VERTICAL PERSPECTIVE PROJECTION OF THE ROTATIONAL ELLIPSOID

Prof. Wagih N. Hanna
Faculty of Engineering, Ain Shams University, Abbassia, Cairo, EGYPT
Comission IV, Working Group 2

KEY WORDS: Cartography, Space, Mapping, Analytical, Geometric.

ABSTRACT

The part of the earth visible from a certain outer point, which is the perspective center is illustrated in the photo from the space camera as a perspective projection. We consider the camera axis to be truly vertical through the center of projection and normal to the surface of the ellipsoid. In fact there are two real normals through any point in the plane of the ellipse. One normal belongs to the nearest part of latitude φ₁, another normal to the other part of latitude φ₂. These latitudes are of different values and signs. The location of any point is given by the geographic coordinates φ, λ as well as its altitude h above the reference ellipsoid. In this paper we deduce analytically the corresponding map coordinates of the perspective projection of the points on the surface of the rotational ellipsoid onto the tangent plane through the two points of intersection of these normals with the ellipsoid. The coordinates are deduced in terms of the geodetic and geocentric coordinates.

1 INTRODUCTION

1.1 Notations Fig.(1), Fig.(2)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>L</td>
<td>The perspective center.</td>
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<tr>
<td>H</td>
<td>The altitude of L.</td>
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<tr>
<td>LO' , LO''</td>
<td>The perpendiculars on the ellipsoid from L.</td>
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<tr>
<td>φ</td>
<td>Latitude.</td>
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<tr>
<td>λ</td>
<td>Longitude.</td>
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<tr>
<td>h</td>
<td>Altitude above the reference ellipsoid.</td>
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<tr>
<td>P(φ, λ,h)</td>
<td>Any Point on the earth P(X,Y,Z).</td>
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<td>P₀(φ₀, λ₀,h₀)</td>
<td>Center of map (minimal distance).</td>
</tr>
<tr>
<td>P′₀(φ′₀, λ′₀,h′₀)</td>
<td>Center of map (maximal distance).</td>
</tr>
<tr>
<td>T, T₀</td>
<td>Tangent plane at P₀, P′₀.</td>
</tr>
<tr>
<td>φ₀, λ₀</td>
<td>The geocentric latitude w.r.t the center of the earth O.</td>
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<tr>
<td>P(x,y)</td>
<td>The map point corresponding to P</td>
</tr>
<tr>
<td>X-X₀, Y-Y₀, Z-Z₀</td>
<td>ΔX, ΔY, ΔZ</td>
</tr>
<tr>
<td>LP₀, LP′₀</td>
<td>d λ, d φ</td>
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1.2 The Geometry of the Rotational Ellipsoid

Consider the earth as a rotational ellipsoid (Reference Ellipsoid) with semi-major axis a and semi minor axis c (axis of rotation). The equation of the rotational ellipsoid in the system shown in (Fig.2) is given by:

\[ \frac{X^2}{a^2} + \frac{Y^2}{c^2} + \frac{Z^2}{b^2} = 1 \]  (1)

The following elliptical parameters (Fig.3) are frequently used in the analytical calculations:

Eccentricity :

\[ e^2 = \frac{a^2 - c^2}{a^2} \]  (2)

Radius of curvature: at a point on the ellipsoid:

\[ N = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \]  (3)

1.3 Coordinates

The position of a point on the surface of the rotational ellipsoid is expressed by either the geocentric cartesian coordinates (X,Y,Z) or by the two dimensional geodetic coordinates (φ,λ), where:

\[ X = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \cos \phi \cos \lambda \]  (4)
\[ Y = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \cos \phi \sin \lambda \]  (5)
\[ Z = \frac{a(1 - e^2)}{\sqrt{1 - e^2 \sin^2 \phi}} \sin \phi \]  (6)

\[ \phi = \arctan \left[ \frac{Z}{(1 - e^2)X^2 + Y^2} \right] \]  (7)
\[ \lambda = \arctan \left( \frac{Y}{X} \right) \]   (8)

