin{bmatrix} [K_1 \oplus R(\alpha_i)]^o \cap K_2^o & [K_1 \oplus R(\alpha_i)]^o \cap \partial K_2 & [K_1 \oplus R(\alpha_i)]^o \cap K_2^- \\ \partial[K_1 \oplus R(\alpha_i)] \cap K_2^o & \partial[K_1 \oplus R(\alpha_i)] \cap \partial K_2 & \partial[K_1 \oplus R(\alpha_i)] \cap K_2^- \\ [K_1 \oplus R(\alpha_i)]^- \cap K_2^o & [K_1 \oplus R(\alpha_i)]^- \cap \partial K_2 & [K_1 \oplus R(\alpha_i)]^- \cap K_2^- \end{bmatrix} \quad [10]$$

$$\Re^9_{(2,1)}(\alpha_j) = \begin{bmatrix} K_1^o \cap [K_2 \oplus R(\alpha_j)]^o & K_1^o \cap \partial[K_2 \oplus R(\alpha_j)] & K_1^o \cap [K_2 \oplus R(\alpha_j)]^- \\ \partial K_1 \cap [K_2 \oplus R(\alpha_j)]^o & \partial K_1 \cap \partial[K_2 \oplus R(\alpha_j)] & \partial K_1 \cap [K_2 \oplus R(\alpha_j)]^- \\ K_1^- \cap [K_2 \oplus R(\alpha_j)]^o & K_1^- \cap \partial[K_2 \oplus R(\alpha_j)] & K_1^o \cap [K_2 \oplus R(\alpha_j)]^- \end{bmatrix} \quad [11]$$

where the $K_i \oplus R(\alpha_t)$ means relevant dilation by the closed angular bearing R with angle α_t, and the $\Re^9_{(i,j)}(\alpha_t)$ means the dynamic 9-intersection for directional relations with parameter α_t from K_i to K_j. Based on equations [10] and [11], we also can derive dynamic topological relations by using the different parameter α_t.

Similarly, according to the derived dynamic topological relations by the dynamic 9-intersections of equations [10] and [11] with different parameter α_t, we also can simply get the Hausdorff direction $\varphi(K_i,K_j)$ (or $\varphi(K_j,K_i)$)between two closed sets K_1 and K_2 by calculating the minimum and maximum dilated angles based on equation [9]:

$$\left.\begin{aligned} &\left.\begin{aligned}\varphi(K_1,K_2) &= \max\{\min(\alpha_i), \min(\alpha_j \pm \pi)\}\\ \varphi(K_2,K_1) &= \varphi(K_1,K_2) \pm \pi\end{aligned}\right\}; \text{ when}\\ &\Re^9_{(1,2)}(\alpha_i) = \underbrace{\begin{bmatrix}100\\010\\001\end{bmatrix}, \begin{bmatrix}000\\010\\001\end{bmatrix}}, \underbrace{\begin{bmatrix}111\\001\\001\end{bmatrix}, \begin{bmatrix}011\\001\\001\end{bmatrix}, \begin{bmatrix}001\\011\\001\end{bmatrix}}, \underbrace{\begin{bmatrix}001\\111\\001\end{bmatrix}, \begin{bmatrix}111\\101\\001\end{bmatrix}, \begin{bmatrix}1{*}1\\{*}11\\001\end{bmatrix}};\\ &\Re^9_{(2,1)}(\alpha_j) = \underbrace{\begin{bmatrix}100\\010\\001\end{bmatrix}, \begin{bmatrix}000\\010\\001\end{bmatrix}}_{equal}, \underbrace{\begin{bmatrix}100\\100\\111\end{bmatrix}, \begin{bmatrix}000\\100\\111\end{bmatrix}, \begin{bmatrix}000\\010\\111\end{bmatrix}}_{contains}, \underbrace{\begin{bmatrix}010\\010\\111\end{bmatrix}, \begin{bmatrix}110\\100\\111\end{bmatrix}, \begin{bmatrix}1{*}0\\{*}10\\111\end{bmatrix}}_{convers}. \end{aligned}\right\} \quad [12]$$

where " * " means either empty (0) or non-empty (1). The binary directional relations derived by [12] are also suitable for different types of spatial objects and their combining types. Some examples are shown in **Fig.5**.

