3D RIGID TRANSFORMATION

Luigi Mussio, Grazia Tucci

Politecnico di Milano
Dipartimento di Ingegneria Idraulica, Ambientale e del Rilevamento
P.zza L. da Vinci, 32 - 20133 Milano - Italia

ISPRS Commission VI, Working Group 3

KEY WORDS: three dimensional GIS, object modeling, topological spatial relation, transformation algorithms.

ABSTRACT

This paper, after an introduction to 3D GIS, deals with 3D models and 3D topology. A special attention is paid to 3D rigid transformations, required by any 3D model.

1. INTRODUCTION

Commercial GISs use space reference and consider elevation data as an attribute.
Indeed surface reconstruction opens partially the GISs to a new dimensions. In fact elevations are treated as quantitative data, but topology remains in 2D. A full achievement of the third dimension is gained by transferring topology from 2D to 3D.
The way towards 3D GISs is remarkable, because 3D GISs permit to manage morphologically complex objects. Some examples are given by oil exploration, idrogeology, geological modeling, environmental monitoring, civil engineering, etc.

2. 3D MODELS (Molenaar, 1994)

There are different approaches in the modeling of morphologically complex objects contained in a GIS:

- object oriented,

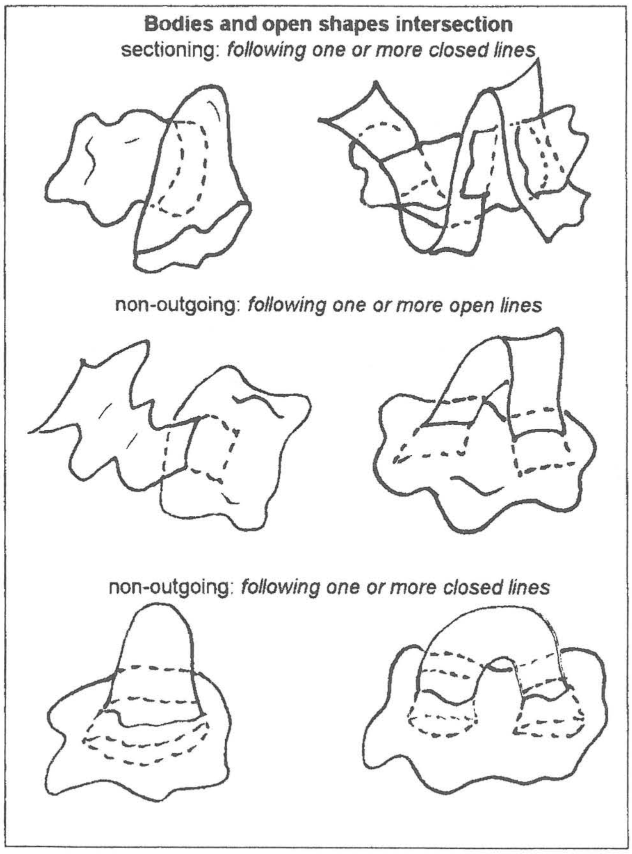

Figure 1. *Examples of topological relationship among spatial objects.*

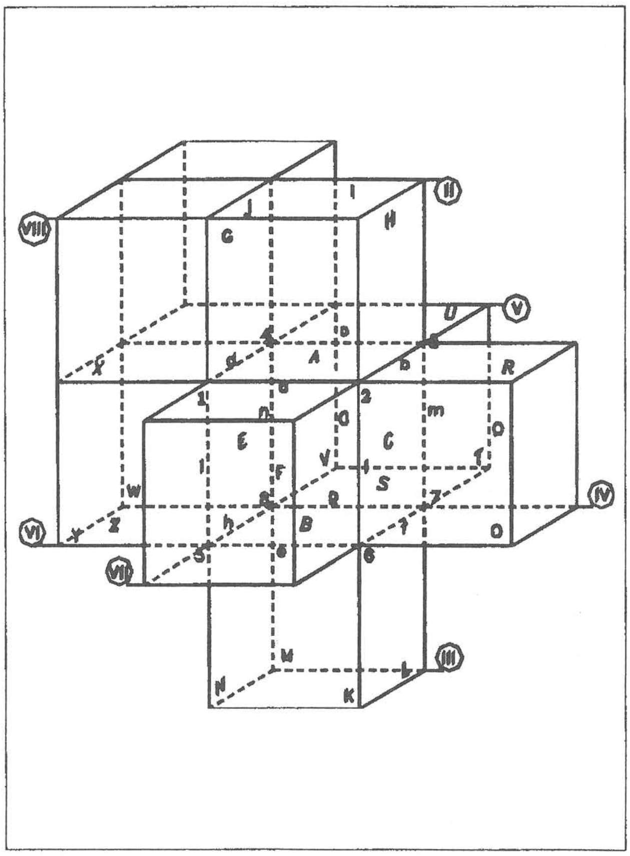

Figure 2. *Sample of raster modelling.*

- vector modeling
- raster modeling represents different models at different levels of complexity.

Object oriented approach is fashinating, because it is able to fully understand fully the form of objects and the interaction among thems. However it is a complex matter in 2D and becomes, more and more, complex in 3D: an approximate calculation of possible cases overcomes two hundred.