The equation of the tangent plane at any point (φ,λ) is given by:

\[ x \cos \phi \cos \lambda + y \cos \phi \sin \lambda + z \sin \phi = \frac{a^2}{N} \]  (9)

Sometimes the points are expressed by the geocentric latitude φ₀, which is the angle subtended by the geocentric line OP and the equator. The relation between the geocentric latitude φ₀ and the geodetic latitude φ is

Fig. (1) Two real normals from one point in the plane of the ellipse

Fig (2) Geometry of the perspective projection (minimal distance)
given by:
\[ \phi_0 = \arctan[(1-e^2)\tan\phi] \]

2 CENTRAL PROJECTION

2.1 Vertical Projection onto a Tangent Plane T (Minimal Distance)

From Fig.(4), we have for the point P:
\[ y_i = LP_o \tan\Delta = H \tan(\Delta - \Delta_o) \]
\[ \tan\Delta = \frac{\Delta X}{\Delta Z} \]
\[ \Delta_2 = \frac{\pi - \phi_0}{2} \]
So we get:
\[ y_i = H = \frac{\Delta X \sin\phi_o - \Delta Z \sin\phi_o}{\Delta Z \sin\phi_o + \Delta X \cos\phi_o} \]
(11)

Also
\[ x_y = y_i \cdot \frac{y_i}{P \cdot L P \sin\Delta_i} \]
where LP = ΔX/sinΔ. So we get
\[ x_i = H = \frac{\Delta Z \sin\phi_o + \Delta X \cos\phi_o}{\Delta Z \sin\phi_o + \Delta X \cos\phi_o} \]
(12)

To derive the mapping equation in terms of the geocentric coordinates (Φ,λ,h) of the point P, P_o, and L we have:

For the point P:
\[ X_p = \pi = (N + h) \cos\phi \cos\lambda \]
\[ Y_p = PS = (N + h) \cos\phi \sin\lambda \]
\[ Z_p = JO = [N(1-e^2) + h] \sin\phi \]
(13)

For the point P_o:
\[ X_p = MO = (N_o + h_o) \cos\phi \]
\[ Y_p = 0 \]
\[ Z_p = P_o M = [N_o (1-e^2) + h_o] \sin\phi \]
(14)

\[ X_o = RO = RM + MO = [H + h_o + N_o] \cos\phi - R_o \cos\phi \]
\[ Y_o = 0 \]
\[ Z_o = LR = [H + h_o + N_o (1-e^2)] \sin\phi_o = R_o \cos\phi \]
(15)

Substituting from the above geocentric coordinates of the points P, P_o, and L into equations (11) and (12), we obtain after some trigonometrical and algebraic reductions the mapping equations:

\[ x_i = \frac{H \cos\lambda}{G - \sin\phi_o - C \cos\phi_c \cos\lambda} \]
(16)
\[ y_i = \frac{F \cos\phi_c - C \sin\phi_c \cos\lambda}{G - \sin\phi_o - C \cos\phi_c \cos\lambda} \]
where
\[ C = \frac{N + h}{a} \]
\[ S = \frac{\pi}{a} \]
\[ F = \frac{e^2 N \sin\phi_c \cos\phi_o}{a} \]
\[ G = \frac{H + h_o + N_o (1-e^2 \sin^2 \phi_o)}{a} \]

After expanding the forms of sine and cosine of (λ,λ_o) and substituting into equations (16), we get the general mapping formulas of the vertical perspective projection of the rotational ellipsoid onto the tangent plane of minimal distance:

\[ x_i = \frac{H \cos\lambda \sin\lambda - C \sin\lambda \cos\lambda}{G - \sin\phi_o - C \cos\phi_c \cos\lambda - C \cos\phi_c \sin\lambda \sin\lambda} \]
(18)
\[ y_i = \frac{F \cos\phi_c - C \sin\phi_c \cos\lambda - \sin\phi_c \sin\lambda \sin\lambda}{G - \sin\phi_o - C \cos\phi_c \cos\lambda - C \cos\phi_c \sin\lambda \sin\lambda} \]
(19)