3.3. Order Relations between Sets

Formal methods for the description of order spatial relations can be based on mathematical theories of partially-ordered sets and lattices. The use of greatest lower bounds and least upper bounds for describing order spatial relations shows that we need a lattice in order to find an answer in all possible cases. Since not every partially-ordered set is a lattice, it is, however, always possible to add elements to a partially-ordered set to create a lattice. The process of normal completion specifies how to find the smallest numbers of elements necessary to add to a partially-ordered set to create a lattice, i.e. to build the minimal containing lattice of a partially-ordered set. The general descriptions of order relations can be found in Kainz et. al. [1993]. In this paper, we emphasize to study the problems of the detail classifications of order relations and their integration with other kind of spatial relations.

Based on metric and topological relations, order spatial relation

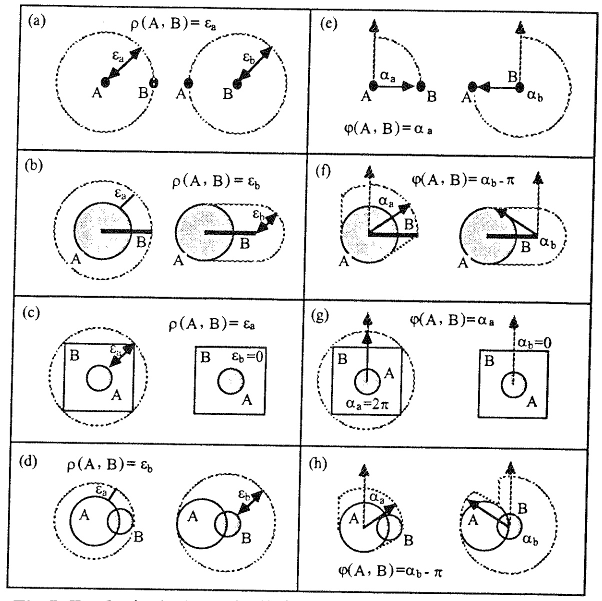

Fig.5. To obtain the Hausdorff distance and directions between two planar sets A and B.

can be classified to the three kinds, i.e. distance orders (such as the areas of regions or the lengths of lines), directional orders (such one spatial object at left/right or in front/behind to another one), and topological orders (such as one spatial object is inside another or its inverse operations whether a spatial object contains another one). Let $\cap$ be the greatest lower bound and U be the least upper bound of the completion lattices, **Fig.6** shows some examples of lattice resulting from the normal completion of order spatial relations measured by distances, directions and topological covers.

4. MODEL EXTENSIONS

4.1. Metric Relations between Subsets

According to the fuzzy set theory [Zadeh, 1965], the concept of distances or directions between subsets are fuzzy, since the spatial objects may contain many subsets, the distances or

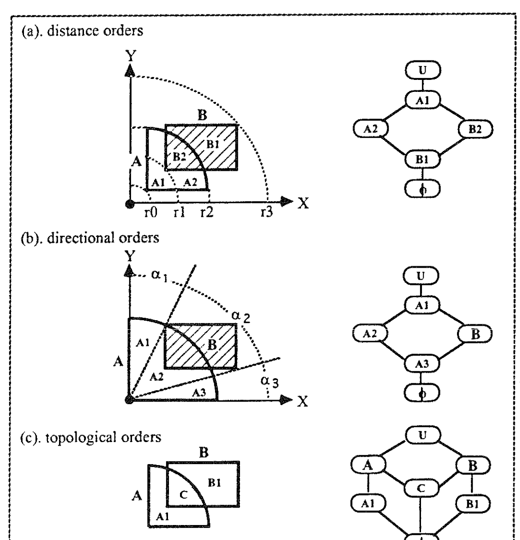

Fig.6. The classification of order spatial relations.

directions between these subsets are difficult to be represented just by a single value. If we define the fuzzy membership values as the covering percentages of generated region areas (or point numbers and line lengths) by the dynamic intersection of sets in R^2, we can find that the Hausdorff metric is just the special case with the fuzzy membership value equal to one. Based on the changes of the covering areas of regions ((or point numbers and line lengths) from an empty set to a complete set, we can estimate the fuzzy membership values from zero to one, then we can quantitatively derive the metric relations between subsets.