Figure 1 shows some representative examples of interaction between bodies and shapes.

On the other hand, raster modeling simplifies too much the problem. The voxels (cube elements occupying regularly the space) waste memory repeating identically useless information, because they are unable to recognise features. The same is true for the pixels in 2D, but the waste is obviously much bigger in 3D.

Figure 2 shows an example of raster modeling.

A reasonable compromise between the two above mentioned approaches is given by vector modeling, assuming a linear approximation of space elements. In this case, points (or vertices), segments (or edges) polygons (or faces) and polyhedrons describe the elements of the space.

In addition some rules permit to study morphologically complex objects from their topology point of view.

- Non-straight lines get separated in segments connected by non characteristic points.
- Non-plane shapes get separated in polygons connected by non characteristic segments and points.
- Non-polyhedral bodies get separated in polyhedrons connected by non characteristic polygons.

Figure 3 shows a view of a complex object and its representation by structural elements.

3. 3D TOPOLOGY (Molenaar, De Hoop, 1994)

An explanation of 3D topology can be presented independently on the chosen approach.

Assuming points, lines, shapes and bodies as primary elements, the following features are defined to better describe the same elements.

- Points haven't features.
- Lines are supposed continuous, open or closed (in the second case, they aren't simply connected, but a single hole is expected only).
- Shapes are supposed again continuous, open or close (in the second case, they aren't simply connected, but a single hole is expected only).
- Bodies are supposed to be continuous and open.