2.2 The mapping equations using geocentric latitude of the perspective center

The mapping equations (18) and (19) can also be modified if the geocentric latitude \( \phi_0 \) of the perspective center \( L \) is given as in the case of some satellite problems. So we have:

\[ \phi_o = (H + h_o + N_o) \cos\phi_o \]
\[ G = \frac{R_o \cos(\phi_0^0 - \phi_o)}{a} \]
(20)

The value of \( \phi_o \) may be found by a rapidly converging iteration, with an initial value of \( \phi_o = \phi_0 \) and using equation (20) to obtain \( R_o \). The new value \( R_o \) is used to obtain the next approximation of \( \phi_o \). The values of \( R_o \) and \( \phi_o \) are iterated until the change in \( \phi_o \) is considered negligible.
Fig (3) Dimensions on the rotational ellipsoid

Fig (4) Polar and equatorial views of the rotational ellipsoid with geometry form perspective center L
2.3 Vertical Projection onto a Tangent Plane $T_1$
(Maximal Distance)

In this case, the perspective center L is at a distance $LP_o=D$ from the tangent plane $T_1$, the relation between $\phi_o$ and $\phi'$ is given by the equation:

$$\phi' = \phi_o + \arcsin \left( \frac{e^2 \sin(2\phi_o)}{1 + \frac{H}{N_o}} \right)$$  \hspace{1cm} (21)

The normal distance $LP_o=D$ between the center L and the plane $T_1$ is given by:

$$D = N_o + \frac{(N_o + H) \cos \phi_o}{\cos \phi'}$$  \hspace{1cm} (22)

The mapping equations can be obtained by the same method explained above. The analogy of equations (16) gives the mapping coordinates relating to the new system:

$$x_i' = \frac{C_s \sin \lambda}{G' - S \sin \phi' - C \cos \phi' \cos \delta \lambda}$$

$$y_i' = \frac{F' + S \cos \phi' - C \sin \phi' \cos \delta \lambda}{G' - S \sin \phi' - C \cos \phi' \cos \delta \lambda}$$ \hspace{1cm} (23)

where

$$F' = e^2 N_o \sin \phi_o \cos \phi' - \frac{(H + h_o + N_o) \sin \Delta \phi}{a}$$

$$G' = -e^2 N_o \sin \phi_o \sin \phi' + \frac{(H + h_o + N_o) \cos \Delta \phi}{a}$$ \hspace{1cm} (24)

C and S are determined from equation (17).

2.4 Inverse Mapping Equations

The inverse mapping equations for the vertical perspective projection of the ellipsoid permit the calculation of geodetic latitude $\phi$ and longitude $\lambda$ for a selected point which has rectangular coordinates $x_i$, and $y_i$.

2.4.1 Geodetic Latitude $\phi$ : Since the mapping equations (18),(19) are linear in $\sin \lambda$ and $\cos \lambda$, then they can be arranged as follows:

$$A_1 \sin \lambda + A_2 \cos \lambda = A_3 S + A_4$$

$$A_5 \sin \lambda + A_6 \cos \lambda = A_7 S + A_8$$ \hspace{1cm} (25)

By eliminating $\lambda$ from equations (25) we get a quadratic equation in $S$, which has two solutions. One of them gives the latitude of the nearest point on the ellipsoid, the other of the hidden point on the other side. Since $Z>0$ then the positive root is considered. So we have:

$$\sin \phi = \frac{a S}{N(1 - e^2) + h}$$ \hspace{1cm} (26)

Then by using an iteration method we get $\phi$.

2.4.2 Geodetic Longitude $\lambda$: After some trigonometric calculations equations (25) can be reduced to:

$$\lambda = \lambda_o + \arctan \left( \frac{x_i (T \sin \phi_o - S)}{S(y_i \sin \phi_o + H \cos \phi_o) - (y_i G - HF)} \right)$$ \hspace{1cm} (27)

where: $T = [H + h_o + N_o(1 - e^2)]/a$

References