For reasons of simplicity the distances and directions between closed sub-regions discussed in this paper only, related models for estimation of fuzzy membership functions are defined as follows:

$$\left.\begin{aligned}\Phi_A(\lambda)&=\frac{A\{[K_1\oplus B(\lambda)]\cap K_2\}+A\{K_1\cap[K_2\oplus B(\lambda)]\}}{A(K_1)+A(K_2)}\\ \Omega_A(\theta)&=\frac{A\{[K_1\oplus R(\theta)]\cap K_2\}+A\{K_1\cap[K_2\oplus R(\theta)]\}}{A(K_1)+A(K_2)}\end{aligned}\right\}\qquad [13]$$

where $A\{*\}$ means covered area sizes by dynamic intersections with the parameters λ and θ for distances and directions separately, the functions $0\le\Phi_A(\lambda)\le 1$ and $0\le\Omega_A(\theta)\le 1$ with the parameters $0\le\lambda\le\rho(K_1,K_2)$ and $0\le\theta\le\varphi(K_1,K_2)\pm\pi$ are called the size distribution functions [Chen, 1991; Serra, 1982]. An examples of distance relations between two spatial regions is shown in **Fig.7**.

4.2. Distance Relations between Sets in Constrained Spaces

In a space, there are often some obstacles such as rivers between objects. In this case, we cannot take a straight path if the path crosses the rivers; we should take the shortest path that does not cross the river except at bridges. The shortest path between a location p_1 and a point p_2 in a space E with a constrained set C is defined by the shortest path among all possible continuous paths connecting p_1 and p_2 that does not intersect the obstacle set C. In mathematics, this shortest path is called the "geodesic line" and its length denoted by the geodesic distance $d_C(p_1,p_2)$. If the obstacle set C cuts apart two points p_1 and p_2 to two separated sets, there is no continuous path linking p_1 and p_2, in this case we define $d_C(p_1,p_2)=+\infty$. As the descriptions in [Lantuejoul and Maisonneuve, 1984], the geodesic distance $d_C(p_1,p_2)$ satisfies all the properties of distance functions, so the space E defined with a geodesic distance d_C is also a metric space. Similarly, the shortest distance between sets K_1 and K_2 in a given space $\mathcal{K}$ with a constrained set C also can be defined by the geodesic Hausdorff distance based on the geodesic morphological operations [Chen, 1991; Lantuejoul and Maisonneuve, 1984].

The geodesic Hausdorff distance between sets is defined on the space $\mathcal{K}$ and the constrained space C where each point is a non empty compact set of R^n. If K_1 and K_2 denote two non empty compact set in R^n (or equivalently two points in $\mathcal{K}$), then the quantity:

$$\rho_C(K_1,K_2)\equiv\inf\{\varepsilon: K_1\subset\oplus D_C^\varepsilon(K_2),\ K_2\subset D_C^\varepsilon(K_1)\}\qquad [14]$$

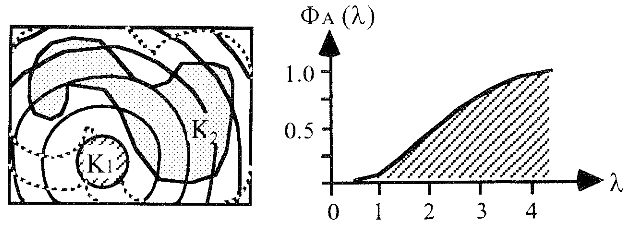

Fig.7. The distance relations between subsets and their size distribution functions.

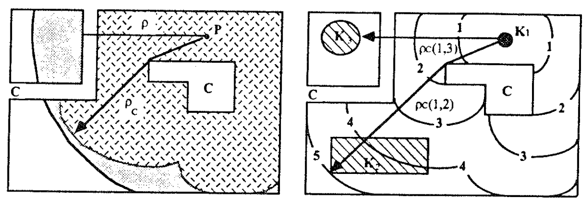

Fig.8. The geodesic Hausdorff distances between sets constrained by set C.

defines a metric ρ_C on $\mathcal{K}$, known as the geodesic Hausdorff metric. From equation [14], D_C^ε is the geodesic dilation denoted by $D_C^\varepsilon(K_i)=\{z\in C^c|B_C(z,\varepsilon)\cap K_i\equiv\cap\}$, $B_C(z,\varepsilon)=\{z\in K|d_C(z,p)\le\varepsilon\}$ is the geodesic ball with a center z and a radius ε, and ρ_C is the radius of the smallest geodesic ball B_C such that both K_1 is contained in the dilated set $D_C^\varepsilon(K_2)$ and K_2 is contained in the dilated set $D_C^\varepsilon(K_1)$.