Notice that all elements are considered finite ; furthermore

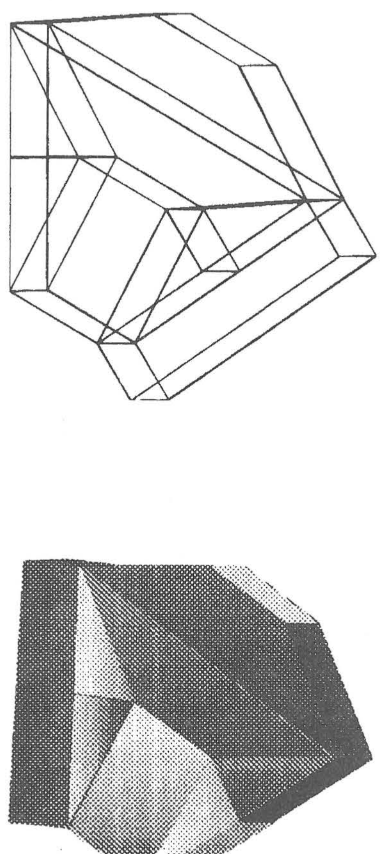

Figure 3. *A view of a complex object.*

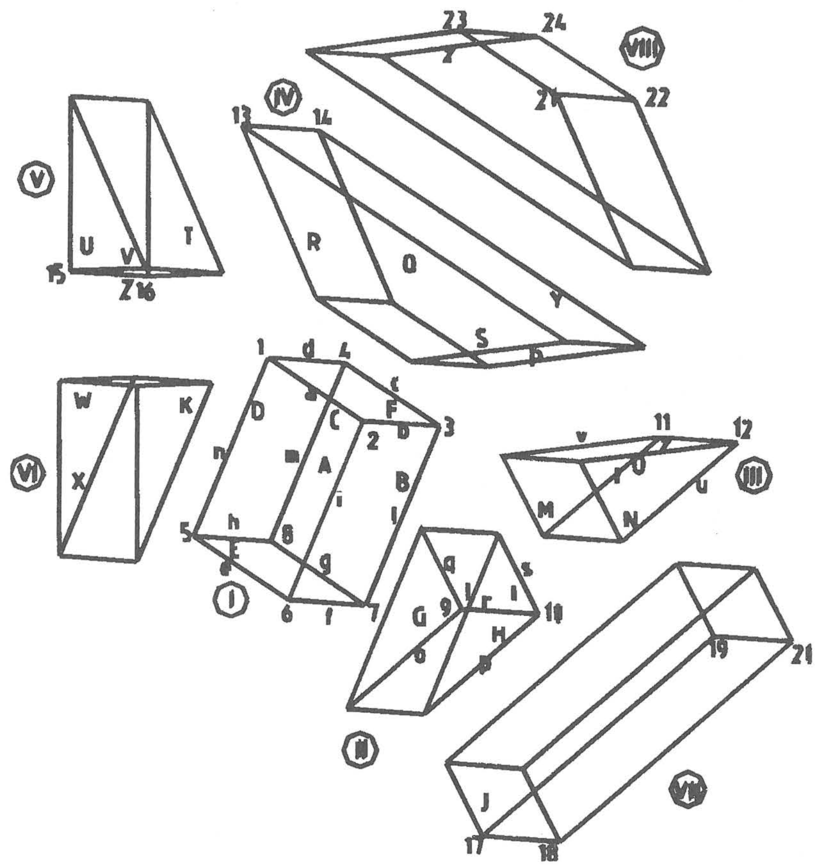

Figure 4. *An exploded view of the structural elements of same object.*

lines have fractal dimension (i.e. occupied space) equal to 1 and shapes have fractal dimension equal to 2. Some famous irregular examples (e.g. Moebius' band) are neglected, because they are useless in this context.
The interaction among the above mentioned primary elements supplies the following list of relationship :

- between bodies,
- among bodies and shapes,
- among bodies and lines,
- among bodies and points,
- between shapes,
- among shapes and lines,
- among shapes and points,
- between lines,
- among lines and points,
- between points.

Figure 4 shows an exploded view of the structural elements of the object presented in figure 3.
Three matrices contain all the topological information.

- Transposed incident matrix transmits information about edges and terminal vertices of each edge.
- Adjacent matrix transmits information about faces and adjacent polyhedrons of each face.
- Cross connection matrix links edges and adjacent faces in a table, which can be compressed by labels and pointers.

Figures 5, 6, 7 illustrate the above mentioned matrices related to the example of figure 4.
The other above mentioned relationships are expressed by direct and transposed or square matrices, which can be

Edges	Terminal vertexes	
a	1	2
b	2	3
c	3	4
d	1	4
e	5	6
f	6	7
g	7	8
h	8	5
i	6	2
l	7	3
m	8	4
n	5	1
o	6	9
p	7	10
q	2	9
r	9	10
s	3	10
t	9	11
u	10	12
v	2	11
z	3	12
....	....	....