In particular case when K_1 and K_2 are reduced to two points, the geodesic Hausdorff distance $\rho_C(K_1,K_2)$ coincides with the point geodesic distance. Similarly, we also can define the dynamic 9-intersection for the spatial relation of geodesic distance and fuzzy geodesic distances between partial separated subsets. The detail description of these formulas is omitted in this paper. An example is shown in **Fig.8** for illustrating the notation of the geodesic and fuzzy geodesic Hausdorff distance between two compact sets.

5. ALGORITHMS AND EXAMPLES

5.1. Algorithms

To determine the Hausdorff distances or directions between spatial objects in layer-based vector or raster GIS environments, the main problem is how to realize the operations of morphological dilation and logical intersection. The dilation by the structure elements of disks or angular bearings can be treated as general and oriented buffer operations separately, and logical intersections can be implemented by overlay operations in GIS environments. Since both buffering and overlay are basic data analyzing functions included in many commercial GIS software, the implementation of the Hausdorff distances or directions changes to a very easy problem. In a raster GIS environment, the main problem is how to considerate the different distance functions when we generate buffer zones. For square grid raster data, generally used distance functions can be separated as 4-connection, 8-connection and quasi-Euclidean distances in 2-D digital spaces. The detail descriptions of algorithms for determining the Hausdorff distances can be found in [Chen et. al, 1995].

5.2. Integrated Reasoning of Spatial Relations

By using the models of spatial relations between sets presented as above, we can integrally derive the compositions among different kinds of spatial relations. It is possible to assess whether it is a consistent query to ask for "all objects K_1 that are farther then 100 meters between K_2 such that K_2 contains K_3 and the distance between K_3 and K_1 is less than 100 meters", we can integrally reason out the compositions between distance and topological relations. by the following steps:

- *since* $\rho(K_1, K_2) \geq 100\ (\mathrm{m})$, $K_1 \oplus B(\rho_{(1,2)})$ *covers* K_2 *;*
- *since* K_2 *contains* K_3, $K_1 \oplus B(\rho_{(1,2)})$ *contains* K_3 *;*
- *so that* $K_1 \oplus B(\rho_{(1,3)})$ *covers (or contains),* K_3 *and* $\rho(K_1, K_3) \leq \rho(K_1, K_2)$ *[see **Fig.9**].*

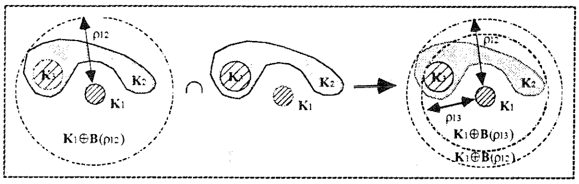

Fig.9. Integrated reasoning the spatial relations of metric and topology.

6. CONCLUSIONS AND OUTLOOKS

As the natural extension of the general 9-intersection which is used for formally deriving topological relations only, the dynamic 9-intersection based on metric topology supplied a general framework for studying different kinds of spatial relations between sets. The presented integrated theory of spatial relations between sets makes a new way for formally deriving complex spatial relations among spatial objects with uncertainties [Chen et. al., 1996], integrally reasoning metric, order and topological spatial relations, and generation of the related standards for transferring spatial relations [Mark et. al.,1995]. Even though the presented approach is only focus on the applications in GIS field, the related results for deriving spatial relations between sets can be also used for many other fields, such as CAD, computer vision, pattern recognition, robot space searching and so on. However, only the theoretical models and algorithms have be presented in this paper, a wide field of practical application for data management and spatial data analysis in 2-D and 3-D GIS environments has not been touched. Therefore, the reported results must be verified and extended in order to be used in different practical environments.

Two main directions for further research shall be pointed here, one is the applications of the presented theoretical models and algorithms in 2-D and 3-D GIS environments for developing the new tools of spatial query and analysis; another one is the extensions of presented theories and models for formally deriving complex spatial relations among spatial objects with multiple representations.

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