Figure 5. *Transposed incidence matrix*

Faces	Adjacent Polyhedron	
A	I	external
B	I	II
C	I	external
D	I	VI
E	I	external
F	I	IV
G	II	external
H	II	VII
I	II	external
L	II	III
M	III	external
N	III	VII
O	III	external
P	III	IV
Q	IV	external
R	IV	V
....	....	....

Figure 6. *Adjacent matrix*

	Edges - Adjacent Faces															
	a	b	c	d	e	f	g	h	i	l	m	n	o	p	q	r
A	1	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0
B	0	1	0	0	0	1	0	0	1	1	0	0	0	0	0	0
C	0	0	1	0	0	0	1	0	0	1	1	0	0	0	0	0
D	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	0
E	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
F	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
G	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0
H	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	1
I	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
L	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1

Figures 7. *Cross-connection matrix related to the examples of figure 4.*

1	Identity		12345678
	Rotation around the faces centers		
	rotation around the z axis		
2	$\pi/2$	rotation	41238567
3	π	rotation	34127856
4	$3/2\pi$	rotation	23416785
	rotation around the x axis		
5	$\pi/2$	rotation	43781265
6	π	rotation	87654321
7	$3/2\pi$	rotation	56218734
	rotation around the y axis		
8	$\pi/2$	rotation	26731584
9	π	rotation	65872143
10	$3/2\pi$	rotation	51486237
	Rotation around the edges midpoints		
11			73268415
12			48513762
13			78436512
14			21563487
15			76583214
16			58761432
	Rotation around the vertices		
17			15624873
18			14852376
19			67325841
20			84375126
21			32674158
22			62157348
23			37842651
24			85147623

All elements are distinct!!!

Figure 8. *S8 Rigid subgroup.*

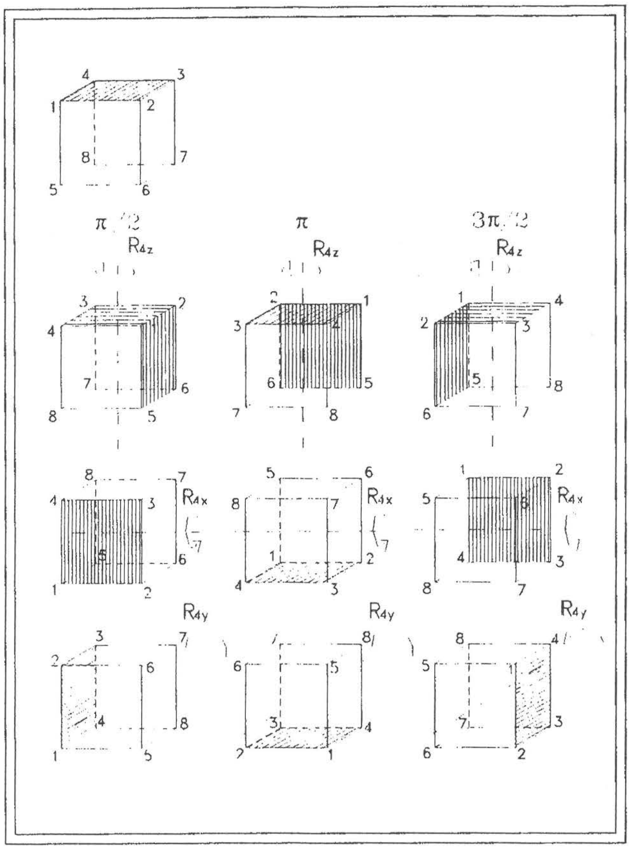

Figure 9a. *Rotation around the face centers.*

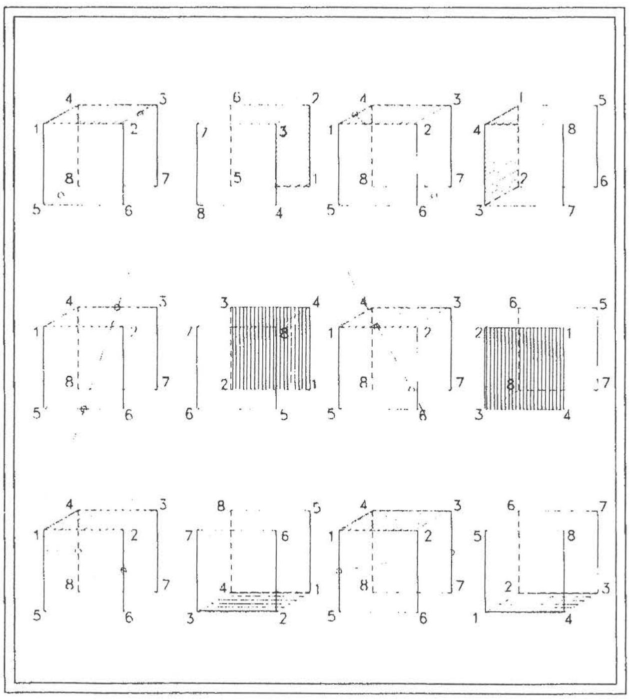

Figure 9b. *Rotation around the edge midpoints.*

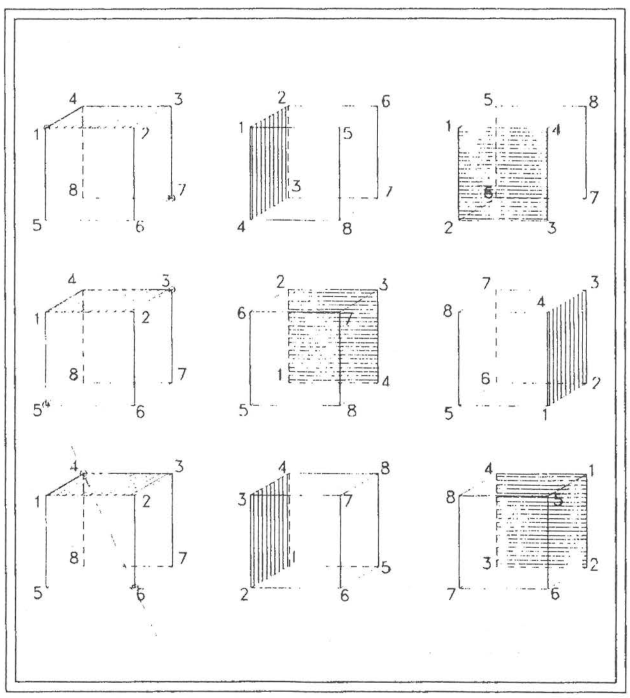

Figure 9c. *Rotation around the vertices.*

obtained easily starting from the three fundamental matrices. More information isn't supplied, but it could appear user friendly.

4. 3D RIGID TRANSFORMATION

Body definition depends on its form and dimension. Body identification depends on its coordinates ; changing from dimensions to coordinates needs a reference frame.

Referring to a reference frame, but for translations and short rotations, bodies are unvaried in matter of symmetry in reference frame space partitions.

- Coordinates and dimensions are identical in the first octant.
- The bodies in the other octant are brought to the first octant by rigid space transformations, according to the symmetry relations contained in the S8 rigid subgroup.

The S8 group involves the permutations of 8 letters and has 8! elements.

There are many different subgroups of the S8 group ; the S8 rigid subgroup has 24 elements presented in figure 8 and represented in figure 9.

The closure of the composition table of the S8 rigid subgroup proofs that this last is a group (see figure 10).

Lets remember that, in algebra, a group (an abelian group) is called one set which owns a composition rule, usually called product (sum), satisfying the following postulates.

- The obtained element, operating by means of the composition rule to two whatever elements in the group, is internal to the same group.
- The composition rule joins the associative property.
- In the group it exists a neutral element which works as a unit (zero), referring to the product (sum).
- Each element in the group has one inverse (opposite) element.

In other words, the group is closed referring to the product (sum).

	I	R4									T2						S3							
		x			y			z			x	x	y	y	z	z	x y z		x y (-z)		x (-y) z		x (-y) (-z)	
		1	2	3	1	2	3	1	2	3	y	-y	z	-z	x	-x	1	2	1	2	1	2	1	2
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	2	3	4	1	20	16	17	21	15	24	19	18	23	22	6	9	14	10	8	13	11	7	5	12
3	3	4	1	2	13	9	14	11	6	12	8	10	5	7	16	15	22	24	21	23	19	17	20	18
4	4	1	2	3	23	15	22	19	16	18	21	24	20	17	9	6	7	12	11	5	8	14	13	10
5	5	21	14	18	6	7	1	23	13	20	22	17	3	9	24	19	8	16	4	11	15	10	12	2
6	6	15	9	16	7	1	5	12	3	11	10	8	14	13	2	4	23	19	18	22	24	20	17	21
7	7	24	13	19	1	5	6	17	14	22	20	23	9	3	21	18	12	4	16	10	2	11	8	15
8	8	17	12	23	21	11	19	9	10	1	3	6	24	18	22	20	16	5	13	2	14	4	15	7
9	9	16	6	15	14	3	13	10	1	8	12	11	7	5	4	2	20	21	24	17	18	23	22	19
10	10	20	11	22	18	12	24	1	8	9	6	3	19	21	23	17	2	14	7	16	5	15	4	13
11	11	22	10	20	19	8	21	6	12	3	1	9	18	24	17	23	15	13	5	4	7	2	16	14
12	12	23	8	17	24	10	18	3	11	6	9	1	21	19	20	22	4	7	14	15	13	16	2	5
13	13	19	7	24	9	14	3	20	5	23	17	22	1	6	18	21	11	15	2	8	16	12	10	4
14	14	18	5	21	3	13	9	22	7	17	23	20	6	1	19	24	10	2	15	12	4	8	11	16
15	15	9	16	6	22	4	23	24	2	21	18	19	20	20	1	3	13	11	12	14	10	5	7	8
16	16	6	15	9	17	2	20	18	4	19	24	21	22	23	3	1	5	8	10	7	12	13	14	11
17	17	12	23	8	2	20	16	14	22	7	13	5	15	4	11	10	18	1	9	24	3	19	21	6
18	18	5	21	14	12	24	10	4	19	16	15	2	11	8	13	7	1	17	22	6	23	9	3	20
19	19	11	22	10	16	17	2	5	23	13	7	14	4	15	12	8	21	9	19	1	6	24	18	3
20	20	7	24	13	8	21	11	16	18	4	2	15	10	12	14	5	6	23	1	20	17	3	9	22
21	21	14	18	5	11	19	8	15	24	2	4	16	12	10	7	13	9	20	7	3	22	1	6	17
22	22	10	20	11	4	23	15	7	17	14	5	13	16	2	8	12	24	3	8	18	1	21	19	9
23	23	8	17	12	15	22	4	13	20	5	14	7	2	16	10	11	19	6	3	21	9	18	24	1
24	24	13	19	7	10	18	12	2	21	15	16	4	8	11	5	14	3	22	17	9	20	6	1	24

Figure 10. *8 Group Table Composition.*

REFERENCES

Molenaar, M.,1994. *Remote Sensing and Geographical Information Processing. Concepts and Application for Land Use Monitoring and Agricolture.* Wageningen Agricultural University.

Molenaar, M., De Hoop, S., 1994. *Advenced Geographic Data Modelling. Spatial Data Modelling and Query Languages for 2D and 3D Applications. Netherlands Geodetic Commissions.* Publications on Geodesy New Series. Number 40.